Nonvanishing criteria for local h-polynomials

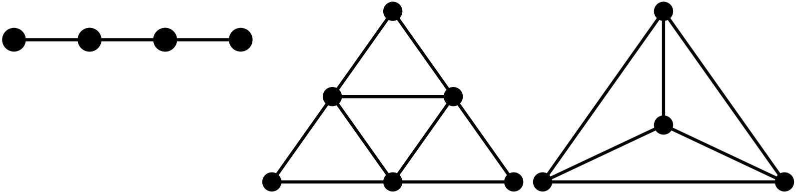
Matt Larson

Joint work with Sam Payne and Alan Stapledon

arxiv: 2209.03553 and 2209.03543

September 17, 2022

Triangulations of simplices



- A triangulation of a simplex gives a map $\sigma \colon \Gamma \to 2^V$.
- For each subset S of V, we can restrict to obtain a triangulation $\sigma|_S \colon \Gamma|_S \to 2^S$.

Local h-polynomial

Definition (Local h-polynomial)

For a triangulation of a simplex $\sigma\colon \Gamma \to 2^V$, define the *local h-polynomial* $\ell(\Gamma, t) = \ell_0 + \ell_1 t + \cdots$ by

$$h(\Gamma, t) = \sum_{S \subseteq V} \ell(\Gamma|_S, t),$$

and $\ell(\Gamma|_\emptyset, t) = 1$.

- We have the formula

$$\ell(\Gamma, t) = \sum_{G \in \Gamma} (-1)^{|V|-|G|} t^{|V|-|\sigma(G)|+|G|} (t-1)^{|\sigma(G)|-|G|}.$$

Properties

Theorem (Stanley 1992)

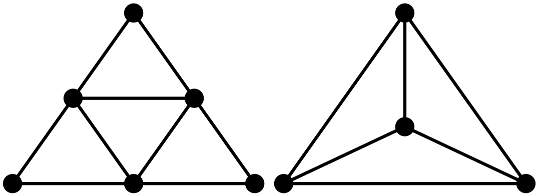
Let $\sigma\colon \Gamma \to 2^V$ be a triangulation of a simplex. Then coefficients of $\ell(\Gamma, t)$ are

- symmetric, i.e., $t^{|V|}\ell(\Gamma, t^{1/|V|}) = \ell(\Gamma, t)$,
- nonnegative, and
- if Γ is a *regular triangulation*, then the coefficients are unimodal.

Examples

$$\ell(\Gamma, t) = \sum_{G \in \Gamma} (-1)^{|V|-|G|} t^{|V|-|\sigma(G)|+|G|} (t-1)^{|\sigma(G)|-|G|}.$$

- If $V \neq \emptyset$, then $\ell_0 = 0$ and ℓ_1 is the number of interior vertices (vertices v of Γ with $\sigma(v) = V$).
- If there are no interior vertices, then ℓ_2 is the number of interior edges minus the number of vertices v of carrier codimension 1.



Applications

Applications of local *h*-polynomials

- Local *h*-polynomials control how the *h*-polynomial of a simplicial *complex* changes under refinement (Stanley 1992).
- Local *h*-polynomials appear when applying the decomposition theorem to a proper toric morphism (Katz-Stapledon 2016, de Cataldo-Migliorini-Mustață 2018).
- Local *h*-polynomials appear when computing the eigenvalues of the monodromy action on the cohomology of the Milnor fiber of a Newton-nondegenerate singularity (Stapledon 2017).
- Local *h*-polynomials have applications to other combinatorial polynomials (Athanasiadis).

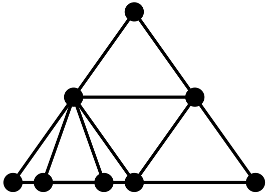
- In the first three applications, it is important to know when local *h*-polynomials vanish.
- In the last three applications, a relative version is necessary.

Prior work on nonvanishing

Theorem (de Moura-Gunther-Payne-Stapledon-Schuchardt 2020)

Let $\sigma \colon \Gamma \to 2^V$ be a triangulation with $\ell(\Gamma, t) = 0$. Assume $\ell_1 = \ell_2 = 0$. Then:

- The *interior edge graph* of Γ is either contractible, or $|V| = 3$ and it has a single cycle.
- If $|V| \leq 4$, then there is an explicit construction of all triangulations with $\ell(\Gamma, t) = 0$.



Local face modules

- For $\theta = \sum_{v \in \Gamma} a_v x^v \in k[\Gamma]$ of degree 1,

$$\mathrm{supp}(\theta) = \{v \colon a_v \neq 0\}.$$

- A linear system of parameters (l.s.o.p.) $\theta_1, \ldots, \theta_{|V|}$ is *special* if for all $v \in \mathrm{supp}(\theta_i)$, $i \in \sigma(v)$. Special l.s.o.p.s exist when k is infinite.
- Given a choice of a special l.s.o.p. $\theta_1, \ldots, \theta_{|V|}$, the *local face module* $L(\Gamma)$ is the image of the ideal of interior faces in $k[\Gamma]/(\theta_1, \ldots, \theta_{|V|})$.

Theorem (Stanley 1992)

The Hilbert series of $L(\Gamma)$ is $\ell(\Gamma, t)$.

Presentation of local face module

Theorem (L.-Payne-Stapledon 2022)

Let $\sigma\colon \Gamma \to 2^V$ be a triangulation, and let $I \subset k[\Gamma]$ be the ideal of interior faces. Then there is an exact sequence

$$K \to I \to L(\Gamma) \to 0,$$

where K is the ideal generated by

$$\left\{\theta_i \cdot x^F : F \text{ is interior } \right\} \cup \left\{\theta_j \cdot x^G : \sigma(G) = V \setminus j\right\}.$$

- We have an explicit resolution of $L(\Gamma)$.
- For any subcomplex $\Gamma' \subset \Gamma$, we have a short exact sequence

$$K \otimes_{k[\Gamma]} k[\Gamma'] \to I \otimes_{k[\Gamma]} k[\Gamma'] \to L(\Gamma) \otimes_{k[\Gamma]} k[\Gamma'] \to 0.$$

Nonvanishing of local h-polynomials

Obstruction to vanishing of $\ell(\Gamma, t)$

Suppose that $V = \{1,2,3,4,5,6\}$, and let $\sigma \colon \Gamma \to 2^V$ be triangulation with a facet $F = \{w_1, \dots, w_6\}$ such that

$$\begin{array}{lll} \sigma(w_1) = \{1\} & \sigma(w_2) = \{2\} & \sigma(w_3) = \{3\} \\ \sigma(w_4) = \{1,4,5\} & \sigma(w_5) = \{2,4,6\} & \sigma(w_6) = \{3,5,6\} \end{array}$$

Then the interior 2-faces of F are $\{w_1, w_5, w_6\}$, $\{w_2, w_4, w_6\}$, $\{w_3, w_4, w_5\}$, and $\{w_4, w_5, w_6\}$. But F has no interior vertices or edges, and it has only three edges with carrier codimension one, namely $\{w_4, w_5\}, \{w_4, w_6\}$, and $\{w_5, w_6\}$. Thus $L(\Gamma)$ is non-zero in degree three.

- We have a short exact sequence

$$K \otimes_{k[\Gamma]} k[F] \to I \otimes_{k[\Gamma]} k[F] \to L(\Gamma) \otimes_{k[\Gamma]} k[F] \to 0.$$

Nonvanishing of local h-polynomials

- Can also attempt to show that $L(\Gamma)$ is nonzero by finding a nonzero map out of $L(\Gamma)$.
- For certain special l.s.o.p.s, the ring $k[\Gamma]/(\theta_1, \ldots, \theta_{|V|})$ is the cohomology ring of a (non-compact) toric variety.
- For each subvariety Y, we have a map $k[\Gamma]/(\theta_1, \ldots, \theta_{|V|}) \to H^*(Y; k)$.

Nonvanishing of local h-polynomials

Theorem (L.-Payne-Stapledon)

Let $\sigma \colon \Gamma \to 2^V$ be a triangulation, and suppose $F = G \sqcup G'$ is a facet of Γ such that $\sigma(G) = \sigma(G') = 2^V$. Then $\ell(\Gamma, t)$ is nonzero.

- Because G is interior, the subvariety corresponding to G is a compact toric variety.
- The image of $x^{G'}$ is nonzero in the cohomology ring of the subvariety corresponding to G.

Applications to eigenvalues of monodromy

- For a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$, the *Igusa p-adic zeta function* $Z_p(s)$ is a rational function in p^{-s} that counts the number of solutions to $f \equiv 0 \pmod{p^k}$.
- The eigenvalues of the monodromy action on the Milnor fiber of $V(f)$ are invariants of the singularities of $V(f)$.

p-adic monodromy conjecture (Denef, Igusa 1980s)

If α is a pole of $Z_p(s)$ for p sufficiently large, then $e^{2\pi i \Re(\alpha)}$ is an eigenvalue of monodromy.

- If f is *Newton-nondegenerate*, then there are combinatorial formulas for both the p-adic zeta function and the eigenvalues of monodromy.
- The formula for the eigenvalues of monodromy involve local h-polynomials.

Applications to eigenvalues of monodromy

Theorem (L.-Payne-Stapledon)

If f is nondegenerate with respect to a simplicial Newton polyhedron, then the p-adic monodromy conjecture is true for f.

- If a certain local h-polynomial is nonzero, then we produce eigenvalues of monodromy.
- If the local h-polynomial vanishes, then get control over the combinatorics of the Newton polyhedron and are able to show cancellation in the p-adic zeta function.

Thank you!

arxiv: 2209.03553 and 2209.03543