Algebraic geometry of delta-matroids

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Maximal isotropic subspaces

- Let $q(x_1, \dots, x_n, x_{\bar{1}}, \dots, x_{\bar{n}}, x_0) = x_1 x_{\bar{1}} + \dots + x_n x_{\bar{n}} + x_0^2$ be the standard quadratic form on k^{2n+1}. Let $L \subset k^{2n+1}$ be a maximal isotropic subspace.
- A maximal isotropic subspace defines a point of the *maximal orthogonal Grassmannian*, $OGr(n; 2n+1)$.
- A point of $OGr(n; 2n+1)$ is determined by its Plücker coordinates corresponding to subsets of $[n, \bar{n}]$ not containing $\{i, \bar{i}\}$, called *maximal admissible subsets*.
- The $\mathbb{G}_m^n$ action on k^{2n+1} by

  $$(t_1, \dots, t_n) \cdot (x_1, \dots, x_0) = (t_1 x_1, \dots, t_n x_n, t_1^{-1} x_{\bar{1}}, \dots, t_n^{-1} x_{\bar{n}}, x_0)$$

  induces an action of $\mathbb{G}_m^n$ on $OGr(n; 2n+1)$.

Torus-orbit closures

- A maximal isotropic subspace $L \subset k^{2n+1}$ gives a point in $OGr(n; 2n+1)$, and we may consider the torus-orbit closure $\overline{\mathbb{G}_m^n \cdot [L]} \subset OGr(n; 2n+1)$.
- The normalization of $\overline{\mathbb{G}_m^n \cdot [L]}$ is a projective toric variety, so its fan is the normal fan of its moment polytope.

Proposition (Gelfand-Serganova)

The moment polytope $P(L)$ of $\overline{\mathbb{G}_m^n \cdot [L]}$ is the convex hull of the points in $\{-1, 1\}^n$ corresponding to nonzero maximal admissible Plücker coordinates of L. All edges of the moment polytope are parallel to e_i or $e_i \pm e_j$.

- Let

$$L = \mathrm{rowspan}\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then $P(L) = \mathrm{Conv}(-e_1 - e_2, e_1 + e_2)$.

Delta-matroids

Definition (delta-matroid)

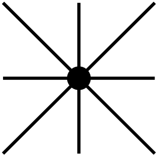
A *delta-matroid* is a polytope whose vertices are contained in $\{-1, 1\}^n$ such that all edges are parallel to e_i or $e_i \pm e_j$ for some i, j.

- Each maximal isotropic $L \subset k^{2n+1}$ *realizes* a delta-matroid $P(L)$.
- There are many other constructions of delta-matroids.

The B_n permutohedral variety

Definition (B_n permutohedral variety)

The B_n *permutohedral variety*, X_{B_n}, is the toric variety whose fan is cut out by the hyperplanes normal to the type B_n roots.



- The normal fan of any delta-matroid is a coarsening of the B_n fan.

Isotropic tautological bundles

Definition (Isotropic tautological bundle)

Give $\mathcal{O}_{X_{B_n}}^{\oplus 2n+1}$ the $\mathbb{G}_m^n$-linearization where

$$(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_0) = (t_1 x_1, \ldots, t_n x_n, t_1^{-1} x_{\bar{1}}, \ldots, t_n^{-1} x_{\bar{n}}, x_0).$$

For $L \subset k^{2n+1}$ a maximal isotropic subspace, define $\mathcal{I}_L$ to the unique $\mathbb{G}_m^n$-equivariant subbundle of $\mathcal{O}_{X_{B_n}}^{\oplus 2n+1}$ whose fiber over the identity is $L \subset k^{2n+1}$.

- $\mathcal{I}_L$ is an anti-nef vector bundle.

Interlace polynomial

Definition (interlace polynomial)

Let P be a delta-matroid. Then

$$\mathrm{Int}_P(v) = \sum_{x \in \{-1,1\}^n} v^{d(x,P)/2}.$$

- The polynomial $\mathrm{Int}_P(v-1)$ has nonnegative coefficients.
- If the delta-matroid is obtained from a matroid, the interlace polynomial is a specialization of the Tutte polynomial.

Log-concavity

Theorem (Eur-Fink-L.-Spink)

Let $L \subset k^{2n+1}$ be a maximal isotropic subspace. Then the coefficients of $(v+1)^n \operatorname{Int}_{P(L)}(\frac{v-1}{v+1})$ are nonnegative, unimodal, and form a log-concave sequence.

- A sequence $a_1, a_2, \ldots, a_n$ of nonnegative integers is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$.

Log-concavity

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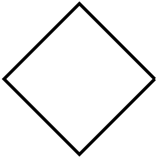
- A sequence $a_1, a_2, \ldots, a_n$ of nonnegative integers is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$.

Definition (cross polytope)

The *cross polytope* is

$$\Diamond_n = \operatorname{Conv}(\pm e_i : i \in [n]).$$



Proof of log-concavity

Theorem (Eur-Fink-L.-Spink)

Let $L \subset k^{2n+1}$ be a maximal isotropic subspace. We have that

$$(v+1)^n \operatorname{Int}_{P(L)}\left(\frac{v-1}{v+1}\right) = \sum_k v^k \int_{\mathbb{P}_{X_{B_n}}(\mathcal{I}_L)} c_1(\mathcal{O}(1))^{2n-1-k}[\Diamond_n]^k$$

- Because $\mathcal{I}_L \subset \mathcal{O}_{X_{B_n}}^{\oplus 2n+1}$,

$$\mathbb{P}_{X_{B_n}}(\mathcal{I}_L) \subset \mathbb{P}_{X_{B_n}}(\mathcal{O}_{X_{B_n}}^{\oplus 2n+1}) = \mathbb{P}^{2n} \times X_{B_n},$$

  so $\mathcal{O}(1)$ is nef.
- Log-concavity follows from the Khovanskii–Teissier inequality.

Beyond realizable delta-matroids

- We can describe the $\mathbb{G}_m^n$-equivariant K-class $[\mathcal{I}_L]$ solely in terms of $P(L)$.
- For any delta-matroid P, we define an equivariant K-class $[\mathcal{I}_P] \in K_{\mathbb{G}_m^n}(X_{B_n})$ such that $[\mathcal{I}_L] = [\mathcal{I}_{P(L)}]$.

Enveloping matroid

Definition (matroid)

A *matroid* is a polytope whose vertices are contained in $\{0, 1\}^n$ such that all edges are parallel to $e_i - e_j$ for some i, j.

Definition (enveloping matroid)

Let $\mathrm{env}\colon \mathbb{R}^{2n} \to \mathbb{R}^n$ be the map

$$\mathrm{env}(x_1, \ldots, x_n, x_{\bar{1}}, \ldots, x_{\bar{n}}) = (x_1 - x_{\bar{1}}, \ldots, x_n - x_{\bar{n}}).$$

A matroid M is an *enveloping matroid* of a delta-matroid P if $\mathrm{env}(M) = P$.

- Realizable delta-matroids have enveloping matroids.
- Not all delta-matroids have enveloping matroids.

Theorem (Eur-Fink-L.-Spink)

Let P be a delta-matroid in $\mathbb{R}^n$ which has an enveloping matroid. Then the coefficients of $(v+1)^n \operatorname{Int}_P(\frac{v-1}{v+1})$ are nonnegative, unimodal, and form a log-concave sequence.

- We conjecture that the hypothesis that P has an enveloping matroid can be removed.

Proof of log-concavity

$$(v+1)^n \operatorname{Int}_{P(L)}\left(\frac{v-1}{v+1}\right) = \sum_k v^k \int_{\mathbb{P}^{2n}\times X_{B_n}} [\mathbb{P}_{X_{B_n}}(\mathcal{I}_L)] c_1(\mathcal{O}(1))^{2n-1-k}[\Diamond_n]^k.$$

- We have

$$[\mathbb{P}_{X_{B_n}}(\mathcal{I}_L)] = \sum_{i=0}^{n} c_i(\mathcal{I}_L^\vee) c_1(\mathcal{O}(1))^{n+1-i} \in H^\bullet(\mathbb{P}^{2n}\times X_{B_n}).$$

- For any delta-matroid P, we define

$$[\mathbb{P}_{X_{B_n}}(\mathcal{I}_P)] := \sum_{i=0}^{n} c_i([\mathcal{I}_P]^\vee) c_1(\mathcal{O}(1))^{n+1-i} \in H^\bullet(\mathbb{P}^{2n}\times X_{B_n}).$$

- When P has an enveloping matroid M, results of Adiprasito–Huh–Katz and Ardila–Denham–Huh can be used that show that $[\mathbb{P}_{X_{B_n}}(\mathcal{I}_P)]$ has Hodge-theoretic properties resembling those of an integral subvariety.

Other results

- We describe $K(X_{B_n})$ in terms of delta-matroids.
- We construct an integral isomorphism $K(X_{B_n}) \to H^\bullet(X_{B_n})$ that satisfies a Hirzebruch–Riemann–Roch-type formula.
- We construct a nef and extremal basis of $H^\bullet(X_{B_n})$.
- We give formulas for the volume and lattice point enumerators of every B_n generalized permutohedron.
- We introduce a Tutte-like invariant of delta-matroids $U_P(u, v)$ which has $U_P(0, v) = \mathrm{Int}_P(v)$.
- We introduce another family of vector bundles associated to certain delta-matroids.
- We prove several specializations of $U_P(u, v)$ have log-concavity properties when P has an enveloping matroid, including the log concavity of the sequence

$$|\{T \subseteq S\colon T \text{ independent in } M \text{ and } S \text{ spanning in } M,\ |S| - |T| = i\}|.$$

Thank you!

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