Kapranov degrees

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Pullbacks of psi classes

- For each $i \in \{1, \ldots, n\}$, set $\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{M}_{0,n})$ to be the first Chern class of the ith cotangent line bundle.
- For $S \subset \{1, \ldots, n\}$ with $|S| \geq 3$, let $f_S \colon \overline{M}_{0,n} \to \overline{M}_{0,S}$ be the forgetful map. We set

$$X_{S,i} := f_S^* \psi_i \quad \text{for } S \ni i.$$

- The **Kapranov degrees** are the intersection numbers

$$\int_{\overline{M}_{0,n}} X_{S_1,i_1} \cdots X_{S_{n-3},i_{n-3}}.$$

- Each $\mathbb{L}_i$ is globally generated, and its complete linear system induces a birational map $\overline{M}_{0,n} \to \mathbb{P}^{n-3}$, the *Kapranov map*.
- Kapranov degrees are the multidegrees of the embedding

$$\overline{M}_{0,n} \hookrightarrow \prod_{S,i} \mathbb{P}^{|S|-3}.$$

Basic properties

- Permutation invariance
- Because $X_{S,i} := f_S^* \psi_i$ is nef, Kapranov degrees are non-negative.
- Because $X_{S \cup T,i} - X_{S,i}$ is effective,

$$\int_{\overline{M}_{0,n}} X_{S_1,i_1} \cdots X_{S_{n-3},i_{n-3}} \leq \int_{\overline{M}_{0,n}} X_{S_1 \cup T_1,i_1} \cdots X_{S_{n-3} \cup T_{n-3},i_{n-3}}.$$

Examples

- Witten:
$$\int_{\overline{M}_{0,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} = \binom{n-3}{a_1, \ldots, a_n}.$$
- If $|S| = 4$, then $f_S \colon \overline{M}_{0,n} \to \overline{M}_{0,4} = \mathbb{P}^1$ is a cross-ratio map, and $X_{S,i} = X_S$ does not depend on the choice of i. If $|S_i| = 4$ for all i, then
$$\int_{\overline{M}_{0,n}} X_{S_1} \cdots X_{S_{n-3}} = \text{ the degree of } \overline{M}_{0,n} \to (\mathbb{P}^1)^{n-3}.$$
This is called the **cross-ratio degree**. Studied by Arkani-Hamed–He–Lam–Thomas, Gallet–Grasegger–Schicho, Goldner, Silversmith.

Examples

- For each k, ℓ, there is a birational map $\overline{M}_{0,n} \to \overline{LM}_n$ such that $X_{S,k}, X_{S,\ell}$ are pulled back if $\{k, \ell\} \subset S$.
- If $i_j \in \{k, \ell\}$ and $\{k, \ell\} \subset S_j$ for all j, then the Kapranov degree can be computed on $\overline{LM}_n$. $\overline{LM}_n$ is a toric variety, and the Kapranov degree is the mixed volume of some simplices. Studied by Postnikov, Eur–Fink–L.–Spink.
- The Kapranov degree

$$\int_{\overline{M}_{0,n}} X_{S_1,1} X_{S_2,1} \cdots X_{S_k,1} \psi_2^{n-3-k}, \text{ where } 2 \in S_i \text{ for all i}$$

can be computed in terms of the characteristic polynomial of a cotransversal matroid (Huh–Katz).
- Different special cases studied by Castravet–Tevelev, Cavalieri–Gillespie–Monin, Gillespie–Griffin–Levinson.
- K-theoretic version studied by Pandharipande, Lee, L.–Li–Payne–Proudfoot.

Recursion

- We have

$$X_{S \cup a, i} = X_{S,i} + \sum_{\{a,i\} \subset P,\, S \setminus \{a,i\} \subset Q} D_{P,Q}.$$

- We obtain a non-negative recursion of the form

$$\int_{\overline{M}_{0,n}} X_{S_1 \cup a, i_1} \cdots X_{S_{n-3}, i_{n-3}} = \int_{\overline{M}_{0,n}} X_{S_1, i_1} \cdots X_{S_{n-3}, i_{n-3}}$$
$$+ \sum_{\{a,i\} \subset P,\, S \setminus \{a,i\} \subset Q} \left(\int_{\overline{M}_{0,P \cup *}} \right) \cdot \left(\int_{\overline{M}_{0,Q \cup *}} \right)$$

Non-vanishing

Theorem (BELL)

We have $\int_{\overline{M}_{0,n}} X_{S_1,i_1} \cdots X_{S_{n-3},i_{n-3}} = 0$ if and only if there is $\emptyset \neq T \subset \{1,\ldots,n-3\}$ such that

$$|T| > |\cup_{j\in T} S_j| - 3.$$

- If there is such a T, then $\prod_{j\in T} X_{S_j,i_j}$ is pulled back from $\overline{M}_{0,\cup_{j\in T} S_j}$ and so vanishes for dimension reasons.
- Characterizes when cross-ratio degrees are positive, answering a question of Silversmith.
- Does not generalize to higher genus: we have

$$\int_{\overline{M}_{1,2}} f_1^*\psi_1 f_2^*\psi_2 = 0.$$

Proof of Theorem

Theorem (BELL)

We have $\int_{\overline{M}_{0,n}} X_{S_1,i_1} \cdots X_{S_{n-3},i_{n-3}} > 0$ if and only, for all $\emptyset \neq T \subset \{1, \ldots, n-3\}$, we have $|\cup_{j \in T} S_j| - 3 \geq |T|$.

- Reduce to the case of cross-ratio degree ($|S_i| = 4$ for all i) using monotonicity.
- Equivalent to understanding: when is the map $(\mathbb{P}^1)^n \dashrightarrow \overline{M}_{0,n} \to (\mathbb{P}^1)^{n-3}$ dominant?
- This happens if (and only if) the Jacobian matrix has full rank at the generic point.

Proof of Theorem

- Recall that $\mathrm{cr}_{1,2,3,4} = \frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)}$.
- We have

$$d\,\mathrm{cr}_{1,2,3,4} = (x_1-x_4)^{-2}(x_2-x_3)^{-2}\det\begin{pmatrix} dx_1 & dx_2 & dx_3 & dx_4 \\ 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{pmatrix}.$$

- Need to check that an $(n-3)\times n$ has full rank; this follows from the **GM-MDS** theorem from information theory.

GM-MDS theorem

Theorem (Lovett, Yildiz–Hassibi 2019)

Set

$$M = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{n-3} & x_2^{n-3} & \cdots & x_n^{n-3} \end{pmatrix}.$$

For $S_1, \ldots, S_{n-3} \subset \{1, \ldots, n\}$, there is an invertible matrix G such that $(GM)_{jk} = 0$ unless $k \in S_j$ if and only if, for all $\emptyset \neq T \subset \{1, \ldots, n-3\}$, we have $|\cup_{j \in T} S_j| - 3 \geq |T|$.

- Holds in positive characteristic as well, which implies that $\overline{M}_{0,n} \to (\mathbb{P}^1)^{n-3}$ is generically étale when it is dominant.

Counting 3-transversals

- For $T \subseteq \{1, \ldots, n-3\}$ and $j \in \{n-2, n-1, n\}$, set $Y_{T,j} = X_{T \cup \{n-2,n-1,n\},j}$.
- A **3-transversal** of $(T_1, j_1), \ldots, (T_{n-3}, j_{n-3})$ is a map $t\colon [n-3] \to \{n-2, n-1, n\}$ such that there is a bijection $m\colon [n-3] \to [n-3]$ satisfying $m(i) \in T_i$ and $t(m(i)) = j_i$ for all $i \in [n-3]$.

Theorem (BELL)

The Kapranov degree $\int_{\overline{M}_{0,n}} Y_{T_1,j_1} \cdots Y_{T_{n-3},j_{n-3}}$ is equal to the number of 3-transversals of $(T_1, j_1), \ldots, (T_{n-3}, j_{n-3})$.

Thank you!

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