RANK FUNCTIONS AND INVARIANTS OF DELTA-MATROIDS

MATT LARSON

Abstract. In this note, we give a rank function axiomatization for delta-matroids and study the corresponding rank generating function. We relate an evaluation of the rank generating function to the number of independent sets of the delta-matroid, and we prove a log-concavity result for that evaluation using the theory of Lorentzian polynomials.

1. Introduction

Let \([n, \overline{n}]\) denote the set \(\{1, \ldots, n, 1, \ldots, \overline{n}\}\), equipped with the obvious involution \(\overline{\cdot}\). Let AdS\(_n\) be the set of admissible subsets of \([n, \overline{n}]\), i.e., subsets \(S\) that contain at most one of \(i\) and \(\overline{i}\) for each \(i \in [n]\). Set \(e_\overline{i} = -e_i \in \mathbb{R}^n\), and for each \(S \in \text{AdS}_n\), set \(e_S = \sum_{a \in S} e_a\).

Definition 1.1. A delta-matroid \(D\) is a collection \(\mathcal{F} \subset \text{AdS}_n\) of admissible sets of size \(n\), called the feasible sets of \(D\), such that the polytope \(P(D) := \text{Conv}\{e_B : B \in \mathcal{F}\}\) has all edges parallel to \(e_i\) or \(e_i \pm e_j\), for some \(i, j\). We say that \(D\) is even if all edges of \(P(D)\) are parallel to \(e_i \pm e_j\).

Delta-matroids were introduced in [Bou87] by replacing the usual basis exchange axiom for matroids with one involving symmetric difference. They were defined independently in [CK88, DH86]. For the equivalence of the definition of delta-matroids in those works with the one given above, and for general properties of delta-matroids, see [BGW03, Chapter 4].

A delta-matroid is even if and only if all sets in \(\{B \cap [n] : B \in \mathcal{F}\}\) have the same parity. Even delta-matroids enjoy nicer properties than arbitrary delta-matroids. For instance, they satisfy a version of the symmetric exchange axiom [Wen93].

There are many constructions of delta-matroids in the literature. Two of the most fundamental come from matroids: given a matroid \(M\) on \([n]\), we can construct a delta-matroid on \([n, \overline{n}]\) whose feasible sets are the sets of the form \(B \cup \overline{B}\), for \(B\) a basis of \(M\). We can also construct a delta-matroid whose feasible sets are the sets of the form \(I \cup \overline{I}\), for \(I\) independent in \(M\). Additionally, there are delta-matroids corresponding to graphs [Duc92], graphs embedded in surfaces [CMNR19, CMNR19b], and points of a maximal orthogonal or symplectic Grassmannian. Delta-matroids arising from points of a maximal orthogonal or symplectic Grassmannian are called realizable. See [EFLS, Section 6.2] for a discussion of delta-matroids associated to points of a maximal orthogonal Grassmannian.

Given \(S, T \in \text{AdS}_n\), we define \(S \sqcup T = \{a \in S \cup T : \overline{a} \notin S \cup T\}\). A function \(g : \text{AdS}_n \to \mathbb{R}\) is called bisubmodular if, for all \(S, T \in \text{AdS}_n\),

\[
f(S) + f(T) \geq f(S \cap T) + f(S \cup T).\]

There is a large literature on bisubmodular functions, beginning with [DW73]. They have been studied both from an optimization perspective [FI05, Fuj17] and from a polytopal perspective [PP94, Fuj14]. Additionally, bisubmodular functions are closely related to jump systems [BC95].
For a delta-matroid $D$, define a function $g_D : \text{AdS}_n \to \mathbb{Z}$ by
$$g_D(S) = \max_{B \in \mathcal{P}} (\vert S \cap B \vert - \vert \overline{S} \cap B \vert).$$

We call $g_D$ the rank function of $D$. Note that $g_D$ may take negative values. The collection of feasible subsets of $D$ is exactly $\{S : g_D(S) = n\}$, so $D$ can be recovered from $g_D$.

**Theorem 1.2.** A function $g : \text{AdS}_n \to \mathbb{Z}$ is the rank function of a delta-matroid if and only if

1. $g(\emptyset) = 0$ (normalization),
2. $|g(S)| \leq 1$ if $|S| = 1$ (boundedness),
3. $g(S) + g(T) \geq g(S \cap T) + g(S \cup T)$ (submodularity), and
4. $g(S) \equiv |S|$ (mod 2) (parity).

Furthermore, $D$ is even if and only if
$$g_D(S) = \frac{g_D(S \cup i) + g_D(S \cup i)}{2} \text{ whenever } |S| = n - 1 \text{ and } \{i, \overline{i}\} \cap S = \emptyset.$$

The function $g_D$, as well as the observation that it is bisubmodular, has appeared before in the literature [Bou88, CK88]. For example, in [Bou88, Theorem 4.1] it is shown that, if $D$ is a maximal symplectic Grassmannian, then $g_D$ can be computed in terms of the rank of a certain matrix. It was known that delta-matroids admit a description in terms of certain bisubmodular functions. However, the precise characterization in Theorem 1.2 does not appear to have been known before. Indeed, Theorem 1.2 answers a special case of [ACEP20, Question 9.4].

In [Bou97, Bou98], Bouchet gave a rank-function axiomatization of delta-matroids in the more general setting of multimatroids. His rank function differs from ours — in Section 2.2 we discuss the relationship between his results and Theorem 1.2.

Basic operations on delta-matroids — like products, deletion, contraction, and projection — can be simply expressed in terms of rank functions. See Section 2.1.

One of the most important invariants of a matroid $M$ of rank $r$ on $[n]$ is its Whitney rank generating function. If $\text{rk}_M$ is the rank function of $M$, then the rank generating function is defined as
$$R_M(u, v) := \sum_{A \subseteq [n]} u^{r - \text{rk}_M(A)} v^{|A| - r}.$$

The more commonly used normalization is the Tutte polynomial, which is $R_M(u - 1, v - 1)$. The characterization of delta-matroids in terms of rank functions allows us to consider an analogously-defined invariant.

**Definition 1.3.** Let $D$ be a delta-matroid on $[n, \pi]$. Then we define
$$U_D(u, v) = \sum_{S \in \text{AdS}_n} u^{-|S|} v^{|S| - g_D(S)}.$$

Note that the bisubmodularity of $g_D$ implies that the restriction of $g_D$ to the subsets of any fixed $S \in \text{AdS}_n$ is submodular. The boundedness of $g_D$ then implies that $|g_D(S)| \leq |S|$. Because of the parity requirement, $|S| - g_D(S)$ is divisible by 2. Therefore $U_D(u, v)$ is indeed a polynomial. The normalization $U_D(u - 1, v - 1)$ is more analogous to the Tutte polynomial, but it can have negative coefficients. However, the polynomial $U_D(u, v - 1)$ has non-negative coefficients (as follows, e.g., from Theorem 3.3).

The $U$-polynomial of a delta-matroid was introduced by Eur, Fink, Spink, and the author in [EFLS Definition 1.4] in terms of a Tutte polynomial-like recursion; see Proposition 3.1 for a proof that Definition 1.3 agrees with the recursive definition considered there. The specialization $U_D(0, v)$ is the interlace polynomial of $D$, which was introduced in [ABS04] for graphs and in [BH14] for general delta-matroids. See [Mor17] for a survey on the properties of the interlace polynomial.
Various Tutte polynomial-like invariants of delta-matroids have been considered in the literature, such as the Bollobás–Riordan polynomial and its specializations [BR01]. In [KMT18], a detailed analysis of delta-matroid polynomials which satisfy a deletion-contraction formula is carried out. Set \( \sigma_D(A) = \frac{|A|}{2} + \frac{a_D(A) + a_D(\bar{A})}{4} \) for \( A \subseteq [n] \). Then in [KMT18], the polynomial
\[
\sum_{A \subseteq [n]} (x-1)^{\sigma_D([n])-\sigma_D(A)}(y-1)^{|A|-\sigma_D(A)}
\]
is shown to be, in an appropriate sense, the universal invariant of delta-matroids which satisfies a deletion-contraction formula. This polynomial is a specialization of the Bollobás–Riordan polynomial. In [EMGM+22], it is shown that this polynomial has several nice combinatorial properties.

**Example 1.4.** [EFLS] Example 5.5 and 5.6] Let \( M \) be a matroid of rank \( r \) on \([n]\), and let \( S = S^+ \cup S^- \in \text{AdS}_n \) be an admissible set with \( S^+, S^- \subset [n] \). Set \( V = \{ i \in [n] : S \cap \{i, \bar{i}\} = \emptyset \} \). Above, we gave two examples of delta-matroids constructed from \( M \).

1. Let \( D \) be the delta-matroid arising from the independent sets of \( M \). Then \( g_D(S) = |S| + 2 \text{rk}_M(S^+) - 2|S^+| \), and
   \[
   U_D(u, v) = (u+1)^{n-r} R_M \left( u + 3, \frac{2u + v + 2}{u + 1} \right).
   \]
2. Let \( D \) be the delta-matroid arising from the bases of \( M \). Then \( g_D(S) = |S| - 2r + 2 \text{rk}_M(S^+ \cup V) - 2|S^+| + 2 \text{rk}_M(S^+) \), and
   \[
   U_D(u, v) = \sum_{T \subseteq S \subseteq [n]} w^{S \setminus T} v^{r - \text{rk}_M(S)+|T|-\text{rk}_M(T)}.
   \]

We study the \( U \)-polynomial as a delta-matroid analogue of the rank generating function of a matroid. For a matroid \( M \), the evaluation \( R_M(u, 0) \) is essentially the \( f \)-vector of the independence complex of the matroid, i.e., it counts the number of independent sets of \( M \) of a given size. The coefficients of the Tutte polynomial \( R_M(u-1, v-1) \) can be interpreted as counting bases of \( M \) according to their internal and external activities, certain statistics that depend on an ordering of the ground set. See [Bac]. This shows that \( R_M(u, -1) \), the (unsigned) characteristic polynomial of \( M \), is essentially the \( f \)-vector of the broken circuit complex of \( M \).

A set \( S \in \text{AdS}_n \) is independent if it is contained in a feasible set of a delta-matroid \( D \). In [Bon07], Bouchet gave an axiomatization of delta-matroids in terms of their independent sets. The independent sets form a simplicial complex, called the independence complex of \( D \). We relate \( U_D(u, 0) \) to the \( f \)-vector of the independence complex of \( D \) (Proposition [4.4]), which gives linear inequalities between the coefficients of \( U_D(u, 0) \). We give a combinatorial interpretation of the coefficients of \( U_D(u, v-1) \) as counting the number of independent sets of \( D \) of a given size according to a delta-matroid version of activity (Theorem [3.3]). This shows that \( U_D(u, -1) \) is essentially the \( f \)-vector of a certain simplicial complex associated to \( D \).

Following a tradition in matroid theory (see, e.g., [Mas72]), and inspired by the ultra log-concavity of \( R_M(u, 0) \) [ALGY, BH20], we make three log-concavity conjectures for \( U_D(u, 0) \). These conjectures state the sequence of the number of independent sets of a delta-matroid of a given size satisfies log-concavity properties.

**Conjecture 1.5.** Let \( D \) be a delta-matroid on \([n, \overline{n}]\), and let \( U_D(u, 0) = a_n + a_{n-1}u + \cdots + a_0u^n \). Then, for any \( k \in \{1, \ldots, n-1\} \),

1. \( a_k^2 \geq \frac{n-k+1}{n-k} a_{k+1}a_{k-1} \),
2. \( a_k^2 \geq \frac{2n-k+1}{2n-k} a_{k+1}a_{k-1} \), and
3. \( a_k^2 \geq \frac{n-k+1}{n-k} a_{k+1}a_{k-1} \).
Conjecture 1.5(1) follows from [EFLS] Conjecture 1.5, and it is proven in [EFLS] Theorem B when \( D \) has an enveloping matroid (see Definition 3.11). This is a technical condition which is satisfied by many commonly occurring delta-matroids, including all realizable delta-matroids and delta-matroids arising from matroids (although not all delta-matroids, see [Bon97] Section 4 and [EFLS] Example 6.11). The proof uses algebro-geometric methods. Here we prove a special case of Conjecture 1.5(2).

**Theorem 1.6.** Let \( D \) be a delta-matroid on \([n, \overline{m}]\) which has an enveloping matroid. Let \( U_D(u, 0) = a_n + a_{n-1}u + \cdots + a_0u^n \). Then, for any \( k \in \{1, \ldots, n-1\} \), \( a_k^2 \geq \frac{2n-k+1}{2n-k} k \cdot a_{k+1}a_{k-1} \), i.e., Conjecture 1.5(2) holds.

Our argument uses the theory of Lorentzian polynomials [BH20]. We strengthen Theorem 1.6 by proving that a generating function for the independent sets of \( D \) is Lorentzian (Theorem 3.14), which implies the desired log-concavity statement. We deduce that this generating function is Lorentzian from the fact that the Potts model partition function of an enveloping matroid is Lorentzian [BH20] Theorem 4.10.

When \( D \) is the delta-matroid arising from the independent sets of a matroid, Conjecture 1.5(3) follows from the ultra log-concavity of the number of independent sets of that matroid [ALGV][BH20]. When \( D \) is the delta-matroid arising from the bases of a matroid \( M \) on \([n]\), which has an enveloping matroid by [EFLS] Proposition 6.10, Theorem 1.6 gives a new log-concavity result. If we set

\[
a_k = \{|T \subset S \cap [n] : T \text{ independent in } M \text{ and } S \text{ spanning in } M, |S \setminus T| = n-k\},
\]

then Theorem 1.6 gives that \( a_k^2 \geq \frac{2n-k+1}{2n-k} k \cdot a_{k+1}a_{k-1} \) for \( k \in \{1, \ldots, n-1\} \).

**Acknowledgements:** We thank Nima Anari, Christopher Eur, Satoru Fujishige, Steven Noble, and Hunter Spink for enlightening conversations, and we thank Christopher Eur, Steven Noble, and Shiyue Li for helpful comments on a previous version of this paper. The author is supported by an NDSEG fellowship.

## 2. Rank functions of delta-matroids

The proof of Theorem 1.2 goes by way of a polytopal description of normalized bisubmodular functions, which we now recall. To a function \( f : \text{AdS}_n \to \mathbb{R} \) with \( f(\emptyset) = 0 \), we associate the polytope

\[
P(f) = \{x : (e_S, x) \leq f(S) \text{ for all non-empty } S \in \text{AdS}_n\}.
\]

By [BC95] Theorem 4.5 or [ACEP20] Theorem 5.2, \( P(f) \) has all edges parallel to \( e_i \) or \( e_i \pm e_j \) if and only if \( f \) is bisubmodular. In this case, \( P(f) \) is a lattice polytope if and only if \( f \) is integer-valued. For a normalized (i.e., \( f(\emptyset) = 0 \)) bisubmodular function \( f \), we can recover \( f \) from \( P(f) \) via the formula

\[
f(S) = \max_{x \in P(f)} (e_S, x).
\]

Under this dictionary, the bisubmodular function corresponding to the dilate \( kP(f) \) is \( kf \), and the bisubmodular function corresponding to the Minkowski sum \( P(f) + P(g) \) is \( f + g \).

**Proof of Theorem 1.2.** By the polyhedral description of normalized bisubmodular functions, for each delta-matroid \( D \) there is a unique normalized bisubmodular function \( g \) such that \( P(D) = P(g) \). We show that the conditions on a normalized bisubmodular function \( g \) for \( P(D) \) to have all vertices in \([-1, 1]^n\) are exactly those given in Theorem 1.2, namely that \( |g(S)| \leq 1 \) when \( |S| = 1 \) and \( g(S) \equiv |S| \pmod{2} \).

The polytope \( P(g) \) has all vertices in \( \{\pm 1\}^n \) if and only if \( \frac{1}{2}(P(g) + (1, \ldots, 1)) \) is a lattice polytope which is contained in \([0, 1]^n\). The normalized bisubmodular function \( h \) corresponding to the point \((1, \ldots, 1)\) takes value \( h(S) = |S^+| - |S^-| \) on an admissible set of the form \( S = S^+ \cup S^- \), with \( S^+, S^- \subset [n] \). The polytope \( \frac{1}{2}(P(g) + (1, \ldots, 1)) \) is \( P(f) \), where \( f \) is the normalized bisubmodular function defined by \( f := \frac{1}{2}(g + h) \). We note that \( P(f) \) is a lattice polytope which is contained in \([0, 1]^n\) if and only if

1. \( f(i) \in \{0, 1\} \) and \( f(\overline{i}) \in \{-1, 0\} \), and
(2) $f$ is integer-valued.

A normalized bisubmodular function $f$ satisfies these conditions if and only if $g$ satisfies the conditions of Theorem 1.2, giving the characterization of rank functions of delta-matroids.

By [ACEP20, Example 5.2.3], the polytope $P(g_D) = P(D)$ has all edges parallel to $e_i \pm e_j$ if and only if $g_D$ satisfies the condition

$$g_D(S) = \frac{g_D(S \cup i) + g_D(S \cup \overline{i})}{2} \text{ whenever } |S| = n - 1 \text{ and } \{i, \overline{i}\} \cap S = \emptyset.$$

This gives the characterization of even delta-matroids. □

2.1. Compatibility with delta-matroid operations. In this section, we consider several operations on delta-matroids, and we show that the rank function behaves in a simple way under these operations. First we consider minor operations on delta-matroids — contraction, deletion, and projection.

**Definition 2.1.** Let $D$ be a delta-matroid on $[n, \overline{n}]$ with feasible sets $F$, and let $i \in [n]$. We say that $i$ is a **loop** of $D$ if no feasible set contains $i$, and we say that $i$ is a **coloop** if every feasible set contains $i$.

1. If $i$ is not a loop of $D$, then the **contraction** $D/i$ is the delta-matroid with feasible sets $B \setminus i$, for $B \in F$ containing $i$.
2. If $i$ is not a coloop of $D$, then the **deletion** $D \setminus i$ is the delta-matroid with feasible sets $B \setminus \overline{i}$, for $B \in F$ containing $i$.
3. The **projection** $D(i)$ is the delta-matroid with feasible sets $B \setminus \{i, \overline{i}\}$ for $B \in F$.
4. If $i$ is a loop or coloop, then set $D/i = D \setminus i = D(i)$.

For $A \subset [n]$, we define $D/A, D \setminus A$, and $D(A)$ to be the delta-matroids on $[n, \overline{n}] \setminus (A \cup \overline{A})$ obtained by successively contracting, deleting, or projecting away from all elements of $A$. Contractions, deletions, and projections at disjoint sets commute with each other, so this is well defined. If $A$ and $B$ are disjoint subsets of $[n]$, then $D/A \setminus B$ is the delta-matroid obtained by contracting $A$ and then deleting $B$, which is the same as first deleting $B$ and then contracting $A$.

First we describe the rank function of projections. The formula is analogous to the formula for the rank function of a matroid deletion.

**Proposition 2.2.** Let $D$ be a delta-matroid on $[n, \overline{n}]$, and let $A \subset [n]$. For each $S \in \text{AdS}_n$ disjoint from $A \cup \overline{A}$, $g_D(A)(S) = g_D(S)$.

**Proof.** As $S$ is disjoint from $A \cup \overline{A}$, $|B \cap S| - |B \cap \overline{S}|$ depends only on $B \setminus (A \cup \overline{A})$. The feasible sets of $D(A)$ are given by $B \setminus (A \cup \overline{A})$ for $B$ a feasible set of $D$. □

The rank functions of the contractions and deletions are described by the following result. The formula is analogous to the formula for the rank function of a matroid contraction.

**Proposition 2.3.** Let $D$ be a delta-matroid on $[n, \overline{n}]$. Let $A, B \subset [n]$ be disjoint subsets, and let $S \in \text{AdS}_n$ be disjoint from $A \cup B \cup A \cup B$. Then $g_D(A \setminus B)(S) = g_D(S \cup A \cup \overline{B}) - g_D(A \cup \overline{B})$.

Before proving this, we will need the following property of delta-matroids. It follows, for instance, from the greedy algorithm description of delta-matroids in [BC95].

**Proposition 2.4.** Let $D$ be a delta-matroid on $[n, \overline{n}]$, and let $S \subset X \in \text{AdS}_n$. Let $F_S$ be the collection of feasible sets $B$ of $D$ that maximize $|S \cap B|$, i.e., have $|S \cap B| = \max_{B' \in F} |S \cap B'|$. Then

$$\max_{B \in F_S} |T \cap B| = \max_{B' \in F} |T \cap B|.$$

First we consider the case when we delete or contract a single element.
Lemma 2.5. Let $D$ be a delta-matroid on $[n, \overline{n}]$, and let $i \in [n]$. Then

1. If $i$ is not a loop, then $g_{D \setminus i}(S) = g_D(S \cup i) - 1$,
2. If $i$ is not a coloop, then $g_{D \setminus i}(S) = g_D(S \cup i) - 1$.

Proof. We do the case of contraction; the case of deletion is identical. Assume that $i$ is not a loop, and let $F_i$ denote the set of feasible sets of $D$ which contain $i$. Note that $F_i$ is non-empty, so it is the collection of feasible sets $B$ of $D$ which maximize $|\{ i \} \cap B|$. For any $S \in \text{AdS}_n$ with $S \cap \{ i, \overline{i} \} = \emptyset$, by Proposition 2.4 we have that

$$\max_{B \in F_i} |(S \cup i) \cap B| = \max_{B \in F_i} |(S \cup i) \cap B|.$$ 

For any $B$, $|(S \cup i) \cap B| - |(S \cup i) \cap B| = 2|\{ i \} \cap B| - |\{ i \} \cap B|$, so we see that

$$\max_{B \in F_i} (|(S \cup i) \cap B| - |(S \cup i) \cap B|) = \max_{B \in F_i} (|(S \cup i) \cap B| - |(S \cup i) \cap B|).$$

The left-hand side is equal to $g_D(S \cup i)$, and the right-hand side is equal to $g_{D \setminus i}(S) + 1$. \hfill \Box

Proof of Proposition 2.5. First note that $g_D(i) = 1$ if $i$ is not a loop and is $-1$ if $i$ is a loop, and similarly $g_D(\overline{i}) = 1$ if $i$ is not a coloop and is $-1$ if $i$ is a coloop. So Lemma 2.5 implies the result holds when $|S| = 1$.

We induct on the size of $A \cup B$. We consider the case of adding an element $i \in [n]$ to $A$; the case of adding it to $B$ is identical. We compute:

$$g_{D \setminus (A \cup B)}(S) = g_{D/(A \cup B)}(S \cup i) - g_{D/A \cup B}(i)$$

$$= g_D(S \cup A \cup \overline{B} \cup i) - g_D(A \cup \overline{B}) - (g_D(A \cup \overline{B} \cup i) - g_D(A \cup \overline{B}))$$

$$= g_D(S \cup (A \cup i) \cup \overline{B}) - g_D((A \cup i) \cup \overline{B}).$$ \hfill \Box

For two non-negative integers $n_1, n_2$, identify the disjoint union of $[n_1]$ and $[n_2]$ with $[n_1 + n_2]$. Given two delta-matroids $D_1, D_2$ on $[n_1]$ and $[n_2]$, let $D_1 \times D_2$ be the delta-matroid on $[n_1 + n_2]$ whose feasible sets are $B_1 \cup B_2$, for $B_j$ a feasible set of $D_j$. Then we have the following description of the rank function of $D_1 \times D_2$.

Proposition 2.6. Let $D_1, D_2$ be delta-matroids on $[n_1]$ and $[n_2]$, and let $S = S_1 \cup S_2$ be an admissible subset of $[n_1 + n_2, \overline{n_1 + n_2}]$, with $S_1 \subset [n_1, \overline{n_1}]$ and $S_2 \subset [n_2, \overline{n_2}]$. Then $g_{D_1 \times D_2}(S) = g_{D_1}(S_1) + g_{D_2}(S_2)$.

Proof. Let $B_1$ be a feasible set of $D_1$ with $g_{D_1}(S_1) = |S_1 \cap B_1| - |\overline{S_1} \cap B_1|$, and let $B_2$ be a feasible set of $D_2$ with $g_{D_2}(S_2) = |S_2 \cap B_2| - |\overline{S_2} \cap B_2|$. Then $B_1 \cup B_2$ maximizes $B \mapsto |S \cap B| - |\overline{S} \cap B|$, and so $g_{D_1 \times D_2}(S) = |S_1 \cap B_1| - |\overline{S_1} \cap B_1| + |S_2 \cap B_2| - |\overline{S_2} \cap B_2| = g_{D_1}(S_1) + g_{D_2}(S_2).$ \hfill \Box

We now study how the rank function behaves under the operation of twisting. Let $W$ be the signed permutation group, the subgroup of the symmetric group on $[n, \overline{n}]$ which preserves $\text{AdS}_n$. In other words, $W$ consists of permutations $w$ such that $w(\overline{i}) = \overline{w(i)}$. As delta-matroids are collections of admissible sets, $W$ acts on the set of delta-matroids on $[n, \overline{n}]$. This action is usually called twisting in the delta-matroid literature.

Proposition 2.7. Let $D$ be a delta-matroid on $[n, \overline{n}]$, and let $w \in W$. Then $g_{w \cdot D}(S) = g_D(w^{-1} \cdot S)$.

Proof. Note that, for $B$ a feasible set of $D$, $|S \cap (w \cdot B)| - |\overline{S} \cap (w \cdot B)| = |(w^{-1} \cdot S) \cap B| - |(w^{-1} \cdot \overline{S}) \cap B|$, which implies the result. \hfill \Box

Let $S \in \text{AdS}_n$ be an admissible set of size $n$. For any delta-matroid $D$ on $[n, \overline{n}]$, let $r$ be the maximal value of $|S \cap B|$. Then $\{ S \cap B : B \in F, |S \cap B| = r \}$ is the set of bases of a matroid on $S$. When $S = [n]$, this is sometimes called the upper matroid of $D$. We describe the rank function of this matroid in terms of the rank function of $D$. 

**Proposition 2.8.** Let $S \in \text{AdS}_n$ be an admissible set of size $n$, and let $D$ be a delta-matroid on $[n,\overline{n}]$ with $r = \max_{B \in \mathcal{F}} |S \cap B|$. The matroid $M$ on $S$ whose bases are \{ $S \cap B : B \in \mathcal{F}$, $|S \cap B| = r$ \} has rank function

$$
\text{rk}_M(T) = \frac{g_D(T) + |T|}{2}.
$$

**Proof.** Let $\mathcal{F}_S$ be the collection of feasible sets $B$ with $|S \cap B| = r$. Then we have that

$$
\text{rk}_M(T) = \max_{B \in \mathcal{F}_S} |T \cap B| \leq \max_{B \in \mathcal{F}} |T \cap B| = \frac{g_D(T) + |T|}{2}.
$$

On the other hand, by Proposition 2.4 there is a feasible set $B$ which maximizes $|T \cap B|$ and has $|S \cap B| = r$, so we have equality. \qed

2.2. **An alternative normalization.** The results of the previous section, particularly Proposition 2.8, suggest that an alternative normalization of the rank function of a delta-matroid has nice properties. Set

$$
h_D(S) := \frac{g_D(S) + |S|}{2}.
$$

The function $h_D(S)$ is integer-valued and bisubmodular, and the polytope it defines is $P(h_D) = \frac{1}{2}(P(D) + \Box)$, where $\Box = [-1,1]^n$ is the cube. This is because the bisubmodular function corresponding to $\Box$ is $S \mapsto |S|$. Note that the function $h_D$ is non-negative and increasing, in the sense that if $S \subset T \in \text{AdS}_n$, then $h_D(S) \leq h_D(T)$. Theorem 1.2 implies the following characterization of the functions arising as $h_D$ for some delta-matroid $D$.

**Corollary 2.9.** A function $h$: $\text{AdS}_n \to \mathbb{Z}$ is equal to $h_D$ for some delta-matroid $D$ if and only if

1. $h(\emptyset) = 0$ (normalization),
2. $h(S) \in \{0,1\}$ if $|S| = 1$ (boundedness),
3. $h(S) + h(T) \geq h(S \cap T) + h(S \cup T) + |S \cap \overline{T}|/2$.

Indeed, these are exactly the conditions we need for $g(S) := 2h(S) - |S|$ to satisfy the conditions in Theorem 1.2.

The function $h_D$ was studied by Bouchet in [Bou97, Bou98] in the more general setting of multimatroids. The following characterization of the functions $h_D$ follows from [Bou97, Proposition 4.2]:

**Proposition 2.10.** A function $h$: $\text{AdS}_n \to \mathbb{Z}$ is equal to $h_D$ for some delta-matroid $D$ if and only if

1. $h(\emptyset) = 0$,
2. $h(S) \leq h(S \cup a) \leq h(S) + 1$ if $S \cup a$ is admissible,
3. $h(S) + h(T) \geq h(S \cap T) + h(S \cup T)$ if $S \cup T$ is admissible, and
4. $h(S \cup i) + h(S \cup \overline{i}) \geq 2h(S) + 1$ if $S \cup \{i,\overline{i}\} = \emptyset$.

In [Bou98, Theorem 2.16], the following characterization of the functions $h_D$ is stated with a reference to an unpublished paper of Allys.

**Proposition 2.11.** A function $h$: $\text{AdS}_n \to \mathbb{Z}$ is equal to $h_D$ for some delta-matroid $D$ if and only if

1. $h(\emptyset) = 0$,
2. $h(S) \leq h(S \cup a) \leq h(S) + 1$ if $S \cup a$ is admissible, and
3. $h(S) + h(T) \geq h(S \cap T) + h(S \cup T) + |S \cap \overline{T}|$.

It is easy to see directly that a function which satisfies the hypotheses of Corollary 2.9 satisfies the hypotheses of Proposition 2.10 or Proposition 2.11. However, the converse does not seem obvious.
3. The $U$-Polynomial

We now study the $U$-polynomial of delta-matroids. We prove the following recursion for $U_D(u, v)$, which was the original definition of the $U$-polynomial in \cite{EFLS} Definition 1.4].

Proposition 3.1. If $n = 0$, the $U_D(u, v) = 1$. For any $i \in [n]$, the $U$-polynomial satisfies

$$U_D(u, v) = \begin{cases} U_D(i(u, v) + U_D(u, v) + uU_D(i)(u, v), & \text{i is neither a loop nor a coloop} \\ (u + v + 1) \cdot U_D(i(u, v), & \text{i is a loop or a coloop.} \end{cases}$$

First we study the behavior of the $U$-polynomial under products.

Lemma 3.2. Let $D_1, D_2$ be delta-matroids on $[n_1, \overline{\pi}_1]$ and $[n_2, \overline{\pi}_2]$. Then $U_{D_1 \times D_2}(u, v) = U_{D_1}(u, v)U_{D_2}(u, v)$.

Proof. We compute:

$$U_{D_1}(u, v)U_{D_2}(u, v) = \left( \sum_{S_1 \in \text{AdS}_{n_1}} u^{n_1 - |S_1|}v^{\frac{|S_1| - g_{D_1}(S_1)}{2}} \right) \left( \sum_{S_2 \in \text{AdS}_{n_2}} u^{n_2 - |S_2|}v^{\frac{|S_2| - g_{D_2}(S_2)}{2}} \right)$$

$$= \sum_{(S_1, S_2)} u^{n_1 + n_2 - |S_1| - |S_2|}v^{\frac{|S_1| + |S_2| - g_{D_1}(S_1) - g_{D_2}(S_2)}{2}}$$

$$= U_{D_1 \times D_2}(u, v),$$

where the third equality is Proposition 2.6. \hfill \square

Proof of Proposition 3.1. If $n = 0$, then the only admissible subset of $[n, \overline{\pi}]$ is the empty set, and $g_D(\emptyset) = 0$, so $U_D(u, v) = 1$. Now choose some $i \in [n]$. If $i$ is a loop or a coloop, then $D$ is the product of $D$ \setminus $i$ with a delta-matroid on 1 element with 1 feasible set. We observe that $U$-polynomial of a delta-matroid on 1 element with 1 feasible set is $u + v + 1$, and so Lemma 3.2 implies the recursion in this case.

If $i$ is neither a loop nor a coloop, the admissible subsets of $[n, \overline{\pi}]$ are partitioned into sets containing $i$, sets containing $\overline{i}$, and sets containing neither $i$ nor $\overline{i}$. If $S$ contains $i$, then $u^{n - |S|}v^{\frac{|S| - g_D(S)}{2}} = u^{n - 1 - |S| - 1}v^{\frac{|S| - g_D(S)}{2}}$. If $S$ contains $\overline{i}$, then $u^{n - |S|}v^{\frac{|S| - g_D(S)}{2}} = u \cdot u^{n - 1 - |S| - 1}v^{\frac{|S| - g_D(S)}{2}}$. Adding these up implies the recursion in this case.

3.1. The Independence Complex of a Delta-Matroid. In this section, we introduce the independence complex of a delta-matroid and use it to study the $U$-polynomial.

Definition 3.3. We say that $S \in \text{AdS}_n$ is independent in $D$ if $g_D(S) = |S|$, or, equivalently, if $S$ is contained in a feasible subset of $D$. The independence complex of $D$ is the simplicial complex on $[n, \overline{\pi}]$ whose facets are given by the feasible sets of $D$.

Let $S \in \text{AdS}_n$, and let $T = \{i \in [n] : S \cap \{i, \overline{i}\} = \emptyset\}$. Note $S$ is independent if and only if $S$ is a feasible set of $D(T)$.

The following result is immediate from the definition of $U_D(u, 0)$.

Proposition 3.4. Let $f_i(D)$ be the number of $i$-dimensional faces of the independence complex of $D$. Then $U_D(u, 0) = f_{n-1}(D) + f_{n-2}(D)u + \cdots + f_1(D)u^n$. 

Note that the \( f \)-vector of a pure simplicial complex, like the independence complex of a delta-matroid, is a pure \( O \)-sequence. Then [Hib89] gives the following inequalities.

**Corollary 3.5.** Let \( U_D(u, 0) = a_n + a_{n-1}u + \cdots + a_0u^n \). Then \( (a_0, \ldots, a_n) \) is the \( f \)-vector of a pure simplicial complex. In particular, \( a_i \leq a_{n-i} \) for \( i \leq n/2 \) and \( a_0 \leq a_1 \leq \cdots \leq a_{\lfloor n/2 \rfloor} \).

Proposition 3.4 is a delta-matroid analogue of the fact that, for a matroid \( M \), the coefficients of \( R_M(u, 0) \), when written backwards, are the face numbers of the independence complex of \( M \). The independence complex of a matroid is shellable [Bjo92], which is reflected in the fact that \( R_M(u - 1, 0) \) has non-negative coefficients. The independence complex of a delta-matroid is not in general shellable or Cohen–Macaulay, and \( U_D(u - 1, 0) \) can have negative coefficients.

Recall that \( \square = [-1, 1]^n \) is the cube. The map \( S \mapsto e_S \) induces a bijection between \( \text{AdS}_n \) and lattice points of \( \square \). We use this to give a polytopal description of the independent sets of \( D \), which will be useful in the sequel.

**Proposition 3.6.** The map \( S \mapsto e_S \) induces a bijection between independent sets of \( D \) and lattice points in \( \frac{1}{2}(P(D) + \square) \).

**Proof.** If \( S \) is independent in \( D \), then there is \( T \in \text{AdS}_n \) such that \( S \cup T \in \mathcal{F} \). Then \( e_S = \frac{1}{2}(e_{S \cup T} + e_{S \cup T}) \), so \( e_S \) lies in \( \frac{1}{2}(P(D) + \square) \).

The correspondence between normalized bisubmodular functions and polytopes gives that

\[
\frac{1}{2}(P(D) + \square) = \left\{ x : \langle e_S, x \rangle \leq \frac{g_D(S) + |S|}{2} \right\}.
\]

If \( S \) is not independent, then \( e_S \) violates the inequality \( \langle e_S, e_S \rangle \leq \frac{g_D(S) + |S|}{2} \), so \( e_S \) does not lie in \( \frac{1}{2}(P(D) + \square) \). \( \square \)

### 3.2. The activity expansion of the U-polynomial

We now discuss an expansion of \( U_D(u, v - 1) \) in terms of a statistic associated to each independent set of a delta-matroid \( D \), similar to the expansion of the Tutte polynomial of a matroid in terms of basis activities. We rely heavily on the work of Morse [Mor19], who gave such an expansion for the interlace polynomial \( U_D(0, v - 1) \). Throughout we fix the ordering \( 1 < 2 < \cdots < n \) on \([n]\). For \( S \in \text{AdS}_n \), let \( \tilde{S} \subset [n] \) denote the unsigned version of \( S \), i.e., the image of \( S \) under the quotient of \([n, \pi]\) by the involution.

**Definition 3.7.** Let \( B \) be a feasible set in a delta-matroid \( D \). We say that \( i \in [n] \) is \( B \)-orientable if the symmetric difference \( B \Delta \{i, \bar{i} \} \) is not a feasible set of \( D \). We say that \( i \) is \( B \)-active if \( i \) is \( B \)-orientable and there is no \( j < i \) with \( B \Delta \{i, j, \bar{j}, \bar{i} \} \) a feasible set of \( D \). For an independent set \( I \) of \( D \), we say that \( i \in I \) is \( I \)-active if \( i \) is \( I \)-active in the projection \( D([n] \setminus \bar{I}) \). Let \( a(I) \) denote the number of \( i \in \bar{I} \) which are \( I \)-active.

**Theorem 3.8.** Let \( D \) be a delta-matroid on \([n, \bar{n}]\). Then

\[
U_D(u, v - 1) = \sum_{I \text{ independent in } D} u^{n - |I|} v^{a(I)}.
\]

**Proof.** By [Mor19, Corollary 5.3], this holds after we evaluate at \( u = 0 \) for any delta-matroid \( D \). By [EFLS, Proposition 5.2], we have that

\[
U_D(u, v - 1) = \sum_{S \subset [n]} u^{n - |S|} U_D([n]\setminus S)(0, v - 1).
\]

The result follows because each independent set \( I \) is a feasible set of exactly one projection of \( D \). \( \square \)
Theorem 3.8 implies that the coefficient of $u^{n-i}$ in $U_D(u, -1)$ counts the number of independent sets of size $i$ with $a(I) = 0$. This is analogous to how the coefficient of $u^{r-i}$ in $R_M(u, -1)$ counts the number of independent sets of external activity zero in a matroid $M$, which form the faces of dimension $i-1$ in the broken circuit complex of $M$ [Bac]. This interpretation in terms of a simplicial complex generalizes to delta-matroids.

**Proposition 3.9.** The independent sets $I$ of $D$ with $a(I) = 0$ form a simplicial complex on $[n, \bar{n}]$.

**Proof.** It suffices to check that if $i$ is not $B$-active for some feasible set $B$ of $D$ and $S \subset [n] \setminus i$, then $i$ is not active for $B \setminus (S \cup \bar{S})$. Because $i$ is not $B$-active, either $B\Delta\{i, i\}$ is feasible (which remains true after we project away from $S$), or there is $j < i$ such that $B\Delta\{i, j, i, j\}$ is feasible. If $j \not\in S$, then this remains true after we project away from $S$. If $j \in S$, then $i$ is not $B \setminus (S \cup \bar{S})$-orientable. □

This complex can be complicated; for instance, its dimension is not easy to predict. The following example shows that the complex defined above need not be pure, so we cannot use it to deduce that $U_D(u, -1)$ is pure O-sequence as in Corollary 3.5.

**Example 3.10.** Let $D$ be the delta-matroid on $[3, \bar{3}]$ with feasible sets $\{1, 2, 3\}$, $\{1, 2, \bar{3}\}$, and $\{1, \bar{2}, 3\}$. Every element of $[3, \bar{3}]$ has no active elements, $\{1, 2\}$, $\{1, \bar{2}\}$, $\{2, \bar{3}\}$, $\{1, \bar{3}\}$, and $\{1, \bar{3}\}$ are the independent sets of size 2 with no active elements, and every feasible set has an active element. The complex defined in Proposition 3.9 has $f$-vector $(1, 6, 6)$, so $U_D(u, -1) = 6u + 6u^2 + u^3$. This complex is not pure because 1 is not contained in any facet.

**3.3. Enveloping matroids.** We now recall the definition of an enveloping matroid of a delta-matroid, which was introduced for algebro-geometric reasons in [EFLS] Section 6. A closely related notion was considered in [Bou97].

For $S \subset [n, \bar{n}]$, let $u_S$ denote the corresponding indicator vector in $\mathbb{R}^{[n, \bar{n}]}$. For a matroid $M$ on $[n, \bar{n}]$, let $P(M) = \text{Conv}\{u_B : B \text{ basis of } M\}$, and let $IP(M) = \text{Conv}\{u_S : S \text{ independent in } M\}$.

**Definition 3.11.** Let $\text{env} : \mathbb{R}^{[n, \bar{n}]} \to \mathbb{R}^n$ be the map given by $(x_1, \ldots, x_n, x_{\bar{1}}, \ldots, x_{\bar{n}}) \mapsto (x_1 - x_{\bar{1}}, \ldots, x_n - x_{\bar{n}})$. Let $D$ be a delta-matroid on $[n, \bar{n}]$, and let $M$ be a matroid on $[n, \bar{n}]$. We say that $M$ is an enveloping matroid for $D$ if $\text{env}(P(M)) = P(D)$.

Note that enveloping matroids necessarily have rank $n$. In [EFLS] Section 6.3], it is shown that many different types of delta-matroids have enveloping matroids, such as realizable delta-matroids, delta-matroids arising from the independent sets or bases of a matroid, and delta-matroids associated to graphs or embedded graphs. We will need the following property of enveloping matroids.

**Proposition 3.12.** Let $M$ be an enveloping matroid for a delta-matroid $D$ on $[n, \bar{n}]$. Let $S \in \text{AdS}_n$ be an admissible set. Then $S$ is independent in $M$ if and only if it is independent in $D$.

**Proof.** If $S \in \text{AdS}_n$, then $\text{env}(us) = es$, and $S$ is the only admissible set with this property. Furthermore, if $S \in \text{AdS}_n$ has size $n$, then $us$ is the only indicator vector of a subset of $[n, \bar{n}]$ of size $n$ which is a preimage of $es$ under env. Because $\text{env}(P(M)) = P(D)$, we see that if $B$ is a feasible set of $D$, then $B$ is a basis for $M$. This implies that the independent sets in $D$ are independent in $M$.

By [EFLS] Lemma 7.6, $\text{env}(IP(M)) = \frac{1}{2}(P(D) + \Box)$. If $S$ is admissible and independent in $M$, then $\text{env}(us) = es \in \frac{1}{2}(P(D) + \Box)$, so by Proposition 3.10 $S$ is independent in $D$. □

**3.4. Lorentzian polynomials.** For a multi-index $m = (m_0, m_1, \ldots)$, let $w^m = w_0^{m_0}w_1^{m_1}\cdots$. A homogeneous polynomial $f(w_0, w_1, \ldots)$ of degree $d$ with real coefficients is said to be strictly Lorentzian if all its coefficients are positive, and the quadratic form obtained by taking $d - 2$ partial derivatives is nondegenerate with exactly one positive eigenvalue. We say that $f$ is Lorentzian if it is a coefficient-wise limit of
strictly Lorentzian polynomials. Lorentzian polynomials enjoy strong log-concavity properties, and the class of Lorentzian polynomials is preserved under many natural operations.

The following lemma is a special case of [RSW, Proposition 3.3]. Alternatively, it can be deduced from the proof of [BH20, Corollary 3.5]. We thank Nima Anari for discussing this lemma with us.

Lemma 3.13. For a polynomial \( f(w_0, w_1, \ldots) = \sum_m c_m w^m \), let

\[
\tilde{f}(w_0, w_1, \ldots) = \sum_{m: m_i \leq 1 \text{ for } i \neq 0} c_m w^m.
\]

If \( f \) is Lorentzian, then \( \tilde{f} \) is Lorentzian.

For \( S \in \text{AdS}_n \), recall that \( \overline{S} \subset [n] \) denotes the unsigned version of \( S \). For a set \( T \), let \( w_T = \prod_{a \in T} w_a \). We now state a strengthening of Theorem 1.6.

Theorem 3.14. Let \( D \) be a delta-matroid on \([n, n]\) which has an enveloping matroid. Then the polynomial

\[
\sum_{S \in \text{AdS}_n} w_0^{2n-|S|} \overline{w_S} \in \mathbb{R}[w_0, w_1, \ldots, w_n]
\]

is Lorentzian.

Remark 3.15. In [EFLS, Theorem 8.1], it is proven that if \( D \) has an enveloping matroid, then the polynomial

\[
\sum_{S \in \text{AdS}_n} \frac{w_0^{|S|} w^{|S|}}{|S|!} \overline{w_S} \in \mathbb{R}[w_0, w_1, \ldots, w_n]
\]

is Lorentzian.

Proof of Theorem 1.6. By [BH20, Theorem 2.10], the specialization

\[
\sum_{S \in \text{AdS}_n} w_0^{2n-|S|} y^{|S|} = \sum_{i=0}^n f_i(D) w_0^{2n-i} y^i
\]

is Lorentzian. By [BH20, Example 2.26], the coefficients of a Lorentzian polynomial in two variables of degree \( 2n \) are log-concave after dividing the coefficient of \( w_0^{2n-i} y^i \) by \( \binom{2n}{i} \), which implies the result.

Proof of Theorem 3.14. Let \( M \) be an enveloping matroid of \( D \). By [BH20, Proof of Theorem 4.14], the polynomial

\[
\sum_{S \in \text{AdS}_n} w_0^{2n-|S|} w_S \in \mathbb{R}[w_0, w_1, \ldots, w_n, w_T, \ldots, w_{\overline{T}}]
\]

is Lorentzian. Setting \( w_1 = w_1 \), by [BH20, Theorem 2.10] the polynomial

\[
\sum_{S \in \text{AdS}_n} w_0^{2n-|S|} w_S \in \mathbb{R}[w_0, w_1, \ldots, w_n]
\]

is Lorentzian. A term \( w_0^{2n-|S|} w_S \) has degree at most 1 in each of the variables \( w_1, \ldots, w_n \) if and only if \( S \) is admissible, in which case it is equal to \( \overline{w_S} \). Therefore, by Lemma 3.13 the polynomial

\[
\sum_{S \in \text{AdS}_n} w_0^{2n-|S|} \overline{w_S} \in \mathbb{R}[w_0, w_1, \ldots, w_n]
\]

is Lorentzian. By Proposition 3.12 this polynomial is equal to the polynomial in Theorem 3.14.
Remark 3.16. Let \((U, \Omega, r)\) be a multimatroid \([\text{Bou97}]\), i.e., \(U\) is a finite set, \(\Omega\) is a partition of \(U\), and \(r\) is a function on partial transversals of \(\Omega\) satisfying certain conditions. An independent set is a partial transversal \(S\) of \(\Omega\) with \(r(S) = |S|\). A multimatroid is called shelterable if \(r\) can be extended to the rank function of a matroid on \(U\). Then the argument used to prove Theorem 1.6 shows that, if \(a_k\) is the number of independent sets of a shelterable multimatroid of size \(k\), then

\[
    a_k^2 \geq \frac{|U| - k + 1}{|U| - k} k^{a_k+1}a_k-1.
\]

References


[Duc92] Alain Duchamp, Delta matroids whose fundamental graphs are bipartite, Linear Algebra Appl. 160 (1992), 99–112. MR1137846


[Fuj17] On the interplay between independent sets of a shelterable multimatroid of size \(k\), then

\[
    a_k^2 \geq \frac{|U| - k + 1}{|U| - k} k^{a_k+1}a_k-1.
\]


Stanford U. Department of Mathematics, 450 Jane Stanford Way, Stanford, CA 94305

Email address: mwlarson@stanford.edu