STRAIGHTENING LAWS FOR CHOW RINGS OF MATROIDS

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Abstract. We give elementary and non-inductive proofs of three fundamental theorems about Chow rings of matroids: the standard monomial basis, Poincaré duality, and the dragon-Hall–Rado formula. Our approach, which also works for augmented Chow rings of matroids, is based on a straightening law. This approach also gives a decomposition of the Chow ring of a matroid into pieces indexed by flats.

1. Introduction

A *matroid* M is a finite nonempty atomic ranked lattice $\mathcal{L}_{\mathrm{M}}$ whose rank function $\mathrm{rk}\colon \mathcal{L}_{\mathrm{M}} \to \mathbb{Z}$ is submodular:

$$\mathrm{rk}(F_1 \vee F_2) + \mathrm{rk}(F_1 \wedge F_2) \leq \mathrm{rk}(F_1) + \mathrm{rk}(F_2) \text{ for all } F_1, F_2 \in \mathcal{L}_{\mathrm{M}}.$$

The minimal element of $\mathcal{L}_{\mathrm{M}}$ is usually denoted $\emptyset$ and the maximal element is the *ground set*, which is usually denoted E. That $\mathcal{L}$ is *atomic* means that every element is the join of the atoms it contains, and that it is *ranked* means that every maximal chain in an interval $[\emptyset, F]$ has the same length, which is $\mathrm{rk}(F)$. The *rank* of a matroid is $\mathrm{rk}(E)$. The elements of $\mathcal{L}_{\mathrm{M}}$ are called *flats*. Let $\overline{\mathcal{L}}_{\mathrm{M}} = \mathcal{L}_{\mathrm{M}} \setminus \{\emptyset\}$.

Definition 1.1. The *Chow ring* $\underline{A}^\bullet(\mathrm{M})$ of M is the ring given by the presentation

$$\underline{A}^\bullet(\mathrm{M}) = \frac{\mathbb{Z}[h_F]_{F \in \overline{\mathcal{L}}_{\mathrm{M}}}}{((h_F - h_{G\vee F})(h_G - h_{G\vee F}) : F, G \in \overline{\mathcal{L}}_{\mathrm{M}}) + (h_a :\ a \text{ atom})}.$$

Chow rings of matroids were first considered in [FY04] as a generalization of Chow rings of the wonderful compactifications of hyperplane arrangement complements, which were introduced in [DCP95]. They play a key role in the proof of log-concavity results for matroids [AHK18, BST23, ADH23]. The above definition is called the simplicial presentation of $\underline{A}^\bullet(\mathrm{M})$. It was first considered in [Yuz02] and then extensively studied in [BES]. See [LLPP24, Appendix A] for a proof of the equivalence between the above definition of $\underline{A}^\bullet(\mathrm{M})$ and the definition used in [FY04]. The Chow ring of a matroid is graded, with each h_F in degree 1. We now state three fundamental results about Chow rings of matroids.

Theorem 1.2. [BES, AHK18, FY04] *Let* M *be a matroid of rank r. Then*

(1) *The monomials*

(SM) $$\{h_{F_1}^{a_1} \cdots h_{F_\ell}^{a_\ell} : \emptyset = F_0 < F_1 < \cdots < F_\ell,\ a_i < \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) \text{ for } i = 1, \ldots, \ell\}$$

form an integral basis for $\underline{A}^\bullet(\mathrm{M})$.

(2) *There is an isomorphism* $\deg\colon \underline{A}^{r-1}(\mathrm{M}) \to \mathbb{Z}$ *given by*

(dHR) $$\deg(h_{F_1} \cdots h_{F_{r-1}}) = \begin{cases} 1 & \text{for all } \emptyset \neq T \subseteq [r-1],\ \mathrm{rk}(\bigvee_{i\in T} F_i) \geq |T| + 1 \\ 0 & \text{otherwise.} \end{cases}$$

(3) *The pairing*

(PD) $$\underline{A}^k(\mathrm{M}) \times \underline{A}^{r-1-k}(\mathrm{M}) \to \mathbb{Z} \text{ given by } (a, b) \mapsto \deg(ab)$$

is unimodular, i.e., it defines an isomorphism $\underline{A}^k(\mathrm{M}) \to \mathrm{Hom}(\underline{A}^{r-1-k}(\mathrm{M}), \mathbb{Z})$.

The augmented Chow ring of a matroid is a variant of the Chow ring of a matroid introduced in [BHM+22]. It plays a key role in the proof of the top-heavy conjecture in [BHM+].

Definition 1.3. The *augmented Chow ring* $A^\bullet(\mathrm{M})$ of M is the ring given by the presentation

$$A^\bullet(\mathrm{M}) = \frac{\mathbb{Z}[h_F]_{F \in \overline{\mathcal{L}}_{\mathrm{M}}}}{((h_F - h_{G\vee F})(h_G - h_{G\vee F}) : F, G \in \overline{\mathcal{L}}_{\mathrm{M}}) + (h_a^2, h_a h_F - h_a h_{F\vee a} : F \in \overline{\mathcal{L}}_{\mathrm{M}},\ a \text{ atom})}.$$

See [LLPP24, Appendix A] for a proof of the equivalence between the above definition and the definition used in [BHM+22]. Note that $\underline{A}^\bullet(\mathrm{M})$ is a quotient of $A^\bullet(\mathrm{M})$. We now state three fundamental results about augmented Chow rings of matroids.

Theorem 1.4. [EL24, BHM+22] *Let* M *be a matroid of rank* r. *Then*

(1) The monomials

$$\text{(SM)} \qquad \{h_{F_1}^{a_1} \cdots h_{F_\ell}^{a_\ell} : \emptyset = F_0 < F_1 < \cdots < F_\ell,\ a_1 \leq \mathrm{rk}(F_1),\ a_i < \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) \text{ for } i = 2, \dots, \ell\}$$

form an integral basis for $A^\bullet(\mathrm{M})$.

(2) There is an isomorphism $\deg\colon A^r(\mathrm{M}) \to \mathbb{Z}$ *given by*

$$\text{(HR)} \qquad \deg(h_{F_1} \cdots h_{F_r}) = \begin{cases} 1 & \text{for all } T \subseteq [r],\ \mathrm{rk}(\bigvee_{i \in T} F_i) \geq |T| \\ 0 & \text{otherwise.} \end{cases}$$

(3) The pairing

$$\text{(PD)} \qquad A^k(\mathrm{M}) \times A^{r-k}(\mathrm{M}) \to \mathbb{Z} \text{ given by } (a, b) \mapsto \deg(ab)$$

is unimodular.

We give elementary and non-inductive proofs of Theorems 1.2 and 1.4. We use only the above definition of a matroid and basic linear algebra. We now discuss the history of the above results.

Theorem $\underline{\mathrm{SM}}$ and SM give *standard monomial* bases for (augmented) Chow rings of matroids. A Gröbner basis for $\underline{A}^\bullet(\mathrm{M})$ was given in [FY04], and this gives a monomial basis for $\underline{A}^\bullet(\mathrm{M})$. In [BES, Corollary 3.3.3], it is shown that this monomial basis is essentially equivalent to the one given in Theorem $\underline{\mathrm{SM}}$. Theorem SM has not appeared explicitly in the literature before, but it is well-known to experts. Using the "free coextension trick", the result of [FY04] can be used to produce a Gröbner basis for $A^\bullet(\mathrm{M})$ as well; see [MM23, Section 5]. After some further manipulations this yields Theorem SM; see [EHL23, Theorem 7.7] for a special case. We note that Theorem $\underline{\mathrm{SM}}$ can be easily used to prove that the Gröbner basis given in [FY04] is indeed a Gröbner basis.

Theorem $\underline{\mathrm{dHR}}$ and HR are known as the *dragon Hall–Rado* and *Hall–Rado* formula, respectively, after the Hall–Rado theorem in matroid theory [Rad42]. Theorem $\underline{\mathrm{dHR}}$ is a generalization of Postnikov's dragon marriage theorem [Pos09, Theorem 9.3], and was proven in [BES, Theorem 5.2.4] using an inductive argument based on [Spe08, Proposition 4.4], which relies on a connection between $\underline{A}^\bullet(\mathrm{M})$ and the permutohedral toric variety. Theorem HR was proven in [EL24, Theorem 1.3] using a polyhedral interpretation of $A^\bullet(\mathrm{M})$. The argument given there can be adapted to prove Theorem $\underline{\mathrm{dHR}}$; see [EL24, Remark 6.3]. See also [EFLS24, Corollary 4.8]. Even the existence of the isomorphism deg, which is called the *degree map*, is nontrivial. It can be constructed using a tropical interpretation of the Chow ring, see [AHK18, Definition 5.9].

Theorem $\underline{\mathrm{PD}}$ and PD state that (augmented) Chow rings of matroids satisfy a version of *Poincaré duality*. Theorem $\underline{\mathrm{PD}}$ was first proven in [AHK18, Theorem 6.19] using an inductive argument. Different inductive proofs have been given in [BHM+22, BDF22]. Non-inductive arguments using Theorem $\underline{\mathrm{SM}}$ have been given in [BES, DR22, PP23]. Theorem PD was proven in [BHM+22, Theorem 1.3(4)] using an inductive argument. It can also be deduced from [AHK18, Theorem 6.19]; see [BHM+22, Remark 4.1].

There is a generalization of the Chow ring of a matroid to take into account a *building set* on the lattice of flats. Yuzvinsky gave an analogue of Theorem SM and Theorem PD for Chow rings of realizable matroids at the *minimal building set* [Yuz97]. Yuzvinsky's argument requires significant effort to adapt it to all matroids. Feichtner and Yuzvinsky give a Gröbner basis, and therefore a standard monomial basis, for the Chow ring of a matroid at any building set [FY04]. These Gröbner basis arguments are further generalized in [BDF22, PP23].

Besides Poincaré duality, (augmented) Chow rings of matroids satisfy the other parts of the *Kähler package*: the Hard Lefschetz theorem and the Hodge–Riemann relations. At the moment, the only proofs of the full Kähler package rely on intricate inductions [AHK18, BHM$^+$22, PP23].

Our approach begins with the augmented Chow ring. We use a "straightening" procedure which allows us to rewrite any monomial in terms of the standard monomials. This implies that the standard monomials span $A^\bullet(\mathrm{M})$, and so $A^r(\mathrm{M})$ has dimension at most 1. We then directly verify that $\deg\colon A^r(\mathrm{M}) \to \mathbb{Z}$ defined in Theorem HR is well-defined and an isomorphism. Finally, we prove Poincaré duality and prove the linear independence of the standard monomials simultaneously by showing that a certain matrix is lower triangular. With some additional arguments, we can deduce Theorem 1.2 because $\underline{A}^\bullet(\mathrm{M})$ is a quotient of $A^\bullet(\mathrm{M})$.

Our approach to Poincaré duality is closely related to the approach in [BES], which is in turn inspired by an argument of Hampe [Ham17] in the case of Boolean matroids. However, there are significant differences. For example, the argument in [BES] relies on Poincaré duality for Boolean matroids.

In Section 2, we prove Theorem 1.4 and then deduce Theorem 1.2 from it. In Section 3, we explain a consequences of our approach: the (augmented) Chow ring of a matroid has a direct sum decomposition indexed by $\overline{\mathcal{L}}_{\mathrm{M}}$. We use this to derive a new recursion for the Hilbert series of $A^\bullet(\mathrm{M})$ and $\underline{A}^\bullet(\mathrm{M})$. This decomposition also gives different proof of Theorem SM. In Section 4, we construct an *algebra with straightening law* related to the Chow ring of a matroid. We use this to give another proof of Theorem SM.

Notation. Throughout, M will be a matroid of rank r. When we consider a monomial $h_{F_1}^{a_1} \cdots h_{F_k}^{a_k}$ in $A^\bullet(\mathrm{M})$ or $\underline{A}^\bullet(\mathrm{M})$, we always assume the a_i are nonzero, but we allow $k = 0$. We do not assume the F_i are distinct unless otherwise stated.

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2. Proof of Theorem 1.2 and Theorem 1.4

2.1. **Straightening monomials.** We begin by using a straightening procedure to prove that the standard monomials, i.e., the elements in Theorem SM, span $A^\bullet(\mathrm{M})$. We then use this to prove Theorem HR.

Proposition 2.1. *The monomials*

$$\{h_{F_1}^{a_1} \cdots h_{F_\ell}^{a_\ell} : \emptyset = F_0 < F_1 < \cdots < F_\ell,\ a_1 \leq \mathrm{rk}(F_1),\ a_i < \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) \text{ for } i = 2, \ldots, \ell\}$$

integrally span $A^\bullet(\mathrm{M})$.

We prepare by proving three lemmas.

Lemma 2.2. *The monomials*

$$\{h_{F_1}^{a_1} \cdots h_{F_\ell}^{a_\ell} : \emptyset = F_0 < F_1 < \cdots < F_\ell\}$$

integrally span $A^\bullet(\mathrm{M})$.

Proof. It suffices to write each monomial of the form $m = h_{G_1}^{b_1} \cdots h_{G_\ell}^{b_\ell}$ in terms of the monomials where the flats used form a chain. Suppose that G_i and G_j are distinct and are both maximal in $\{G_1, \dots, G_\ell\}$. Then we can use the relation

$$h_{G_i} h_{G_j} = h_{G_i} h_{G_i \vee G_j} + h_{G_j} h_{G_i \vee G_j} - h_{G_i \vee G_j}^2 \tag{1}$$

to write m as a sum of monomials where there are fewer distinct maximal elements in the set of flats used in each monomial. Repeating this, we can write m as a sum of monomials where, in each monomial, the set of flats used has a unique maximal element.

We can therefore assume that G_ℓ is maximal. If G_i and G_j are distinct maximal elements in $\{G_1, \dots, G_{\ell-1}\}$, then we use the relation (1). As $G_\ell \geq G_i \vee G_j$, G_ℓ will still be maximal in each of the resulting terms. Repeating this argument gives the desired result. □

Lemma 2.3. *Suppose that G covers F in $\overline{\mathcal{L}}_{\mathrm{M}}$, i.e., $F \leq G$ and $\mathrm{rk}(G) = \mathrm{rk}(F) + 1$. Then $h_F h_G = h_G^2$.*

Proof. Because $\mathcal{L}_{\mathrm{M}}$ is atomic, there is an atom a such that $G = F \vee a$. By the defining relations in $A(\mathrm{M})$, we have that $(h_a - h_G)(h_F - h_G) = 0$ and $h_a h_F = h_a h_G$. The result follows. □

Proof of Proposition 2.1. By Lemma 2.2, it suffices to show that each monomial $m = h_{F_1}^{a_1} \cdots h_{F_\ell}^{a_\ell}$, where $\emptyset = F_0 < F_1 < \cdots < F_\ell$, is either equal to a standard monomial or vanishes. If m is not standard, then either $a_1 > \mathrm{rk}(F_1)$ or $a_i \geq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1})$ for some $i \geq 2$.

Suppose $a_1 > \mathrm{rk}(F_1)$. Choose a chain of covers $\emptyset = G_0 < G_1 < \cdots < G_{\mathrm{rk}(F_1)} = F_1$. Applying Lemma 2.3 repeatedly, we have that

$$h_{F_1}^{a_1} = h_{G_{\mathrm{rk}(F_1)-1}}^{a_1 - 1} h_{F_1} = \cdots = h_{G_1}^{a_1 - \mathrm{rk}(F_1)+1} h_{G_2} \cdots h_{F_1}.$$

As G_1 is an atom and $a_1 - \mathrm{rk}(F_1) + 1 \geq 2$, we see that $m = 0$ in this case.

Suppose $a_i \geq \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1})$ for some $i \geq 2$. Choose a chain of covers $F_{i-1} = G_0 < G_1 < \cdots < G_{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1})} = F_i$. Applying Lemma 2.3 repeatedly, we have that

$$h_{F_{i-1}}^{a_{i-1}} h_{F_i}^{a_i} = h_{F_{i-1}}^{a_{i-1}} h_{G_1}^{a_i - \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) + 1} h_{G_2} \cdots h_{G_{\mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) - 1}} h_{F_i} = h_{F_i}^{a_{i-1} + a_i}.$$

This rewriting decreases the number of flats in the chain. Applying these two operations shows that m is either equal to a standard monomial or vanishes. □

We say that a multiset $\{F_1, \dots, F_r\}$ of flats satisfies the *Hall–Rado* condition if, for all $T \subseteq [r]$, $\mathrm{rk}(\bigvee_{i \in T} F_i) \geq |T|$. We say that T *witnesses* the failure of the Hall–Rado condition if $\mathrm{rk}(\bigvee_{i \in T} F_i) < |T|$. We now prove Theorem HR.

Proof of Theorem HR. By Proposition 2.1, $A^r(\mathrm{M})$ is spanned by h_E^r. Note that $\{E, \dots, E\}$ satisfies the Hall–Rado condition, so if deg is well-defined then it is an isomorphism. We construct deg by defining a linear map from the degree r part of $\mathbb{Z}[h_F]_{F \in \overline{\mathcal{L}}_{\mathrm{M}}}$ to $\mathbb{Z}$ using the formula in Theorem HR and showing that it descends to $A^\bullet(\mathrm{M})$. It therefore suffices to prove that if $m = h_{F_1} \cdots h_{F_{r-2}}$ is a monomial in the degree $r-2$ part of $\mathbb{Z}[h_F]_{F \in \overline{\mathcal{L}}_{\mathrm{M}}}$, then the degree vanishes if we multiply m by any of the defining relations of $A^\bullet(\mathrm{M})$.

We first do the relation $h_a^2 = 0$, for a an atom. Set $F_{r-1} = F_r = a$. Then $\{F_1, \dots, F_r\}$ does not satisfy the Hall–Rado condition because $1 = \mathrm{rk}(F_{r-1} \vee F_r) < 2$.

We now do the relation $h_a h_F = h_a h_{F \vee a}$, for a an atom and $F \in \overline{\mathcal{L}}_{\mathrm{M}}$. Set $F_{r-1} = a, F_r = F$, and $F'_r = F \vee a$. We need to show that $\{F_1, \dots, F_r\}$ satisfies the Hall–Rado condition if and only if $\{F_1, \dots, F_{r-1}, F'_r\}$ does. If $\{F_1, \dots, F_r\}$ satisfies the Hall–Rado condition, then so does $\{F_1, \dots, F_{r-1}, F'_r\}$ because $F'_r \geq F_r$. Suppose

$\{F_1, \ldots, F_r\}$ fails the Hall–Rado condition, so there is some $T \subseteq [r]$ such that $\mathrm{rk}(\bigvee_{i \in T} F_i) < |T|$. We see that T witnesses that $\{F_1, \ldots, F_r'\}$ also fails the Hall–Rado condition unless $r \in T$ and

$$\mathrm{rk}(a \vee \bigvee_{i \in T} F_i) = |T| \text{ and } \mathrm{rk}(a \vee \bigvee_{i \in T} F_i) = \mathrm{rk}(\bigvee_{i \in T} F_i) + 1.$$

In this case, taking $T' = T \cup \{r-1\}$ shows that $\{F_1, \ldots, F_r'\}$ also fails the Hall–Rado condition.

Finally, we do the relation $(h_{F_{r-1}} - h_{F_{r-1} \vee F_r})(h_{F_r} - h_{F_{r-1} \vee F_r}) = 0$, for $F_{r-1}, F_r \in \overline{\mathcal{L}}_{\mathrm{M}}$. Set $S_0 = \{F_1, \ldots, F_r\}$, $S_1 = \{F_1, \ldots, F_{r-1}, F_{r-1} \vee F_r\}$, $S_2 = \{F_1, \ldots, F_{r-1} \vee F_r, F_r\}$, and $S_3 = \{F_1, \ldots, F_{r-1} \vee F_r, F_{r-1} \vee F_r\}$. If S_0 satisfies the Hall–Rado condition, then so do S_1, S_2, and S_3. There are then two cases which we must prove are impossible.

Case 1: S_0 fails Hall–Rado, and S_1, S_2, S_3 satisfy Hall–Rado.
Let $T \subseteq [r]$ witness the failure of Hall–Rado for S_0. If $r-1 \notin T$, then T witnesses the failure of Hall–Rado for S_2. If $r \notin T$, then T witnesses the failure of Hall–Rado for S_1. But if $\{r-1, r\} \subseteq T$, then T witnesses the failure of Hall–Rado for S_3.

Case 2: S_0, S_1, S_2 fail Hall–Rado, and S_3 satisfies Hall–Rado.
Let $T_1 \subseteq [r]$ witness that S_1 fails the Hall–Rado condition. We must have $r-1 \in T_1$ and $r \notin T_2$, as otherwise it would contradict our hypothesis. We can also assume that $T_1 \setminus \{r-1\}$ does not witness the failure of Hall–Rado for S_3, so we must have $\mathrm{rk}(\bigvee_{i \in T_1} F_i) = |T_1| - 1$. Similarly, we can find $T_2 \subseteq [r]$ with $r \in T_2, r-1 \notin T_2$, and $\mathrm{rk}(\bigvee_{i \in T_2} F_i) = |T_2| - 1$. By the submodularity of the rank function, we have that

$$\mathrm{rk}\left((\bigvee_{i \in T_1} F_i) \wedge (\bigvee_{i \in T_2} F_i) \right) + \mathrm{rk}\left(\bigvee_{i \in T_1 \cup T_2} F_i \right) \leq \mathrm{rk}\left(\bigvee_{i \in T_1} F_i \right) + \mathrm{rk}\left(\bigvee_{i \in T_2} F_i \right). \tag{2}$$

Set $H = \bigvee_{i \in T_1 \cap T_2} F_i$, so $H \leq (\bigvee_{i \in T_1} F_i) \wedge (\bigvee_{i \in T_2} F_i)$. We may assume that $\mathrm{rk}(H) \geq |T_1 \cap T_2|$, as otherwise $T_1 \cap T_2$ witnesses the failure of Hall–Rado for S_3. By (2), we get that $\mathrm{rk}(\bigvee_{i \in T_1 \cup T_2} F_i) \leq |T_1 \cup T_2| - 2$. But then $T_1 \cup T_2$ witnesses the failure of Hall–Rado for S_3. ☐

2.2. **Maps between Chow rings.** For the proof of Theorem PD and SM, we will use some maps considered in [BHM+22, Section 2.6]. For a matroid M of rank r and a flat $G \in \mathcal{L}_{\mathrm{M}}$, let M^G be the matroid whose lattice of flats is the interval $[\emptyset, G]$, and let M_G be the matroid whose lattice of flats is the interval $[G, E]$. It is easily checked that these are indeed matroids. We will use G to denote the minimal element of M_G and the maximal element of M^G.

We say that a flat is *nonempty* if it is not minimal and that it is *proper* if it is not maximal. Choose a proper flat G of M. Let $\mathcal{A}$ denote the set of atoms of M which are not contained in G. Set $h_\emptyset = 0$, and let

$$x_G = -\sum_{S \subseteq \mathcal{A}} (-1)^{|S|} h_{G \vee \bigvee_{a \in S} a} \in A^\bullet(\mathrm{M}).$$

Similarly, if G is a proper nonempty flat, we set $x_G = -\sum_{S \subseteq \mathcal{A}} (-1)^{|S|} h_{G \vee \bigvee_{a \in S} a} \in \underline{A}^\bullet(\mathrm{M})$. We will always make clear whether we think of x_G as living in $A^\bullet(\mathrm{M})$ or $\underline{A}^\bullet(\mathrm{M})$.

Lemma 2.4. *Let G be a proper flat of M. There is a surjective ring homomorphism $\varphi_G \colon A^\bullet(\mathrm{M}) \to A^\bullet(\mathrm{M}^G) \otimes \underline{A}^\bullet(\mathrm{M}_G)$ given by $\varphi_G(h_F) = h_F \otimes 1$ if $F \leq G$, and $\varphi_G(h_F) = 1 \otimes h_{F \vee G}$ otherwise. The kernel of φ_G is*

$$(h_F : F \text{ covers } G) + (h_H - h_K : K, H \not\leq G \text{ and } H \vee G = K \vee G).$$

When $G = \emptyset$, we interpret $A^\bullet(\mathrm{M}^\emptyset)$ as $\mathbb{Z}$, so $\varphi_\emptyset$ maps $A^\bullet(\mathrm{M})$ to $\underline{A}^\bullet(\mathrm{M})$.

Proof of Lemma 2.4. Note that $A^\bullet(\mathrm{M})$ is a quotient of $\mathbb{Z}[h_F]_{F \in \overline{\mathcal{L}}_{\mathrm{M}}}$. When we impose the second set of relations in the above ideal, we obtain $\mathbb{Z}[h_F]_{F \in \overline{\mathcal{L}}_{\mathrm{M}},\, F \leq G \text{ or } F > G}$ Note that $A^\bullet(\mathrm{M}^G) \otimes \underline{A}^\bullet(\mathrm{M}_G)$ is a quotient of this ring, and the image of the ideal defining $A^\bullet(\mathrm{M})$ is the ideal defining $A^\bullet(\mathrm{M}^G) \otimes \underline{A}^\bullet(\mathrm{M}_G)$. ☐

Similarly, we have the following lemma, whose proof is identical to Lemma 2.4.

Lemma 2.5. *Let G be a proper nonempty flat of* M. *There is a surjective ring homomorphism $\varphi_G\colon \underline{A}^\bullet(\mathrm{M}) \to \underline{A}^\bullet(\mathrm{M}^G) \otimes \underline{A}^\bullet(\mathrm{M}_G)$ given by $\varphi_G(h_F) = h_F \otimes 1$ if $F \leq G$, and $\varphi_G(h_F) = 1 \otimes h_{F\vee G}$ otherwise. The kernel of φ_G is*

$$(h_F : F \text{ covers } G) + (h_H - h_K : K, H \not\leq G \text{ and } H \vee G = K \vee G).$$

Lemma 2.6. *Let $F, G \in \overline{\mathcal{L}}_{\mathrm{M}}$, and suppose that G covers $F \wedge G$. Then $h_G h_F = h_G h_{G\vee F}$ in $A^\bullet(\mathrm{M})$.*

Proof. Because G covers $F \wedge G$, there is an atom a such that $G = (F \wedge G) \vee a$. Then $G \vee F = F \vee a$, so we have

$$h_G h_F = h_G h_{G\vee F} + h_F h_{G\vee F} - h_{G\vee F}^2.$$

We conclude by noting that, by Lemma 2.3, we have

$$h_F h_{G\vee F} = h_F h_{F\vee a} = h_{F\vee a}^2 = h_{G\vee F}^2.$$ □

Lemma 2.7. *Let G be a proper flat. The kernel of $\varphi_G\colon A^\bullet(\mathrm{M}) \to A^\bullet(\mathrm{M}^G) \otimes \underline{A}^\bullet(\mathrm{M}_G)$ is contained in the annihilator $\mathrm{ann}(x_G)$. Similarly, if G is a proper nonempty flat, the kernel of $\varphi_G\colon \underline{A}^\bullet(\mathrm{M}) \to \underline{A}^\bullet(\mathrm{M}^G) \otimes \underline{A}^\bullet(\mathrm{M}_G)$ is contained in $\mathrm{ann}(x_G)$.*

Proof. We do the augmented case; the non-augmented case be proved similarly or deduced by applying $\varphi_\emptyset$. We first show that $x_G \cdot h_F = 0$ if F covers G, i.e., $F = G \vee a$ for some atom $a \notin G$. Let $\mathcal{A}$ be the set of atoms not contained in G. Then

$$x_G \cdot h_{G\vee a} = -\sum_{S\subseteq\mathcal{A}} (-1)^{|S|} h_{G\vee\bigvee_{b\in S} b} h_{G\vee a}.$$

Note that $(G \vee \bigvee_{b\in S} b) \wedge (G \vee a)$ is either covered by $G \vee a$ or is $G \vee a$. In either case, by Lemma 2.6, we have $h_{G\vee\bigvee_{b\in S} b} h_{G\vee a} = h^2_{G\vee a\vee\bigvee_{b\in S} b}$. We see that the terms in the sum indexed by those S which contain a cancel with the terms where S does not contain a, and so the sum is 0.

We now show that $x_G(h_H - h_K) = 0$ if $K, H \not\leq G$ and $H \vee G = K \vee G$. For $S \subseteq \mathcal{A}$, we have

$$h_{G\vee\bigvee_{b\in S} b} h_H = h_{G\vee\bigvee_{b\in S} b} h_{H\vee G\vee\bigvee_{b\in S} b} + h_H h_{H\vee G\vee\bigvee_{b\in S} b} - h^2_{H\vee G\vee\bigvee_{b\in S} b}.$$

Because $H \vee G = K \vee G$, the first and third term on the right-hand side cancel with the corresponding term in $x_G h_K$. We get that

$$x_G(h_H - h_K) = -\sum_{S\subseteq\mathcal{A}} (-1)^{|S|} h_H h_{H\vee G\vee\bigvee_{b\in S} b} + \sum_{S\subseteq\mathcal{A}} (-1)^{|S|} h_K h_{K\vee G\vee\bigvee_{b\in S} b}.$$

Because $H \not\leq G$, we may choose an atom $a \leq H$ with $a \not\leq G$. Then the terms in the first sum indexed by those S which contain a cancel with the terms corresponding to S that do not contain a, so the first sum is 0. Similarly, the second sum is 0. □

In particular, the map $A^\bullet(\mathrm{M}) \to A^\bullet(\mathrm{M})/\mathrm{ann}(x_G)$ factors through φ_G, and similarly in the non-augmented setting. This will be a useful aid in computations.

2.3. **Projection formulas and dragon-Hall–Rado.** We now show that the maps constructed in the previous section are compatible with degree maps. Along the way, we prove Theorem dHR. First we prove that the standard monomials span $\underline{A}^\bullet(\mathrm{M})$.

Proposition 2.8. *The monomials*

$$\{h_{F_1}^{a_1} \cdots h_{F_\ell}^{a_\ell} : \emptyset = F_0 < F_1 < \cdots < F_\ell,\ a_i < \mathrm{rk}(F_i) - \mathrm{rk}(F_{i-1}) \text{ for } i = 1, \ldots, \ell\}$$

integrally span $\underline{A}^\bullet(\mathrm{M})$.

Proof. The map $\varphi_\emptyset \colon A^\bullet(\mathrm{M}) \to \underline{A}^\bullet(\mathrm{M})$ is surjective, so by Proposition 2.1, it suffices to show that $h_F^{\mathrm{rk}(F)} = 0$ in $\underline{A}^\bullet(\mathrm{M})$. By Lemma 2.3, $h_F^{\mathrm{rk}(F)}$ is divisible by h_a for any atom a contained in F, and so it is 0 in $\underline{A}^\bullet(\mathrm{M})$. □

We now prove Theorem dHR. We deduce it from Theorem HR, although one can also argue analogously to the proof of Theorem HR.

Proof of Theorem dHR. By Proposition 2.8, $\underline{A}^{r-1}(\mathrm{M})$ is spanned by h_E^{r-1}, so if the degree map exists then it is an isomorphism. By Lemma 2.7, there is a surjective ring homomorphism $\underline{A}^\bullet(\mathrm{M}) \to A^\bullet(\mathrm{M})/\operatorname{ann}(x_\emptyset)$. Note that $A^\bullet(\mathrm{M})/\operatorname{ann}(x_\emptyset)$ is identified with the ideal $(x_\emptyset)$, with degree shifted by 1. We define the degree map via the composition

$$\deg \colon \underline{A}^{r-1}(\mathrm{M}) \to A^{r-1}(\mathrm{M})/\operatorname{ann}(x_\emptyset) \to A^r(\mathrm{M}) \to \mathbb{Z},$$

where the second map is multiplication by $x_\emptyset$. In order to prove Theorem dHR, it suffices to show that

$$\begin{aligned}\deg(x_\emptyset h_{F_1} \cdots h_{F_{r-1}}) &= -\sum_{S \subseteq \mathcal{A}} (-1)^{|S|} \deg(h_{\bigvee_{a\in S} a} h_{F_1} \cdots h_{F_{r-1}}) \\ &= \begin{cases} 1 & \text{for all } \emptyset \neq T \subseteq [r-1],\ \mathrm{rk}(\bigvee_{i\in T} F_i) \geq |T|+1 \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Suppose that $\{F_1, \ldots, F_{r-1}\}$ satisfies the dragon-Hall–Rado condition. If S is nonempty, $\{\bigvee_{a\in S} a, F_1, \ldots, F_{r-1}\}$ satisfies the Hall–Rado condition. We see that every term in the sum is 1 except for $S = \emptyset$, so the sum is 1.

Now suppose that a multiset $\{F_1, \ldots, F_{r-1}\}$ fails the dragon-Hall–Rado condition. Let

$$\mathcal{S} = \{S \subseteq \mathcal{A} : \{F_1, \ldots, F_{r-1}, \bigvee_{a\in S} a\} \text{ fails Hall–Rado}\}.$$

Clearly $\mathcal{S}$ is downward closed: if $T \subseteq S \in \mathcal{S}$, then $T \in \mathcal{S}$. Because $\{F_1, \ldots, F_{r-1}\}$ fails dragon-Hall–Rado, there is some i such that $\{a \in \mathcal{A} : a \leq F_i\}$ is contained in $\mathcal{S}$.

Let $I_1, I_2 \in \mathcal{S}$. We claim that $I_1 \cup I_2 \in \mathcal{S}$. If there is a witness to the failure of Hall–Rado for $F_1, \ldots, F_{r-1}, \bigvee_{a\in I_1} a$ which does not contain $\bigvee_{a\in I_1} a$, then this is immediate. Choose $T_1, T_2 \subseteq [r-1]$ such that $\{F_j : j \in T_1\} \cup \{\bigvee_{a\in I_1} a\}$ and $\{F_j : j \in T_2\} \cup \{\bigvee_{a\in I_2} a\}$ witness the failure of Hall–Rado, so

$$\mathrm{rk}(\bigvee_{a\in I_1} a \vee \bigvee_{j\in T_1} F_j) < |T_1| + 1,$$

and similarly for T_2. By the monotonicity of the rank function, we have that

$$\mathrm{rk}((\bigvee_{a\in I_1} a \vee \bigvee_{j\in T_1} F_j) \wedge (\bigvee_{a\in I_2} a \vee \bigvee_{j\in T_2} F_j)) \geq \mathrm{rk}(\bigvee_{j\in T_1\cap T_2} F_j) \geq |T_1 \cap T_2|,$$

where the last inequality is by the assumption that every witness to the failure of Hall–Rado contains $\bigvee_{a\in I_1} a$. By the submodularity of the rank function function, we have that

$$|T_1 \cap T_2| + \mathrm{rk}(\bigvee_{a\in I_1\cup I_2} a \vee \bigvee_{j\in T_1\cup T_2} F_j) \leq \mathrm{rk}(\bigvee_{a\in I_1} a \vee \bigvee_{j\in T_1} F_j) + \mathrm{rk}(\bigvee_{a\in I_2} a \vee \bigvee_{j\in T_2} F_j).$$

This implies that $\mathrm{rk}(\bigvee_{a\in I_1\cup I_2} a \vee \bigvee_{j\in T_1\cup T_2} F_j) < |T_1 \cup T_2| + 1$, so $I_1 \cup I_2 \in \mathcal{S}$, as desired.

Therefore $\mathcal{S}$ contains a maximal element, so it is a Boolean lattice of size at least 2. We see that the sum is zero. □

Let G be a proper flat of M. The tensor product of the degree maps gives an isomorphism $\deg \colon A^{\mathrm{rk}(G)}(\mathrm{M}^G) \otimes \underline{A}^{r-1-\mathrm{rk}(G)}(\mathrm{M}_G) \to \mathbb{Z}$. If G is nonempty, there is an isomorphism $\deg \colon \underline{A}^{\mathrm{rk}(G)-1}(\mathrm{M}^G) \otimes \underline{A}^{r-1-\mathrm{rk}(G)}(\mathrm{M}_G) \to \mathbb{Z}$.

It will be convenient to extend by zero the degree maps to the entirety of $A^\bullet(\mathrm{M})$, $\underline{A}^\bullet(\mathrm{M})$ and so on. The following lemmas will be critical to the proof of Theorem PD and $\underline{\text{PD}}$.

Lemma 2.9. *Let $y \in A^{r-1}(\mathrm{M})$, and let G be a proper flat. Then*

$$\deg(\varphi_G(y)) = \deg(x_G \cdot y).$$

Proof. Lemma 2.7 implies that the right-hand side only depends on $\varphi_G(y)$. As the degree $r-1$ part of $A^\bullet(\mathrm{M}^G) \otimes \underline{A}^\bullet(\mathrm{M}_G)$ is $\mathbb{Z}$, the maps $y \mapsto \deg(\varphi_G(y))$ and $y \mapsto \deg(x_G \cdot y)$ are equal up to a constant.

Let $y = h_G^{\mathrm{rk}(G)} h_E^{r-1-\mathrm{rk}(G)}$. We see from Theorem HR and Theorem $\underline{\text{dHR}}$ that $\deg(\varphi_G(y)) = 1$. Let $\mathcal{A}$ be the set of atoms of M not contained in G. We have

$$\deg(x_G \cdot y) = -\sum_{S \subseteq \mathcal{A}} (-1)^{|S|} \deg(h_{G \vee \bigvee_{a \in S} a} h_G^{\mathrm{rk}(G)} h_E^{r-1-\mathrm{rk}(G)}).$$

The term $S = \emptyset$ vanishes because it does not satisfy the Hall–Rado condition; all other terms are 1, so the sum is 1. □

Lemma 2.10. *Let $a \in \underline{A}^\bullet(\mathrm{M})$, and let G be a proper nonempty flat. Then*

$$\deg(\varphi_G(a)) = \deg(x_G \cdot a).$$

Proof. This can be proved as in the proof of Lemma 2.9. Alternatively, we can choose a lift $\tilde{a} \in A^\bullet(\mathrm{M})$ such that $\varphi_\emptyset(\tilde{a}) = a$ and apply Lemma 2.9 twice to $x_\emptyset \cdot x_G \cdot \tilde{a}$. □

We will apply Lemma 2.10 iteratively. This will require the following lemmas.

Lemma 2.11. *Let G be a proper flat, let $H > G$, and consider $x_H \in A^\bullet(\mathrm{M})$. Then $\varphi_G(x_H) = 1 \otimes x_H \in A^\bullet(\mathrm{M}^G) \otimes \underline{A}^\bullet(\mathrm{M}_G)$.*

Proof. Let $\mathcal{A}$ be the set of atoms of M not contained in H, and let $\mathcal{A}'$ be the set of atoms of M_G not contained in H. There is a map $p\colon \mathcal{A} \to \mathcal{A}'$ given by $a \mapsto G \vee a$. Note that for atom $a \in \mathcal{A}'$ and any nonempty subset T of $p^{-1}(a)$, $G \vee \bigvee_{b \in T} b = G \vee a$. It therefore suffices to show that, for any $S \subseteq \mathcal{A}'$, we have

$$(-1)^{|S|} h_{H \vee \bigvee_{a \in S} a} = \sum_{a \in S,\, \emptyset \neq T_a \subseteq p^{-1}(a)} (-1)^{\sum |T_a|} h_{H \vee \bigvee_{a \in S} a} \in \underline{A}^\bullet(\mathrm{M}_G),$$

where the sum is over all choices of nonempty subsets T_a of $p^{-1}(a)$ for each $a \in S$. Let $n_1, \ldots, n_{|S|}$ be the sizes of the sets $p^{-1}(a)$ for $a \in S$. Note that each n_i is positive. Then the coefficient of $h_{H \vee \bigvee_{a \in S} a}$ on the right-hand side is

$$((-1+1)^{n_1} - 1) \cdot ((-1+1)^{n_2} - 1) \cdot \ldots \cdot ((-1+1)^{n_{|S|}} - 1) = (-1)^{|S|}. \qquad \square$$

The non-augmented version of the previous lemma can be proved in the same way, or it can be deduced by applying $\varphi_\emptyset$.

Lemma 2.12. *Let G be a proper flat, let $H > G$, and consider $x_H \in \underline{A}^\bullet(\mathrm{M})$. Then $\varphi_G(x_H) = 1 \otimes x_H \in \underline{A}^\bullet(\mathrm{M}^G) \otimes \underline{A}^\bullet(\mathrm{M}_G)$.*

Remark 2.13. One can additionally show that, if $H < G$, $\varphi_G(x_H) = x_H \otimes 1$, and that $\varphi_G(x_H) = 0$ if H and G and incomparable. See [BHM[+]22, Proposition 2.17].

2.4. **Poincaré duality and linear independence.** Now that we have access to Lemma 2.9 and Lemma 2.10, we can begin our proof of Theorem PD. Our strategy is closely related to [BES, Proposition 3.3.10], which is based on [Ham17, Proposition 3.2]. Let $m = h_{F_{i_1}}^{a_1} \cdots h_{F_{i_k}}^{a_k}$ be a standard monomial for $A^\bullet(\mathrm{M})$. Extend the chain $F_{i_1} < \cdots < F_{i_k}$ to a maximal chain of flats $\emptyset = F_0 < F_1 < \cdots < F_r = E$. Let $\mathcal{G}_m$ be the collection of flats obtained by removing from this chain the a_j flats below F_{i_j} for each j and removing E. Because m is a standard monomial, $\{F_{i_1}, \ldots, F_{i_k}\} \setminus \{E\} \subseteq \mathcal{G}_m$. We do this process and obtain a collection of flats $\mathcal{G}_m$ for each standard monomial m. We call $\mathcal{G}_m$ the *essential flats* of m.

We will now prove the key propositions that allow us to prove Theorem SM and Theorem PD. See Example 2.16 for an example illustrating their proofs.

Proposition 2.14. *Let $m \in A^\ell(\mathrm{M})$ be a standard monomial, and let $\mathcal{G}_m = \{G_1 < \cdots < G_k\}$ be the essential flats. Then $\deg(m \cdot x_{G_1} \cdots x_{G_k}) = 1$.*

Proof. Set $G_0 = \emptyset$ and set $G_{k+1} = E$. Note that possibly $G_1 = \emptyset$ as well. Applying Lemma 2.9, Lemma 2.10, Lemma 2.12, and Lemma 2.11, we can write the degree as a degree in $A^\bullet(\mathrm{M}^{G_1}_{G_0}) \otimes \underline{A}^\bullet(\mathrm{M}^{G_2}_{G_1}) \otimes \cdots \otimes \underline{A}^\bullet(\mathrm{M}^{G_{k+1}}_{G_k})$. Here if $\mathrm{rk}(G_{i+1}) = \mathrm{rk}(G_i) + 1$, then we interpret $\underline{A}^\bullet(\mathrm{M}^{G_{i+1}}_{G_i})$ as $\mathbb{Z}$, and similarly if $\mathrm{rk}(G_1) = 0$.

The only terms which are not $\mathbb{Z}$ are $\underline{A}^\bullet(\mathrm{M}^{G_{i+1}}_{G_i})$ if $\mathrm{rk}(G_{i+1}) - \mathrm{rk}(G_i) > 1$ and $A^\bullet(\mathrm{M}^{G_1}_{G_0})$ if $\mathrm{rk}(G_1) > 0$. From the construction of $\mathcal{G}_m$, we see that, if $i \neq 0$, then $h_{G_{i+1}}^{\mathrm{rk}(G_{i+1}) - \mathrm{rk}(G_i) - 1}$ appears in m. If $i = 0$ and $\mathrm{rk}(G_1) > 0$, then $h_{G_1}^{\mathrm{rk}(G_1)}$ appears in m. In the first case, after applying φ_{G_i} for all $G_i \in \mathcal{G}_m$, $h_{G_{i+1}}^{\mathrm{rk}(G_{i+1}) - \mathrm{rk}(G_i) - 1}$ lands in top degree in $\underline{A}^\bullet(\mathrm{M}^{G_{i+1}}_{G_i})$. In the second case, $h_{G_1}^{\mathrm{rk}(G_1)}$ lands in top degree in $A^\bullet(\mathrm{M}^{G_1}_{G_0})$. By Theorem HR and <u>dHR</u>, we see that the degree is 1. ☐

For a standard monomial $m = h_{F_1}^{a_1} \cdots h_{F_k}^{a_k}$, we set $\delta(m) = (\sum_{\mathrm{rk}(F_i) \leq 1} a_i, \sum_{\mathrm{rk}(F_i) \leq 2} a_i, \ldots, \sum_{\mathrm{rk}(F_i) \leq r} a_i)$.

Proposition 2.15. *Let $m \in A^\ell(\mathrm{M})$ be a standard monomial, and let $\mathcal{G}_m = \{G_1 < \cdots < G_k\}$ be the essential flats. Let $m' \in A^\ell(\mathrm{M})$ be a standard monomial which has $\deg(m' \cdot \prod_{G \in \mathcal{G}_m} x_G) \neq 0$. Then either $m = m'$ or $\delta(m') > \delta(m)$ lexicographically.*

Proof. Set $G_0 = \emptyset$ and set $G_{k+1} = E$. As in the proof of Proposition 2.14, we can write the degree as a degree in $A^\bullet(\mathrm{M}^{G_1}_{G_0}) \otimes \underline{A}^\bullet(\mathrm{M}^{G_2}_{G_1}) \otimes \cdots \otimes \underline{A}^\bullet(\mathrm{M}^{G_{k+1}}_{G_k})$. As before, the top degree of $\underline{A}^\bullet(\mathrm{M}^{G_{i+1}}_{G_i}) = \mathbb{Z}$ is $\mathrm{rk}(G_{i+1}) - \mathrm{rk}(G_i) - 1$. Also, the top degree of $A^\bullet(\mathrm{M}^{G_1}_{G_0})$ is $\mathrm{rk}(G_1)$.

Let $m' = h_{F_1}^{a_1} \cdots h_{F_\ell}^{a_\ell}$. Let G_j be the least element of $\mathcal{G}_m$ with $G_j \geq F$. After applying φ_{G_i} for all $i \in \mathcal{G}_m$, $h_{F_i}^{a_i}$ is mapped to $1 \otimes \cdots \otimes h_{G_j \vee F_i}^{a_i} \otimes \cdots \otimes 1$. In particular, for each $i > 0$ with $\mathrm{rk}(G_{i+1}) - \mathrm{rk}(G_i) > 1$, $\deg(m' \cdot \prod_{G \in \mathcal{G}_m} x_G)$ vanishes unless there are flats $F_j, \ldots, F_p$ appearing in m' with $a_j + \cdots + a_p = \mathrm{rk}(G_{i+1}) - \mathrm{rk}(G_i) - 1$ and $F_q \leq G_{i+1}$, $F_q \not\leq G_i$ for each $q = j, \ldots, p$. Similarly, if $\deg(m' \cdot \prod_{G \in \mathcal{G}_m} x_G)$ is nonzero and $\mathrm{rk}(G_1) > 0$, then there must be $F_1, \ldots, F_p$ appearing in m' with $a_1 + \cdots + a_p = \mathrm{rk}(G_1)$ and $F_q \leq G_1$ for each $q = 1, \ldots, p$. Adding these conditions up, this implies that the degree vanishes if $\delta(m') < \delta(m)$ or if $\delta(m) = \delta(m')$ and $m \neq m'$. ☐

Example 2.16. Let M be the Boolean matroid of rank 6, i.e., $\mathcal{L}_\mathrm{M}$ is the Boolean lattice on 6 elements. Let $F_i = \{1, \ldots, i\}$ for $i = 0, 1, \ldots, 6$. Let $m = h_{F_2} h_{F_5}^2$. We may take $\mathcal{G}_m = \{F_0, F_2, F_5\}$. We apply φ_{F_0}, then φ_{F_2}, and then φ_{F_5} to write $\deg(m \cdot x_{F_0} x_{F_2} x_{F_5})$ as a degree in

$$\underline{A}^\bullet(\mathrm{M}^{F_2}) \otimes \underline{A}^\bullet(\mathrm{M}^{F_5}_{F_2}) \otimes \underline{A}^\bullet(\mathrm{M}_{F_5}) = \underline{A}^\bullet(\mathrm{M}^{F_2}) \otimes \underline{A}^\bullet(\mathrm{M}^{F_5}_{F_2}).$$

We have $\varphi_{F_5} \circ \varphi_{F_2} \circ \varphi_{F_0}(h_{F_2}) = h_{F_2} \otimes 1$ and $\varphi_{F_5} \circ \varphi_{F_2} \circ \varphi_{F_0}(h_{F_5}^2) = 1 \otimes h_{F_5}^2$, so the degree is 1.

Let m' be a standard monomial where the rank of the smallest flat appearing is at least 3, so $\delta(m') < \delta(m)$ lexicographically. Then, for each h_G appearing in m', we have

$$\varphi_{F_5} \circ \varphi_{F_2} \circ \varphi_{F_0}(h_{F_2}) = \begin{cases} 1 \otimes h_{G \vee F_2} & G \leq F_5 \\ 0 & G \not\leq F_5, \end{cases}$$

In particular, no term appearing in m' maps to something of the form $h_F \otimes 1$. This implies that $\deg(m' \cdot x_{F_0} x_{F_2} x_{F_5}) = 1$.

Proof of Theorem PD and SM. Fix $0 \leq k \leq r$. Choose a total order $<$ on the set of standard monomials of degree k such that $m < m'$ if $\delta(m) < \delta(m')$ lexicographically. For each standard monomial m, we have an element $d(m) \coloneqq \prod_{G \in \mathcal{G}_m} x_G \in A^{r-k}(\mathrm{M})$. By Proposition 2.14 and Proposition 2.15, the matrix whose rows and columns are labeled by standard monomials of degree k, and whose entry indexed by (m, m') is $\deg(m \cdot d(m'))$, is lower triangular with 1's on the diagonal. This implies that the standard monomials of degree k are linearly independent, so, by Proposition 2.1, they are a basis.

We also see that $\dim A^k(\mathrm{M}) \leq \dim A^{r-k}(\mathrm{M})$. Replacing k by $r-k$, we see that $\dim A^k(\mathrm{M}) = \dim A^{r-k}(\mathrm{M})$, and so the $d(m)$ rationally span $A^{r-k}(\mathrm{M}) \otimes \mathbb{Q}$. Because the determinant of the pairing between $A^k(\mathrm{M})$ and the subgroup of $A^{r-k}(\mathrm{M})$ spanned by the $d(m)$ is 1, we see that the $d(m)$ must integrally span $A^{r-k}(\mathrm{M})$, which proves Theorem PD. □

In order to prove Theorem $\underline{\text{PD}}$ and $\underline{\text{SM}}$, we will need an analogue of Proposition 2.14 and 2.15 for non-augmented Chow rings. We will deduce these from their augmented versions.

For a standard monomial $m = h_{F_{i_1}}^{a_1} \cdots h_{F_{i_k}}^{a_k}$ for $\underline{A}^\bullet(\mathrm{M})$, we define $\mathcal{G}_m$ in the same way as in the augmented setting: extend the chain $F_{i_1} < \cdots < F_{i_k}$ to a maximal chain of flats $\emptyset = F_0 < F_1 < \cdots < F_r = E$. Let $\mathcal{G}_m$ be collection of flats obtained by removing from this chain the a_j flats below F_{i_j} for each j and removing E. Because m is a standard monomial for $\underline{A}^\bullet(\mathrm{M})$, $\emptyset \in \mathcal{G}_m$. We define $\delta(m)$ in the same way as for standard monomials for $A^\bullet(\mathrm{M})$.

Proposition 2.17. *Let m be a standard monomial of $\underline{A}^\bullet(\mathrm{M})$. Then*

(1) we have that $\deg(m \cdot \prod_{\emptyset \neq G \in \mathcal{G}_m} x_G) = 1$.
(2) for each standard monomial m' for $\underline{A}^\bullet(\mathrm{M})$ with $\deg(m' \cdot \prod_{G \in \mathcal{G}_m} x_G) \neq 0$, either $m = m'$ or $\delta(m') > \delta(m)$ lexicographically.

Proof. Let $m = h_{F_1}^{a_1} \cdots h_{F_k}^{a_k}$. By Proposition 2.9, we have that the degree $\deg(m \cdot \prod_{\emptyset \neq G \in \mathcal{G}_m} x_G)$ in $\underline{A}^\bullet(\mathrm{M})$ is equal to the degree in $A^\bullet(\mathrm{M})$ of $h_{F_1}^{a_1} \cdots h_{F_k}^{a_k}$ times $\prod_{G \in \mathcal{G}_m} x_G$. The result then follows from Proposition 2.14 and Proposition 2.15. □

Proof of Theorem $\underline{\text{PD}}$ and $\underline{\text{SM}}$. Fix $0 \leq k \leq r$. Choose a total order $<$ on the set of standard monomials of degree k such that $m < m'$ if $\delta(m) < \delta(m')$ lexicographically. For each standard monomial m, we have an element $d(m) \coloneqq \prod_{\emptyset \neq G \in \mathcal{G}_m} x_G \in \underline{A}^{r-1-k}(\mathrm{M})$. By Proposition 2.17, the matrix whose rows and columns are labeled by standard monomials of degree k, and whose entry indexed by (m, m') is $\deg(m \cdot d(m'))$, is lower triangular with 1's on the diagonal. As in the proof of Theorem PD and SM, this implies the linear independence of the standard monomials and Poincaré duality. □

3. Gradings by $\mathcal{L}_\mathrm{M}$

One corollary of our approach is the existence of a "grading" of $A^\bullet(\mathrm{M})$ by $\mathcal{L}_\mathrm{M}$, which we now study. Special cases of this decomposition were used in [EHKR10, Section 5.1] and [Rai10]. For a flat F, let $A^\bullet(\mathrm{M})_F$ be the span of the monomials $h_{G_1}^{a_1} \cdots h_{G_k}^{a_k}$, where $G_1 \vee \cdots \vee G_k = F$. For example, $A^\bullet(\mathrm{M})_\emptyset = \mathrm{span}(1)$.

Proposition 3.1. *We have a direct sum decomposition*

$$A^\bullet(\mathrm{M}) = \bigoplus_{F \in \mathcal{L}_\mathrm{M}} A^\bullet(\mathrm{M})_F.$$

Proof. There is clearly such a decomposition for $\mathbb{Z}[h_F]_{F \in \overline{\mathcal{L}}_\mathrm{M}}$, and the relations in $A^\bullet(\mathrm{M})$ respect this decomposition. □

Lemma 3.2. *Let F be a proper nonempty flat of M. There is a graded ring isomorphism $\bigoplus_{G \le F} A^\bullet(\mathrm{M}^F)_G \xrightarrow{\sim} \bigoplus_{G \le F} A^\bullet(\mathrm{M})_G$ given by $h_G \mapsto h_G$. In particular, the graded abelian groups $A^\bullet(\mathrm{M}^F)_F$ and $A^\bullet(\mathrm{M})_F$ are isomorphic.*

Proof. Note that the subring $\bigoplus_{G \le F} A^\bullet(\mathrm{M})_F$ of $A^\bullet(\mathrm{M})$ is generated by h_G for $G \le F$, and the relations in $\bigoplus_{G \le F} A^\bullet(\mathrm{M}^F)_F$ and $\bigoplus_{G \le F} A^\bullet(\mathrm{M})_F$ are the same. □

If $\mathrm{rk}(\mathrm{M}) > 0$, the *truncation* $\mathrm{Tr}\,\mathrm{M}$ is the matroid whose lattice of flats $\mathcal{L}_{\mathrm{Tr}\,\mathrm{M}}$ is obtained by removing the flats F with $\mathrm{rk}(F) = \mathrm{rk}(E) - 1$. There is a surjective ring homomorphism $A^\bullet(\mathrm{M}) \to A^\bullet(\mathrm{Tr}\,\mathrm{M})$ given by $h_F \mapsto h_E$ if $\mathrm{rk}(F) = \mathrm{rk}(E) - 1$ and $h_F \mapsto h_F$ otherwise. The kernel of this map is $(h_E - h_F : \mathrm{rk}(F) = \mathrm{rk}(E) - 1)$.

Lemma 3.3. *Let M be a matroid of rank $r > 0$. There is an isomorphism of graded $A^\bullet(\mathrm{M})$-modules $A^\bullet(\mathrm{Tr}\,\mathrm{M})[-1] \xrightarrow{\sim} A^\bullet(\mathrm{M})_E$ given by $1 \mapsto h_E$.*

Proof. By Lemma 2.2 and its proof, $A^\bullet(\mathrm{M})_E$ is the ideal generated by h_E. We have an identification of $A^\bullet(\mathrm{M})$-modules $A^\bullet(\mathrm{M})/\operatorname{ann}(h_E)[-1] \xrightarrow{\sim} (h_E)$ given by multiplication by h_E.

We claim that the kernel of the map $A^\bullet(\mathrm{M}) \to A^\bullet(\mathrm{Tr}\,\mathrm{M})$ is $\operatorname{ann}(h_E)$, which concludes the proof. If F is a flat of M with $\mathrm{rk}(F) = r - 1$, then $h_E(h_F - h_E) = 0$ by Lemma 2.3, so $\operatorname{ann}(h_E)$ contains the kernel. Note that h_E^{r-1} is nonzero in $A^\bullet(\mathrm{M})/\operatorname{ann}(h_E)$ by Theorem HR. Poincaré duality for $A^\bullet(\mathrm{Tr}\,\mathrm{M})$ then implies that the surjective map $A^\bullet(\mathrm{Tr}\,\mathrm{M}) \to A^\bullet(\mathrm{M})/\operatorname{ann}(h_E)$ is an isomorphism because it is an isomorphism in degree $r - 1$. Indeed, Poincaré duality implies that every nonzero ideal of $A^\bullet(\mathrm{Tr}\,\mathrm{M})$ intersects $A^{r-1}(\mathrm{Tr}\,\mathrm{M})$ nontrivially. □

Combining Lemma 3.2 with Lemma 3.3 gives that, if F is a nonempty flat, then $A^\bullet(\mathrm{M})_F \xrightarrow{\sim} A^\bullet(\mathrm{Tr}\,\mathrm{M}^F)[-1]$ as graded abelian groups. In particular, $A^\bullet(\mathrm{M})_F$ vanishes above degree $\mathrm{rk}(F)$ and is 1-dimensional in degree $\mathrm{rk}(F)$. By Theorem SM, we see that $A^{\mathrm{rk}(F)}(\mathrm{M})_F$ is spanned by $h_F^{\mathrm{rk}(F)}$. In particular, a monomial $h_{G_1}^{a_1} \cdots h_{G_k}^{a_k}$, with $a_1 + \cdots + a_k = \mathrm{rk}(F)$ and $G_i \le F$ for each i, is either 0 or equal to $h_F^{\mathrm{rk}(F)}$.

The *graded Möbius algebra* $H^\bullet(\mathrm{M})$ of a matroid M is a ring which is $\bigoplus_{F \in \mathcal{L}_\mathrm{M}} y_F \cdot \mathbb{Z}$ as an abelian group, with multiplication $y_F \cdot y_G = y_{F \vee G}$ if $\mathrm{rk}(F) + \mathrm{rk}(G) = \mathrm{rk}(F \vee G)$ and $y_F \cdot y_G = 0$ otherwise. Note that $H^\bullet(\mathrm{M})$ is graded, with y_F in degree $\mathrm{rk}(F)$, and that $H^\bullet(\mathrm{M})$ is generated in degree 1. A detailed study of modules over the graded Möbius algebra is central to the proof of the top-heavy conjecture in [BHM+]. One of the key results is the following realization of $H^\bullet(\mathrm{M})$ as a subring of $A^\bullet(\mathrm{M})$. We give a simple proof.

Proposition 3.4. [BHM+22, Proposition 2.15] *There is an injective ring homomorphism $H^\bullet(\mathrm{M}) \to A^\bullet(\mathrm{M})$, defined by sending y_a to h_a for each atom of $\mathcal{L}_\mathrm{M}$.*

Proof. Let $a_1, \dots, a_{\mathrm{rk}(F)}$ be atoms with $\bigvee_{i=1}^{\mathrm{rk}(F)} a_i = F$. By Theorem HR, $\deg(h_{a_1} \cdots h_{a_{\mathrm{rk}(F)}} h_E^{r-\mathrm{rk}(F)}) = 1$. In particular, by the discussion above, we have that $h_{a_1} \cdots h_{a_{\mathrm{rk}(F)}} = h_F^{\mathrm{rk}(F)}$. By the direct sum decomposition in Proposition 3.1, the subalgebra generated by the h_a for a an atom has a basis given by $\{h_F^{\mathrm{rk}(F)}\}_{F \in \mathcal{L}_\mathrm{M}}$. We therefore see that this algebra is isomorphic to $H^\bullet(\mathrm{M})$. □

For a matroid M, let $\mathrm{H}_{\mathrm{M}}(t)$ be the Hilbert series of $A^\bullet(\mathrm{M})$, and let $\underline{\mathrm{H}}_{\mathrm{M}}(t)$ be the Hilbert series of $\underline{A}^\bullet(\mathrm{M})$. These polynomials, which are sometimes called (augmented) Chow polynomials, have been extensively studied in [JKU21] and especially [FMSV22], where the authors derive several recursive relations between them. The analysis in this section immediately generalizes to $\underline{A}^\bullet(\mathrm{M})$, and this gives new recursions for $\mathrm{H}_{\mathrm{M}}(t)$ and $\underline{\mathrm{H}}_{\mathrm{M}}(t)$.

Corollary 3.5. *We have that*

$$\mathrm{H}_{\mathrm{M}}(t) = 1 + \sum_{F \in \overline{\mathcal{L}}_{\mathrm{M}}} t \cdot \mathrm{H}_{\mathrm{Tr}\,\mathrm{M}^F}(t) \quad \textit{and} \quad \underline{\mathrm{H}}_{\mathrm{M}}(t) = 1 + \sum_{F \in \mathcal{L}_{\mathrm{M}},\, \mathrm{rk}(F) \geq 2} t \cdot \underline{\mathrm{H}}_{\mathrm{Tr}\,\mathrm{M}^F}(t).$$

Using Lemma 3.3, we give a second proof of Theorem SM; Theorem $\underline{\mathrm{SM}}$ can be proved similarly. Note that the proof of Lemma 3.3 used Poincaré duality for $A^\bullet(\mathrm{Tr}\,\mathrm{M})$.

Proof of Theorem SM. We have the decomposition

$$A^\bullet(\mathrm{M}) = \mathbb{Z} \oplus \bigoplus_{F \in \overline{\mathcal{L}}_{\mathrm{M}}} A^\bullet(\mathrm{Tr}\,\mathrm{M}^F)[-1].$$

By induction, we have a standard monomial basis for each summand on the right-hand side. In the above decomposition, a monomial $h_{G_1}^{a_1} \cdots h_{G_k}^{a_k}$ in $A^\bullet(\mathrm{Tr}\,\mathrm{M}^F)$ is mapped to the monomial $h_{G_1}^{a_1} \cdots h_{G_k}^{a_k} \cdot h_F$ in $A^\bullet(\mathrm{M})$. As $h_{G_1}^{a_1} \cdots h_{G_k}^{a_k}$ is standard in $A^\bullet(\mathrm{Tr}\,\mathrm{M}^F)$ if and only if $h_{G_1}^{a_1} \cdots h_{G_k}^{a_k} \cdot h_F$ in $A^\bullet(\mathrm{M})$, this implies the result. □

Remark 3.6. The geometry of the decomposition in Proposition 3.1 is explained in [Rai10, Section 2]. For each $F \in \overline{\mathcal{L}}_{\mathrm{M}}$, there is an idempotent projection $A^\bullet(\mathrm{M}) \to A^\bullet(\mathrm{M})$ given by $h_G \mapsto h_G$ if $G \leq F$, and otherwise $h_G \mapsto 0$. This map factors through $A^\bullet(\mathrm{M}^F)$, and, when M is realizable, it arises from a retraction of the augmented wonderful variety of a realization whose image is the augmented wonderful variety of a realization of M^F. These projections commute, and $A^\bullet(\mathrm{M})_F$ is the set of elements of $A^\bullet(\mathrm{M})$ which are fixed by the projection associated to F and killed by the projection associated to G for all $G < F$.

4. Algebras with straightening laws

In this section, we construct an algebra with straightening law which is closely related to the (augmented) Chow ring of a matroid. Algebras with straightening laws, also known as *ordinal Hodge algebras* [DCEP82], are certain algebras which are equipped with a standard monomial basis. We follow [BV88] for conventions on algebras with straightening laws.

Definition 4.1. Let $B^\bullet$ be a graded algebra over a ring R, and let $(\Pi, \leq)$ be a finite poset equipped with an injection $\Pi \to B^\bullet$ which identifies Π with a subset of $B^\bullet$. Assume that $B^0 = R$, and that the elements of Π are homogeneous of positive degree. We say that $B^\bullet$ is an *algebra with straightening law* over Π if

(1) the *standard monomials* $\{y_1^{a_1} \cdots y_k^{a_k} : y_1 \leq \cdots \leq y_k \in \Pi\}$ form an R-basis for $B^\bullet$, and
(2) for each $x, y \in \Pi$ incomparable, when we express xy in terms of the standard monomial basis $xy = \sum a_\mu \mu$, where $a_\mu \in R$ and μ is a standard monomial, each μ with $a_\mu \neq 0$ contains a factor of some $z \in \Pi$ with $z < x$ and $z < y$.

We will work in a more general setting than matroids. Let $\mathcal{L}$ be a finite *meet-semilattice*, i.e., a finite partially ordered set where any two elements x, y have a greatest lower bound $x \wedge y$. There is a minimal element $\hat{0}$ of $\mathcal{L}$. Our main example will be $\overline{\mathcal{L}}_{\mathrm{M}}^{\mathrm{op}}$, i.e., the inverted poset of flats of a matroid with the empty set removed. Here the minimal element is E.

Theorem 4.2. *Let $\mathcal{L}$ be a finite meet-semilattice, and let*

$$B^\bullet(\mathcal{L}) = \frac{\mathbb{Z}[h_x]_{x\in\mathcal{L}}}{((h_x - h_{x\wedge y})(h_y - h_{x\wedge y}) : x, y \in \mathcal{L})},\ \textit{with } h_x \textit{ in degree } 1.$$

Then $B^\bullet(\mathcal{L})$ is an algebra with straightening law over $\mathcal{L}$.

When $\mathcal{L} = \overline{\mathcal{L}}_{\mathrm{M}}^{\mathrm{op}}$, then

$$B^\bullet(\mathcal{L}) = \frac{\mathbb{Z}[h_F]_{F\in\overline{\mathcal{L}}_{\mathrm{M}}}}{((h_F - h_{G\vee F})(h_G - h_{G\vee F}) : F, G \in \overline{\mathcal{L}}_{\mathrm{M}})}.$$

There is a quotient map from $B^\bullet(\mathcal{L})$ to $A^\bullet(\mathrm{M})$ and $\underline{A}^\bullet(\mathrm{M})$. In particular, the straightening procedure used in the proof of Lemma 2.2 is a shadow of the straightening law on $B^\bullet(\mathcal{L})$. This is made precise in the proof of Theorem $\underline{\mathrm{SM}}$ at the end of this section.

The order complex of $\mathcal{L}$ is the simplicial complex whose faces are given by chains in $\mathcal{L}$. Let $C^\bullet(\mathcal{L})$ denote the Stanley–Reisner ring of the order complex of $\mathcal{L}$, with variables $\{s_x : x \in \mathcal{L}\}$. The theory of algebras with straightening laws shows that $B^\bullet(\mathcal{L})$ has a Gröbner degeneration to $C^\bullet(\mathcal{L})$. Note that $B^\bullet(\mathcal{L})$ is itself isomorphic to the Stanley–Reisner ring of the order complex of $\mathcal{L}$, via the map which sends h_x to $\sum_{y\leq x} s_y$. Note that this is not an isomorphism of algebras with straightening laws when $C^\bullet(\mathcal{L})$ is considered with the injection $\mathcal{L} \to C^\bullet(\mathcal{L})$ by $x \mapsto s_x$.

The proof of Theorem 4.2 is similar to the geometric argument used to show that the homogeneous coordinate ring of a Schubert variety in the Grassmannian is an algebra with straightening law, see [DCEP82, Proposition 1.3]. We prepare for the proof of Theorem 4.2 with a lemma. We thank Aldo Conca for explaining the proof to us.

Lemma 4.3. *The element $h_{\hat{0}}$ is a non zero-divisor in $B^\bullet(\mathcal{L})$.*

Proof. Choose an ordering $x_1, \ldots, x_n$ of the elements of $\mathcal{L}$ where $x_n = \hat{0}$. For $i = 1, \ldots, n-1$, set $u_i = h_{x_i} - h_{x_{i+1}}$. Then the elements $u_1, \ldots, u_{n-1}, h_{x_n}$ form a basis for the degree 1 part of the polynomial ring $\mathbb{Z}[h_F]_{F\in\hat{\mathcal{L}}}$. After we change to this basis, none of the elements of ideal defining $B^\bullet(\mathcal{L})$ involve $h_{x_n} = h_{\hat{0}}$. As the ideal of $B^\bullet(\mathcal{L})$ is not the unit ideal because it is graded, $h_{\hat{0}}$ is a non zero-divisor. □

Proof of Theorem 4.2. If $x, y \in \mathcal{L}$ are incomparable, then the relation

$$h_x h_y = h_x h_{x\wedge y} + h_y h_{x\wedge y} - h_{x\wedge y}^2$$

shows that Definition 4.1(2) is satisfied. The argument in Lemma 2.2 shows that $B^\bullet(\mathcal{L})$ is spanned by standard monomials, so it suffices to show that the standard monomials are linearly independent. Adjoin a maximal element $\hat{1}$ to $\mathcal{L}$ to form $\hat{\mathcal{L}}$. Let $B^\bullet(\mathcal{L})_x$ be the span of monomials $h_{y_1}^{a_1} \cdots h_{y_k}^{a_k}$ such that $y_1 \wedge \cdots \wedge y_k = x$. For example, $B^\bullet(\mathcal{L})_{\hat{1}} = \mathrm{span}(1)$. As in the proof of Proposition 3.1, there is a direct sum decomposition

$$B^\bullet(\mathcal{L}) = \bigoplus_{x\in\hat{\mathcal{L}}} B^\bullet(\mathcal{L})_x.$$

It therefore suffices to show that the standard monomials $h_{x_1}^{a_1} \cdots h_{x_k}^{a_k}$, with $x_1 \leq \cdots \leq x_k$, are linearly independent in $B^\bullet(\mathcal{L})_{x_1}$.

Let $\hat{\mathcal{L}}_x$ be the interval $[x, \hat{1}]$ in $\hat{\mathcal{L}}$. We see that $\hat{\mathcal{L}}_x \setminus \hat{1}$ is a meet semilattice, and, as in the proof of Lemma 3.2, we have

$$B^\bullet(\mathcal{L})_x \xrightarrow{\sim} B^\bullet(\hat{\mathcal{L}}_x \setminus \hat{1})_{\hat{0}}.$$

In particular, by induction it suffices to show that the standard monomials where $h_{\hat{0}}$ appears are linearly independent. If there was a linear dependence among the standard monomials where $h_{\hat{0}}$ appears, then that would imply that $h_{\hat{0}}$ is a zero-divisor, which contradicts Lemma 4.3. □

One could alternatively establish the linear independence of the standard monomials using the isomorphism $C^\bullet(\mathcal{L}) \to B^\bullet(\mathcal{L})$.

Using Theorem 4.2 in the case $\mathcal{L} = \overline{\mathcal{L}}_{\mathrm{M}}^{\mathrm{op}}$, we can give another proof of Theorem $\underline{\mathrm{SM}}$. One can prove Theorem SM using a similar but more lengthy argument.

Proof of Theorem $\underline{SM}$. We will use Theorem 4.2 to construct a linear endomorphism of ψ of $\mathbb{Z}[h_F]_{F \in \overline{\mathcal{L}}_{\mathrm{M}}}$ whose image is the span of the standard monomials and whose kernel is the ideal defining $\underline{A}^\bullet(\mathrm{M})$. This gives an (abelian group) direct sum decomposition of the polynomial ring, which implies that the standard monomials form a basis for $\underline{A}^\bullet(\mathrm{M})$.

Let $C \subseteq \mathbb{Z}[h_F]_{F \in \overline{\mathcal{L}}_{\mathrm{M}}}$ be the linear span of $\{h_{F_1}^{a_1} \cdots h_{F_\ell}^{a_\ell} : \emptyset < F_1 < \cdots < F_\ell\}$. Theorem 4.2 gives a surjective linear map $\psi_1 \colon \mathbb{Z}[h_F]_{F \in \overline{\mathcal{L}}_{\mathrm{M}}} \to C$: we consider the image of an element of $\mathbb{Z}[h_F]_{F \in \overline{\mathcal{L}}_{\mathrm{M}}}$ inside of $B^\bullet(\overline{\mathcal{L}}_{\mathrm{M}}^{\mathrm{op}})$ and then express it in terms of the standard monomial basis there.

The proofs of Proposition 2.1 and Proposition 2.8 give a map ψ_2 from C to the linear span of the standard monomials: the proof of Proposition 2.1 shows that each monomial corresponding to a chain of flats is either 0 in $\underline{A}^\bullet(\mathrm{M})$ or is equal to a particular standard monomial (which is independent of the choices involved). We define ψ to be $\psi_2 \circ \psi_1$. The well-definedness of ψ implies that the procedure described in the proof of Proposition 2.1 and Proposition 2.8 which rewrites any monomial in terms of the standard monomials for $\underline{A}^\bullet(\mathrm{M})$ is well-defined, i.e., independent of the choices involved.

It is clear that ψ surjects onto the span of the standard monomials; we need to show that the ideal defining $\underline{A}^\bullet(\mathrm{M})$ is in the kernel of ψ. By construction, the kernel of ψ is contained in the ideal defining $\underline{A}^\bullet(\mathrm{M})$. As ψ is linear, it suffices to prove that ψ kills the product of any monomial m with a generator of the ideal defining $\underline{A}^\bullet(\mathrm{M})$.

By construction, $\psi_1(m \cdot (h_F - h_{F\vee G})(h_G - h_{F \vee G})) = 0$ for any incomparable flats F, G, so ψ kills $m \cdot (h_F - h_{F\vee G})(h_G - h_{F\vee G})$ as well.

We need to check that, for any atom a and monomial m, we have $\psi(m \cdot h_a) = 0$. We apply the procedure used to compute ψ_1 (as described in Lemma 2.2) to $m \cdot h_a$, i.e., we find a pair of flats $\{F, G\}$, where h_F and h_G appear in $m \cdot h_a$, which are incomparable and which are maximal with these properties. We then use the relation $h_F h_G = h_F h_{F\vee G} + h_G h_{F \vee G} - h_{F\vee G}^2$. If $a \notin \{F, G\}$, then all resulting terms are divisible by h_a. Note that applying ψ_2 kills any term where h_a appear.

It therefore suffices to understand the case when there is a flat F such that h_F appears in $m \cdot h_a$, F is incomparable with a, and for all G with h_G appearing $m \cdot h_a$, either $G = a$, $G \le F$, or $G \ge F \vee a$. Define m' by $m = m' \cdot h_F$ We use the relation

$$m \cdot h_a h_F = m \cdot h_a h_{F\vee a} + m \cdot h_F h_{F \vee a} - m \cdot h_{F\vee a}^2.$$

The terms appearing after further straightening of $m \cdot h_F h_{F\vee a}$ will be the same as those in $m \cdot h_{F\vee a}^2$, except with $h_F h_{F\vee a}$ replaced by $h_{F\vee a}^2$. But these terms will cancel when we apply ψ_2. □

References

[ADH23] Federico Ardila, Graham Denham, and June Huh, *Lagrangian geometry of matroids*, J. Amer. Math. Soc. **36** (2023), no. 3, 727–794. MR4583774

[AHK18] Karim Adiprasito, June Huh, and Eric Katz, *Hodge theory for combinatorial geometries*, Ann. of Math. (2) **188** (2018), no. 2, 381–452. MR3862944

[BDF22] Christin Bibby, Graham Denham, and Eva Maria Feichtner, *A Leray model for the Orlik-Solomon algebra*, Int. Math. Res. Not. IMRN **24** (2022), 19105–19174. MR4523245

[BES] Spencer Backman, Christopher Eur, and Connor Simpson, *Simplicial generation of Chow rings of matroids*, J. Eur. Math. Soc. (JEMS) (to appear).

[BHM+22] Tom Braden, June Huh, Jacob P. Matherne, Nicholas Proudfoot, and Botong Wang, *A semi-small decomposition of the Chow ring of a matroid*, Adv. Math. **409** (2022), Paper No. 108646. MR4477425

[BHM+] Tom Braden, June Huh, Jacob Matherne, Nicholas Proudfoot, and Botong Wang, *Singular Hodge theory for combinatorial geometries*. arXiv:2010.06088v4.

[BST23] Andrew Berget, Hunter Spink, and Dennis Tseng, *Log-concavity of matroid h-vectors and mixed Eulerian numbers*, Duke Math. J. **172** (2023), no. 18, 3475–3520. MR4718433

[BV88] Winfried Bruns and Udo Vetter, *Determinantal rings*, Lecture Notes in Mathematics, vol. 1327, Springer-Verlag, Berlin, 1988. MR953963

[DCEP82] Corrado De Concini, David Eisenbud, and Claudio Procesi, *Hodge algebras*, Astérisque, vol. 91, Société Mathématique de France, Paris, 1982. With a French summary. MR680936

[DCP95] Corrado De Concini and Claudio Procesi, *Wonderful models of subspace arrangements*, Selecta Math. (N.S.) **1** (1995), no. 3, 459–494. MR1366622

[DR22] Jeshu Dastidar and Dustin Ross, *Matroid psi classes*, Selecta Math. (N.S.) **28** (2022), no. 3, Paper No. 55, 38. MR4405747

[EFLS24] Christopher Eur, Alex Fink, Matt Larson, and Hunter Spink, *Signed permutohedra, delta-matroids, and beyond*, Proc. Lond. Math. Soc. (3) **128** (2024), no. 3, Paper No. e12592, 54. MR4723151

[EHKR10] Pavel Etingof, André Henriques, Joel Kamnitzer, and Eric M. Rains, *The cohomology ring of the real locus of the moduli space of stable curves of genus 0 with marked points*, Ann. of Math. (2) **171** (2010), no. 2, 731–777. MR2630055

[EHL23] Christopher Eur, June Huh, and Matt Larson, *Stellahedral geometry of matroids*, Forum Math. Pi **11** (2023), Paper No. e24. MR4653766

[EL24] Christopher Eur and Matt Larson, *Intersection theory of polymatroids*, Int. Math. Res. Not. IMRN **5** (2024), 4207–4241. MR4714340

[FMSV22] Luis Ferroni, Jacob P. Matherne, Matthew Stevens, and Lorenzo Vecchi, *Hilbert-Poincaré series of matroid Chow rings and intersection cohomology*, 2022. arXiv:2212.03190v2.

[FY04] Eva Maria Feichtner and Sergey Yuzvinsky, *Chow rings of toric varieties defined by atomic lattices*, Invent. Math. **155** (2004), no. 3, 515–536. MR2038195

[Ham17] Simon Hampe, *The intersection ring of matroids*, J. Combin. Theory Ser. B **122** (2017), 578–614. MR3575220

[JKU21] David Jensen, Max Kutler, and Jeremy Usatine, *The motivic zeta functions of a matroid*, J. Lond. Math. Soc. (2) **103** (2021), no. 2, 604–632. MR4230913

[LLPP24] Matt Larson, Shiyue Li, Sam Payne, and Nicholas Proudfoot, *K-rings of wonderful varieties and matroids*, Adv. Math. **441** (2024), Paper No. 109554. MR4710863

[MM23] Matthew Mastroeni and Jason McCullough, *Chow rings of matroids are Koszul*, Math. Ann. **387** (2023), no. 3-4, 1819–1851. MR4657438

[Pos09] Alexander Postnikov, *Permutohedra, associahedra, and beyond*, Int. Math. Res. Not. IMRN **6** (2009), 1026–1106. MR2487491

[PP23] Roberto Pagaria and Gian Marco Pezzoli, *Hodge Theory for Polymatroids*, Int. Math. Res. Not. IMRN **23** (2023), 20118–20168. MR4675068

[Rad42] Richard Rado, *A theorem on independence relations*, Quart. J. Math. Oxford Ser. **13** (1942), 83–89. MR8250

[Rai10] Eric M. Rains, *The homology of real subspace arrangements*, J. Topol. **3** (2010), no. 4, 786–818. MR2746338

[Spe08] David E. Speyer, *Tropical linear spaces*, SIAM J. Discrete Math. **22** (2008), no. 4, 1527–1558. MR2448909

[Yuz02] Sergey Yuzvinsky, *Small rational model of subspace complement*, Trans. Amer. Math. Soc. **354** (2002), no. 5, 1921–1945. MR1881024

[Yuz97] ———, *Cohomology bases for the De Concini-Procesi models of hyperplane arrangements and sums over trees*, Invent. Math. **127** (1997), no. 2, 319–335. MR1427621

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