RANK FUNCTIONS AND INVARIANTS OF DELTA-MATROIDS

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Abstract. In this note, we give a rank function axiomatization for delta-matroids and study the corresponding rank generating function. We relate an evaluation of the rank generating function to the number of independent sets of the delta-matroid, and we prove a log-concavity result for that evaluation using the theory of Lorentzian polynomials.

1. Introduction

Let $[n, \overline{n}]$ denote the set $\{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$, equipped with the obvious involution $\overline{(\cdot)}$. Let AdS_n be the set of *admissible subsets* of $[n, \overline{n}]$, i.e., subsets S that contain at most one of i and $\overline{i}$ for each $i \in [n]$. Set $e_{\overline{i}} := -e_i \in \mathbb{R}^n$, and for each $S \in \mathrm{AdS}_n$, set $e_S = \sum_{a \in S} e_a$.

Definition 1.1. A *delta-matroid* D is a collection $\mathcal{F} \subset \mathrm{AdS}_n$ of admissible sets of size n, called the *feasible sets* of D, such that the polytope

$$P(D) := \mathrm{Conv}\{e_B : B \in \mathcal{F}\}$$

has all edges parallel to e_i or $e_i \pm e_j$, for some i, j. We say that D is *even* if all edges of $P(D)$ are parallel to $e_i \pm e_j$.

Delta-matroids were introduced in [Bou87] by replacing the usual basis exchange axiom for matroids with one involving symmetric difference. They were defined independently in [CK88, DH86]. For the equivalence of the definition of delta-matroids in those works with the one given above, and for general properties of delta-matroids, see [BGW03, Chapter 4].

A delta-matroid is even if and only if all sets in $\{B \cap [n] : B \in \mathcal{F}\}$ have the same parity. Even delta-matroids enjoy nicer properties than arbitrary delta-matroids. For instance, they satisfy a version of the symmetric exchange axiom [Wen93].

There are many constructions of delta-matroids in the literature. Two of the most fundamental come from matroids: given a matroid M on $[n]$, we can construct a delta-matroid on $[n, \overline{n}]$ whose feasible sets are the sets of the form $B \cup \overline{B^c}$, for B a basis of M. We can also construct a delta-matroid whose feasible sets are the sets of the form $I \cup \overline{I^c}$, for I independent in M. Additionally, there are delta-matroids corresponding to graphs [Duc92], graphs embedded in surfaces [CMNR19, CMNR19b], and points of a maximal orthogonal or symplectic Grassmannian. Delta-matroids arising from points of a maximal orthogonal or symplectic Grassmannian are called *realizable*. See [EFLS, Section 6.2] for a discussion of delta-matroids associated to points of a maximal orthogonal Grassmannian.

Given $S, T \in \mathrm{AdS}_n$, we define $S \sqcup T = \{a \in S \cup T : \overline{a} \notin S \cup T\}$. A function $g\colon \mathrm{AdS}_n \to \mathbb{R}$ is called *bisubmodular* if, for all $S, T \in \mathrm{AdS}_n$,

$$f(S) + f(T) \geq f(S \cap T) + f(S \sqcup T).$$

There is a large literature on bisubmodular functions, beginning with [DW73]. They have been studied both from an optimization perspective [FI05, Fuj17] and from a polytopal perspective [FP94, Fuj14]. Additionally, bisubmodular functions are closely related to jump systems [BC95].

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For a delta-matroid D, define a function $g_D\colon \mathrm{AdS}_n \to \mathbb{Z}$ by
$$g_D(S) = \max_{B \in \mathcal{F}}(|S \cap B| - |\overline{S} \cap B|).$$
We call g_D the *rank function* of D. Note that g_D may take negative values. The collection of feasible subsets of D is exactly $\{S : g_D(S) = n\}$, so D can be recovered from g_D.

Theorem 1.2. *A function $g\colon \mathrm{AdS}_n \to \mathbb{Z}$ is the rank function of a delta-matroid if and only if*

(1) $g(\emptyset) = 0$ (normalization),
(2) $|g(S)| \leq 1$ if $|S| = 1$ (boundedness),
(3) $g(S) + g(T) \geq g(S \cap T) + g(S \sqcup T)$ (bisubmodularity), and
(4) $g(S) \equiv |S| \pmod 2$ (parity).

Furthermore, D is even if and only if
$$g_D(S) = \frac{g_D(S \cup i) + g_D(S \cup \overline{i})}{2} \text{ whenever } |S| = n-1 \text{ and } \{i, \overline{i}\} \cap S = \emptyset.$$

The function g_D, as well as the observation that it is bisubmodular, has appeared before in the literature [Bou88, CK88]. For example, in [Bou88, Theorem 4.1] it is shown that, if D is represented by a point of the maximal symplectic Grassmannian, then g_D can be computed in terms of the rank of a certain matrix. It was known that delta-matroids admit a description in terms of certain bisubmodular functions. However, the precise characterization in Theorem 1.2 does not appear to have been known before. Indeed, Theorem 1.2 answers a special case of [ACEP20, Question 9.4].

In [Bou97, Bou98], Bouchet gave a rank-function axiomatization of delta-matroids in the more general setting of multimatroids. His rank function differs from ours — in Section 2.2, we discuss the relationship between his results and Theorem 1.2.

Basic operations operations on delta-matroids — like products, deletion, contraction, and projection — can be simply expressed in terms of rank functions. See Section 2.1.

One of the most important invariants of a matroid M of rank r on $[n]$ is its *Whitney rank generating function*. If rk_M is the rank function of M, then the rank generating function is defined as
$$R_M(u, v) := \sum_{A \subset [n]} u^{r - \mathrm{rk}_M(A)} v^{|A| - \mathrm{rk}_M(A)}.$$
The more commonly used normalization is the *Tutte polynomial*, which is $R_M(u-1, v-1)$. The characterization of delta-matroids in terms of rank functions allows us to consider an analogously-defined invariant.

Definition 1.3. Let D be a delta-matroid on $[n, \overline{n}]$. Then we define
$$U_D(u, v) = \sum_{S \in \mathrm{AdS}_n} u^{n - |S|} v^{\frac{|S| - g_D(S)}{2}}.$$

Note that the bisubmodularity of g_D implies that the restriction of g_D to the subsets of any fixed $S \in \mathrm{AdS}_n$ is submodular. The boundedness of g_D then implies that $|g_D(S)| \leq |S|$. Because of the parity requirement, $|S| - g_D(S)$ is divisible by 2. Therefore $U_D(u, v)$ is indeed a polynomial. The normalization $U_D(u-1, v-1)$ is more analogous to the Tutte polynomial, but it can have negative coefficients. However, the polynomial $U_D(u, v-1)$ has non-negative coefficients (as follows, e.g., from Theorem 3.8).

The U-polynomial of a delta-matroid was introduced by Eur, Fink, Spink, and the author in [EFLS, Definition 1.4] in terms of a Tutte polynomial-like recursion; see Proposition 3.1 for a proof that Definition 1.3 agrees with the recursive definition considered there. The specialization $U_D(0, v)$ is the *interlace polynomial* of D, which was introduced in [ABS04] for graphs and in [BH14] for general delta-matroids. See [Mor17] for a survey on the properties of the interlace polynomial.

Various Tutte polynomial-like invariants of delta-matroids have been considered in the literature, such as the Bollobás–Riordan polynomial and its specializations [BR01]. In [KMT18], a detailed analysis of delta-matroid polynomials which satisfy a deletion-contraction formula is carried out. Set $\sigma_D(A) = \frac{|A|}{2} + \frac{g_D(A)+g_D(\bar{A})}{4}$ for $A \subset [n]$. Then in [KMT18], the polynomial

$$\sum_{A \subset [n]} (x-1)^{\sigma_D([n])-\sigma_D(A)}(y-1)^{|A|-\sigma_D(A)}$$

is shown to be, in an appropriate sense, the universal invariant of delta-matroids which satisfies a deletion-contraction formula. This polynomial is a specialization of the Bollobás–Riordan polynomial. In [EMGM+22], it is shown that this polynomial has several nice combinatorial properties.

Example 1.4. [EFLS, Example 5.5 and 5.6] Let M be a matroid of rank r on $[n]$, and let $S = S^+ \cup \overline{S^-} \in \mathrm{AdS}_n$ be an admissible set with $S^+, S^- \subset [n]$. Set $V = \{i \in [n] : S \cap \{i, \bar{i}\} = \emptyset\}$. Above, we gave two examples of delta-matroids constructed from M.

(1) Let D be the delta-matroid arising from the independent sets of M. Then $g_D(S) = |S| + 2\,\mathrm{rk}_M(S^+) - 2|S^+|$, and
$$U_D(u,v) = (u+1)^{n-r} R_M\left(u+3, \frac{2u+v+2}{u+1}\right).$$
(2) Let D be the delta-matroid arising from the bases of M. Then $g_D(S) = |S| - 2r + 2\,\mathrm{rk}_M(S^+ \cup V) - 2|S^+| + 2\,\mathrm{rk}_M(S^+)$, and
$$U_D(u,v) = \sum_{T \subset S \subset [n]} u^{|S \setminus T|} v^{r-\mathrm{rk}_M(S)+|T|-\mathrm{rk}_M(T)}.$$

We study the U-polynomial as a delta-matroid analogue of the rank generating function of a matroid. For a matroid M, the evaluation $R_M(u,0)$ is essentially the f-vector of the independence complex of the matroid, i.e., it counts the number of independent sets of M of a given size. The coefficients of the Tutte polynomial $R_M(u-1, v-1)$ can be interpreted as counting bases of M according to their internal and external activities, certain statistics that depend on an ordering of the ground set. See [Bac]. This shows that $R_M(u,-1)$, the (unsigned) characteristic polynomial of M, is essentially the f-vector of the broken circuit complex of M.

A set $S \in \mathrm{AdS}_n$ is *independent* if it is contained in a feasible set of a delta-matroid D. In [Bou97], Bouchet gave an axiomatization of delta-matroids in terms of their independent sets. The independent sets form a simplicial complex, called the *independence complex* of D. We relate $U_D(u,0)$ to the f-vector of the independence complex of D (Proposition 3.4), which gives linear inequalities between the coefficients of $U_D(u,0)$. We give a combinatorial interpretation of the coefficients of $U_D(u, v-1)$ as counting the number of independent sets of D of a given size according to a delta-matroid version of activity (Theorem 3.8). This shows that $U_D(u,-1)$ is essentially the f-vector of a certain simplicial complex associated to D.

Following a tradition in matroid theory (see, e.g., [Mas72]), and inspired by the ultra log-concavity of $R_M(u,0)$ [ALGV, BH20], we make three log-concavity conjectures for $U_D(u,0)$. These conjectures state the sequence of the number of independent sets of a delta-matroid of a given size satisfies log-concavity properties.

Conjecture 1.5. *Let D be a delta-matroid on $[n, \overline{n}]$, and let $U_D(u,0) = a_n + a_{n-1}u + \cdots + a_0 u^n$. Then, for any $k \in \{1, \ldots, n-1\}$,*

(1) $a_k^2 \geq \frac{n-k+1}{n-k} a_{k+1} a_{k-1}$,
(2) $a_k^2 \geq \frac{2n-k+1}{2n-k} \frac{k+1}{k} a_{k+1} a_{k-1}$, *and*
(3) $a_k^2 \geq \frac{n-k+1}{n-k} \frac{k+1}{k} a_{k+1} a_{k-1}$.

Conjecture 1.5(1) follows from [EFLS, Conjecture 1.5], and it is proven in [EFLS, Theorem B] when D has an *enveloping matroid* (see Definition 3.11). This is a technical condition which is satisfied by many commonly occurring delta-matroids, including all realizable delta-matroids and delta-matroids arising from matroids (although not all delta-matroids, see [Bou97, Section 4] and [EFLS, Example 6.11]). The proof uses algebro-geometric methods. Here we prove a special case of Conjecture 1.5(2).

Theorem 1.6. *Let D be a delta-matroid on $[n,\overline{n}]$ which has an enveloping matroid. Let $U_D(u,0) = a_n + a_{n-1}u + \cdots + a_0u^n$. Then, for any $k \in \{1,\dots,n-1\}$, $a_k^2 \geq \frac{2n-k+1}{2n-k}\frac{k+1}{k}a_{k+1}a_{k-1}$, i.e., Conjecture 1.5(2) holds.*

Our argument uses the theory of Lorentzian polynomials [BH20]. We strengthen Theorem 1.6 by proving that a generating function for the independent sets of D is Lorentzian (Theorem 3.14), which implies the desired log-concavity statement. We deduce that this generating function is Lorentzian from the fact that the Potts model partition function of an enveloping matroid is Lorentzian [BH20, Theorem 4.10].

When D is the delta-matroid arising from the independent sets of a matroid, Conjecture 1.5(3) follows from the ultra log-concavity of the number of independent sets of that matroid [ALGV, BH20]. When D is the delta-matroid arising from the bases of a matroid M on $[n]$, which has an enveloping matroid by [EFLS, Proposition 6.10], Theorem 1.6 gives a new log-concavity result. If we set

$$a_k = |\{T \subset S \subset [n] : T \text{ independent in } M \text{ and } S \text{ spanning in } M,\ |S \setminus T| = n-k\}|,$$

then Theorem 1.6 gives that $a_k^2 \geq \frac{2n-k+1}{2n-k}\frac{k+1}{k}a_{k+1}a_{k-1}$ for $k \in \{1,\dots,n-1\}$.

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2. Rank functions of delta-matroids

The proof of Theorem 1.2 goes by way of a polytopal description of normalized bisubmodular functions, which we now recall. To a function $f\colon \mathrm{AdS}_n \to \mathbb{R}$ with $f(\emptyset) = 0$, we associate the polytope

$$P(f) = \{x : \langle e_S, x\rangle \leq f(S) \text{ for all non-empty } S \in \mathrm{AdS}_n\}.$$

By [BC95, Theorem 4.5] (or [ACEP20, Theorem 5.2]), $P(f)$ has all edges parallel to e_i or $e_i \pm e_j$ if and only if f is bisubmodular. In this case, $P(f)$ is a lattice polytope if and only if f is integer-valued. For a normalized (i.e., $f(\emptyset) = 0$) bisubmodular function f, we can recover f from $P(f)$ via the formula

$$f(S) = \max_{x \in P(f)} \langle e_S, x\rangle.$$

Under this dictionary, the bisubmodular function corresponding to the dilate $kP(f)$ is kf, and the bisubmodular function corresponding to the Minkowski sum $P(f) + P(g)$ is $f + g$.

Proof of Theorem 1.2. By the polyhedral description of normalized bisubmodular functions, for each delta-matroid D there is a unique normalized bisubmodular function g such that $P(D) = P(g)$. We show that the conditions on a normalized bisubmodular function g for $P(g)$ to have all vertices in $\{-1,1\}^n$ are exactly those given in Theorem 1.2, namely that $|g(S)| \leq 1$ when $|S| = 1$ and $g(S) \equiv |S| \pmod 2$.

The polytope $P(g)$ has all vertices in $\{\pm 1\}^n$ if and only if $\frac{1}{2}(P(g) + (1,\dots,1))$ is a lattice polytope which is contained in $[0,1]^n$. The normalized bisubmodular function h corresponding to the point $(1,\dots,1)$ takes value $h(S) = |S^+| - |S^-|$ on an admissible set of the form $S = S^+ \cup \overline{S^-}$, with $S^+, S^- \subset [n]$. The polytope $\frac{1}{2}(P(g) + (1,\dots,1))$ is $P(f)$, where f is the normalized bisubmodular function defined by $f := \frac{1}{2}(g+h)$. We note that $P(f)$ is a lattice polytope which is contained in $[0,1]^n$ if and only if

(1) $f(i) \in \{0,1\}$ and $f(\overline{i}) \in \{-1,0\}$, and

(2) f is integer-valued.

A normalized bisubmodular function f satisfies these conditions if and only if g satisfies the conditions of Theorem 1.2, giving the characterization of rank functions of delta-matroids.

By [ACEP20, Example 5.2.3], the polytope $P(g_D) = P(D)$ has all edges parallel to $e_i \pm e_j$ if and only if g_D satisfies the condition

$$g_D(S) = \frac{g_D(S \cup i) + g_D(S \cup \overline{i})}{2} \text{ whenever } |S| = n-1 \text{ and } \{i, \overline{i}\} \cap S = \emptyset.$$

This gives the characterization of even delta-matroids. □

2.1. **Compatibility with delta-matroid operations.** In this section, we consider several operations on delta-matroids, and we show that the rank function behaves in a simple way under these operations. First we consider minor operations on delta-matroids — contraction, deletion, and projection.

Definition 2.1. Let D be a delta-matroid on $[n, \overline{n}]$ with feasible sets $\mathcal{F}$, and let $i \in [n]$. We say that i is a *loop* of D if no feasible set contains i, and we say that i is a *coloop* if every feasible set contains i.

(1) If i is not a loop of D, then the *contraction* D/i is the delta-matroid with feasible sets $B \setminus i$, for $B \in \mathcal{F}$ containing i.
(2) If i is not a coloop of D, then the *deletion* $D \setminus i$ is the delta-matroid with feasible sets $B \setminus \overline{i}$, for $B \in \mathcal{F}$ containing $\overline{i}$.
(3) The *projection* $D(i)$ is the delta-matroid with feasible sets $B \setminus \{i, \overline{i}\}$ for $B \in \mathcal{F}$.
(4) If i is a loop or coloop, then set $D/i = D \setminus i = D(i)$.

For $A \subset [n]$, we define $D/A, D \setminus A$, and $D(A)$ to be the delta-matroids on $[n, \bar{n}] \setminus (A \cup \bar{A})$ obtained by successively contracting, deleting, or projecting away from all elements of A. Contractions, deletions, and projections at disjoint sets commute with each other, so this is well defined. If A and B are disjoint subsets of $[n]$, then $D/A \setminus B$ is the delta-matroid obtained by contracting A and then deleting B, which is the same as first deleting B and then contracting A.

First we describe the rank function of projections. The formula is analogous to the formula for the rank function of a matroid deletion.

Proposition 2.2. *Let D be a delta-matroid on $[n, \overline{n}]$, and let $A \subset [n]$. For each $S \in \mathrm{AdS}_n$ disjoint from $A \cup \overline{A}$, $g_{D(A)}(S) = g_D(S)$.*

Proof. As S is disjoint from $A \cup \overline{A}$, $|B \cap S| - |B \cap \overline{S}|$ depends only on $B \setminus (A \cup \overline{A})$. The feasible sets of $D(A)$ are given by $B \setminus (A \cup \overline{A})$ for B a feasible set of D. □

The rank functions of the contractions and deletions are described by the following result. The formula is analogous to the formula for the rank function of a matroid contraction.

Proposition 2.3. *Let D be a delta-matroid on $[n, \overline{n}]$. Let $A, B \subset [n]$ be disjoint subsets, and let $S \in \mathrm{AdS}_n$ be disjoint from $A \cup B \cup \overline{A} \cup \overline{B}$. Then $g_{D/A \setminus B}(S) = g_D(S \cup A \cup \overline{B}) - g_D(A \cup \overline{B})$.*

Before proving this, we will need the following property of delta-matroids. It follows, for instance, from the greedy algorithm description of delta-matroids in [BC95].

Proposition 2.4. *Let D be a delta-matroid on $[n, \overline{n}]$, and let $S \subset T \in \mathrm{AdS}_n$. Let $\mathcal{F}_S$ be the collection of feasible sets B of D that maximize $|S \cap B|$, i.e., have $|S \cap B| = \max_{B' \in \mathcal{F}} |S \cap B'|$. Then*

$$\max_{B \in \mathcal{F}_S} |T \cap B| = \max_{B \in \mathcal{F}} |T \cap B|.$$

First we consider the case when we delete or contract a single element.

Lemma 2.5. *Let D be a delta-matroid on $[n,\overline{n}]$, and let $i \in [n]$. Then*

(1) If i is not a loop, then $g_{D/i}(S) = g_D(S \cup i) - 1$,
(2) If i is not a coloop, then $g_{D\setminus i}(S) = g_D(S \cup \overline{i}) - 1$, and

Proof. We do the case of contraction; the case of deletion is identical. Assume that i is not a loop, and let $\mathcal{F}_i$ denote the set of feasible sets of D which contain i. Note that $\mathcal{F}_i$ is non-empty, so it is the collection of feasible sets B of D which maximize $|\{i\} \cap B|$. For any $S \in \mathrm{AdS}_n$ with $S \cap \{i, \overline{i}\} = \emptyset$, by Proposition 2.4 we have that

$$\max_{B \in \mathcal{F}} |(S \cup i) \cap B| = \max_{B \in \mathcal{F}_i} |(S \cup i) \cap B|.$$

For any B, $|(S \cup i) \cap B| - |\overline{(S \cup i)} \cap B| = 2|(S \cup i) \cap B| - |S \cup i|$, so we see that

$$\max_{B \in \mathcal{F}} (|(S \cup i) \cap B| - |\overline{(S \cup i)} \cap B|) = \max_{B \in \mathcal{F}_i} (|(S \cup i) \cap B| - |\overline{(S \cup i)} \cap B|).$$

The left-hand side is equal to $g_D(S \cup i)$, and the right-hand side is equal to $g_{D/i}(S) + 1$. □

Proof of Proposition 2.3. First note that $g_D(i) = 1$ if i is not a loop and is -1 if i is a loop, and similarly $g_D(\overline{i}) = 1$ if i is not a coloop and is -1 is i is a coloop. So Lemma 2.5 implies the result holds when $|S| = 1$.

We induct on the size of $A \cup B$. We consider the case of adding an element $i \in [n]$ to A; the case of adding it to B is identical. We compute:

$$\begin{aligned} g_{D/(A\cup i)\setminus B}(S) &= g_{D/A\setminus B}(S \cup i) - g_{D/A\setminus B}(i) \\ &= g_D(S \cup A \cup \overline{B} \cup i) - g_D(A \cup \overline{B}) - (g_D(A \cup \overline{B} \cup i) - g_D(A \cup \overline{B})) \\ &= g_D(S \cup (A \cup i) \cup \overline{B}) - g_D((A \cup i) \cup \overline{B}). \end{aligned}$$

□

For two non-negative integers n_1, n_2, identify the disjoint union of $[n_1]$ and $[n_2]$ with $[n_1 + n_2]$. Given two delta-matroids D_1, D_2 on $[n_1]$ and $[n_2]$, let $D_1 \times D_2$ be the delta-matroid on $[n_1 + n_2]$ whose feasible sets are $B_1 \cup B_2$, for B_j a feasible set of D_j. Then we have the following description of the rank function of $D_1 \times D_2$.

Proposition 2.6. *Let D_1, D_2 be delta-matroids on $[n_1]$ and $[n_2]$, and let $S = S_1 \cup S_2$ be an admissible subset of $[n_1 + n_2, \overline{n_1 + n_2}]$, with $S_1 \subset [n_1, \overline{n}_1]$ and $S_2 \subset [n_2, \overline{n}_2]$. Then $g_{D_1 \times D_2}(S) = g_{D_1}(S_1) + g_{D_2}(S_2)$.*

Proof. Let B_1 be a feasible set of D_1 with $g_{D_1}(S_1) = |S_1 \cap B_1| - |\overline{S_1} \cap B_1|$, and let B_2 be a feasible set of D_2 with $g_{D_2}(S_2) = |S_2 \cap B_2| - |\overline{S_2} \cap B_2|$. Then $B_1 \cup B_2$ maximizes $B \mapsto |S \cap B| - |\overline{S} \cap B|$, and so $g_{D_1 \times D_2}(S) = |S_1 \cap B_1| - |\overline{S_1} \cap B_1| + |S_2 \cap B_2| - |\overline{S_2} \cap B_2| = g_{D_1}(S_1) + g_{D_2}(S_2)$. □

We now study how the rank function behaves under the operation of *twisting*. Let W be the *signed permutation group*, the subgroup of the symmetric group on $[n, \overline{n}]$ which preserves AdS_n. In other words, W consists of permutations w such that $w(\overline{i}) = \overline{w(i)}$. As delta-matroids are collections of admissible sets, W acts on the set of delta-matroids on $[n, \bar{n}]$. This action is usually called twisting in the delta-matroid literature.

Proposition 2.7. *Let D be a delta-matroid on $[n, \overline{n}]$, and let $w \in W$. Then $g_{w\cdot D}(S) = g_D(w^{-1} \cdot S)$.*

Proof. Note that, for B a feasible set of D, $|S \cap (w \cdot B)| - |\overline{S} \cap (w \cdot B)| = |(w^{-1} \cdot S) \cap B| - |\overline{(w^{-1} \cdot S)} \cap B|$, which implies the result. □

Let $S \in \mathrm{AdS}_n$ be an admissible set of size n. For any delta-matroid D on $[n, \overline{n}]$, let r be the maximal value of $|S \cap B|$. Then $\{S \cap B \colon B \in \mathcal{F}, |S \cap B| = r\}$ is the set of bases of a matroid on S. When $S = [n]$, this is sometimes called the upper matroid of D. We describe the rank function of this matroid in terms of the rank function of D.

Proposition 2.8. *Let $S \in \mathrm{AdS}_n$ be an admissible set of size n, and let D be a delta-matroid on $[n, \overline{n}]$ with $r = \max_{B \in \mathcal{F}} |S \cap B|$. The matroid M on S whose bases are $\{S \cap B : B \in \mathcal{F}, |S \cap B| = r\}$ has rank function*

$$\mathrm{rk}_M(T) = \frac{g_D(T) + |T|}{2}.$$

Proof. Let $\mathcal{F}_S$ be the collection of feasible sets B with $|S \cap B| = r$. Then we have that

$$\mathrm{rk}_M(T) = \max_{B \in \mathcal{F}_S} |T \cap B| \leq \max_{B \in \mathcal{F}} |T \cap B| = \frac{g_D(T) + |T|}{2}.$$

On the other hand, by Proposition 2.4 there is a feasible set B which maximizes $|T \cap B|$ and has $|S \cap B| = r$, so we have equality. □

2.2. **An alternative normalization.** The results of the previous section, particularly Proposition 2.8, suggest that an alternative normalization of the rank function of a delta-matroid has nice properties. Set

$$h_D(S) := \frac{g_D(S) + |S|}{2}.$$

The function $h_D(S)$ is integer-valued and bisubmodular, and the polytope it defines is $P(h_D) = \frac{1}{2}(P(D) + \square)$, where $\square = [-1, 1]^n$ is the cube. This is because the bisubmodular function corresponding to $\square$ is $S \mapsto |S|$. Note that the function h_D is non-negative and increasing, in the sense that if $S \subset T \in \mathrm{AdS}_n$, then $h_D(S) \leq h_D(T)$. Theorem 1.2 implies the following characterization of the functions arising as h_D for some delta-matroid D.

Corollary 2.9. *A function $h \colon \mathrm{AdS}_n \to \mathbb{Z}$ is equal to h_D for some delta-matroid D if and only if*

(1) $h(\emptyset) = 0$ (normalization),
(2) $h(S) \in \{0, 1\}$ if $|S| = 1$ (boundedness),
(3) $h(S) + h(T) \geq h(S \cap T) + h(S \sqcup T) + |S \cap \overline{T}|/2$.

Indeed, these are exactly the conditions we need for $g(S) := 2h(S) - |S|$ to satisfy the conditions in Theorem 1.2.

The function h_D was studied by Bouchet in [Bou97, Bou98] in the more general setting of multimatroids. The following characterization of the functions h_D follows from [Bou97, Proposition 4.2]:

Proposition 2.10. *A function $h \colon \mathrm{AdS}_n \to \mathbb{Z}$ is equal to h_D for some delta-matroid D if and only if*

(1) $h(\emptyset) = 0$,
(2) $h(S) \leq h(S \cup a) \leq h(S) + 1$ if $S \cup a$ is admissible,
(3) $h(S) + h(T) \geq h(S \cap T) + h(S \cup T)$ if $S \cup T$ is admissible, and
(4) $h(S \cup i) + h(S \cup \overline{i}) \geq 2h(S) + 1$ if $S \cap \{i, \overline{i}\} = \emptyset$.

In [Bou98, Theorem 2.16], the following characterization of the functions h_D is stated with a reference to an unpublished paper of Allys.

Proposition 2.11. *A function $h \colon \mathrm{AdS}_n \to \mathbb{Z}$ is equal to h_D for some delta-matroid D if and only if*

(1) $h(\emptyset) = 0$,
(2) $h(S) \leq h(S \cup a) \leq h(S) + 1$ if $S \cup a$ is admissible, and
(3) $h(S) + h(T) \geq h(S \cap T) + h(S \sqcup T) + |S \cap \overline{T}|$.

It is easy to see directly that a function which satisfies the hypotheses of Corollary 2.9 satisfies the hypotheses of Proposition 2.10 or Proposition 2.11. However, the converse does not seem obvious.

3. The U-polynomial

We now study the U-polynomial of delta-matroids. We prove the following recursion for $U_D(u,v)$, which was the original definition of the U-polynomial in [EFLS, Definition 1.4].

Proposition 3.1. *If $n=0$, the $U_D(u,v)=1$. For any $i\in[n]$, the U-polynomial satisfies*

$$U_D(u,v)=\begin{cases} U_{D/i}(u,v)+U_{D\setminus i}(u,v)+uU_{D(i)}(u,v), & i \text{ is neither a loop nor a coloop}\\ (u+v+1)\cdot U_{D\setminus i}(u,v), & i \text{ is a loop or a coloop.}\end{cases}$$

First we study the behavior of the U-polynomial under products.

Lemma 3.2. *Let D_1, D_2 be delta-matroids on $[n_1,\overline{n}_1]$ and $[n_2,\overline{n}_2]$. Then $U_{D_1\times D_2}(u,v)=U_{D_1}(u,v)U_{D_2}(u,v)$.*

Proof. We compute:

$$\begin{aligned} U_{D_1}(u,v)U_{D_2}(u,v) &= \left(\sum_{S_1\in \mathrm{AdS}_{n_1}} u^{n_1-|S_1|}v^{\frac{|S_1|-g_{D_1}(S_1)}{2}}\right)\left(\sum_{S_2\in \mathrm{AdS}_{n_2}} u^{n_2-|S_2|}v^{\frac{|S_2|-g_{D_2}(S_2)}{2}}\right)\\ &= \sum_{(S_1,S_2)} u^{n_1+n_2-|S_1|-|S_2|}v^{\frac{|S_1|+|S_2|-g_{D_1}(S_1)-g_{D_2}(S_2)}{2}}\\ &= \sum_{(S_1,S_2)} u^{n_1+n_2-|S_1|-|S_2|}v^{\frac{|S_1|+|S_2|-g_{D_1\times D_2}(S_1\cup S_2)}{2}}\\ &= U_{D_1\times D_2}(u,v), \end{aligned}$$

where the third equality is Proposition 2.6. □

Proof of Proposition 3.1. If $n=0$, then the only admissible subset of $[n,\overline{n}]$ is the empty set, and $g_D(\emptyset)=0$, so $U_D(u,v)=1$. Now choose some $i\in[n]$.

First suppose that i is neither a loop nor a coloop. The admissible subsets of $[n,\overline{n}]$ are partitioned into sets containing i, sets containing $\overline{i}$, and sets containing neither i nor $\overline{i}$. If S contains i, then $u^{n-|S|}v^{\frac{|S|-g_D(S)}{2}}=u^{n-1-|S\setminus i|}v^{\frac{|S\setminus i|-g_{D/i}(S\setminus i)}{2}}$. If S contains $\overline{i}$, then $u^{n-|S|}v^{\frac{|S|-g_D(S)}{2}}=u^{n-1-|S\setminus i|}v^{\frac{|S\setminus \overline{i}|-g_{D\setminus i}(S\setminus \overline{i})}{2}}$. If S contains neither i not $\overline{i}$, then $u^{n-|S|}v^{\frac{|S|-g_D(S)}{2}}=u\cdot u^{n-1-|S|}v^{\frac{|S|-g_{D(i)}(S)}{2}}$. Adding these up implies the recursion in this case.

If i is a loop or a coloop, then D is the product of $D\setminus i$ with a delta-matroid on 1 element with 1 feasible set. We observe that U-polynomial of a delta-matroid on 1 element with 1 feasible set is $u+v+1$, and so Lemma 3.2 implies the recursion in this case. □

3.1. The independence complex of a delta-matroid.

In this section, we introduce the independence complex of a delta-matroid and use it to study the U-polynomial.

Definition 3.3. We say that $S\in \mathrm{AdS}_n$ is *independent* in D if $g_D(S)=|S|$, or, equivalently, if S is contained in a feasible subset of D. The *independence complex* of D is the simplicial complex on $[n,\overline{n}]$ whose facets are given by the feasible sets of D.

Let $S\in\mathrm{AdS}_n$, and let $T=\{i\in[n] : S\cap\{i,\overline{i}\}=\emptyset\}$. Note S is independent if and only if S is a feasible set of $D(T)$.

The following result is immediate from the definition of $U_D(u,0)$.

Proposition 3.4. *Let $f_i(D)$ be the number of i-dimensional faces of the independence complex of D. Then $U_D(u,0)=f_{n-1}(D)+f_{n-2}(D)u+\cdots+f_{-1}(D)u^n$.*

Note that the f-vector of a pure simplicial complex, like the independence complex of a delta-matroid, is a *pure O-sequence*. Then [Hib89] gives the following inequalities.

Corollary 3.5. *Let $U_D(u,0) = a_n + a_{n-1}u + \cdots a_0 u^n$. Then $(a_0, \ldots, a_n)$ is the f-vector of a pure simplicial complex. In particular, $a_i \le a_{n-i}$ for $i \le n/2$ and $a_0 \le a_1 \le \cdots \le a_{\lfloor \frac{n+1}{2} \rfloor}$.*

Proposition 3.4 is a delta-matroid analogue of the fact that, for a matroid M, the coefficients of $R_M(u,0)$, when written backwards, are the face numbers of the independence complex of M. The independence complex of a matroid is shellable [Bjö92], which is reflected in the fact that $R_M(u-1,0)$ has non-negative coefficients. The independence complex of a delta-matroid is not in general shellable or Cohen–Macaulay, and $U_D(u-1,0)$ can have negative coefficients.

Recall that $\square = [-1,1]^n$ is the cube. The map $S \mapsto e_S$ induces a bijection between AdS_n and lattice points of $\square$. We use this to give a polytopal description of the independent sets of D, which will be useful in the sequel.

Proposition 3.6. *The map $S \mapsto e_S$ induces a bijection between independent sets of D and lattice points in $\frac{1}{2}(P(D) + \square)$.*

Proof. If S is independent in D, then there is $T \in \mathrm{AdS}_n$ such that $S \cup T \in \mathcal{F}$. Then $e_S = \frac{1}{2}(e_{S \cup T} + e_{S \cup \overline{T}})$, so e_S lies in $\frac{1}{2}(P(D) + \square)$.

The correspondence between normalized bisubmodular functions and polytopes gives that

$$\frac{1}{2}(P(D) + \square) = \left\{ x : \langle e_S, x \rangle \le \frac{g_D(S) + |S|}{2} \right\}.$$

If S is not independent, then e_S violates the inequality $\langle e_S, e_S \rangle \le \frac{g_D(S)+|S|}{2}$, so e_S does not lie in $\frac{1}{2}(P(D) + \square)$. □

3.2. **The activity expansion of the U-polynomial.** We now discuss an expansion of $U_D(u, v-1)$ in terms of a statistic associated to each independent set of a delta-matroid D, similar to the expansion of the Tutte polynomial of a matroid in terms of basis activities. We rely heavily on the work of Morse [Mor19], who gave such an expansion for the interlace polynomial $U_D(0, v-1)$. Throughout we fix the ordering $1 < 2 < \cdots < n$ on $[n]$. For $S \in \mathrm{AdS}_n$, let $\underline{S} \subset [n]$ denote the unsigned version of S, i.e., the image of S under the quotient of $[n, \overline{n}]$ by the involution.

Definition 3.7. Let B be a feasible set in a delta-matroid D. We say that $i \in [n]$ is *B-orientable* if the symmetric difference $B \Delta \{i, \bar{i}\}$ is not a feasible set of D. We say that i is *B-active* if i is B-orientable and there is no $j < i$ with $B \Delta \{i, j, \bar{i}, \bar{j}\}$ a feasible set of D. For an independent set I of D, we say that $i \in \underline{I}$ is *I-active* if i is I-active in the projection $D([n] \setminus \underline{I})$. Let $a(I)$ denote the number of $i \in \underline{I}$ which are I-active.

Theorem 3.8. *Let D be a delta-matroid on $[n, \bar{n}]$. Then*

$$U_D(u, v-1) = \sum_{I \text{ independent in } D} u^{n-|I|} v^{a(I)}.$$

Proof. By [Mor19, Corollary 5.3], this holds after we evaluate at $u = 0$ for any delta-matroid D. By [EFLS, Proposition 5.2], we have that

$$U_D(u, v-1) = \sum_{S \subset [n]} u^{n-|S|} U_{D([n] \setminus S)}(0, v-1).$$

The result follows because each independent set I is a feasible set of exactly one projection of D. □

Theorem 3.8 implies that the coefficient of u^{n-i} in $U_D(u,-1)$ counts the number of independent sets of size i with $a(I)=0$. This is analogous to how the coefficient of u^{r-i} in $R_M(u,-1)$ counts the number of independent sets of external activity zero in a matroid M, which form the faces of dimension $i-1$ in the broken circuit complex of M [Bac]. This interpretation in terms of a simplicial complex generalizes to delta-matroids.

Proposition 3.9. *The independent sets I of D with $a(I)=0$ form a simplicial complex on $[n,\bar{n}]$.*

Proof. It suffices to check that if i is not B-active for some feasible set B of D and $S \subset [n] \setminus i$, then i is not active for $B \setminus (S \cup \bar{S})$. Because i is not B-active, either $B\Delta\{i,\bar{i}\}$ is feasible (which remains true after we project away from S), or there is $j<i$ such that $B\Delta\{i,j,\bar{i},\bar{j}\}$ is feasible. If $j \notin S$, then this remains true after we project away from S. If $j \in S$, then i is not $B \setminus (S \cup \bar{S})$-orientable. □

This complex can be complicated; for instance, its dimension is not easy to predict. The following example shows that the complex defined above need not be pure, so we cannot use it to deduce that $U_D(u,-1)$ is pure O-sequence as in Corollary 3.5.

Example 3.10. Let D be the delta-matroid on $[3,\bar{3}]$ with feasible sets $\{1,\bar{2},\bar{3}\}$, $\{\bar{1},2,\bar{3}\}$, and $\{\bar{1},\bar{2},3\}$. Every element of $[3,\bar{3}]$ has no active elements, $\{\bar{1},2\}$, $\{\bar{1},\bar{2}\}$, $\{\bar{2},3\}$, $\{\bar{2},\bar{3}\}$, $\{\bar{1},3\}$, and $\{\bar{1},\bar{3}\}$ are the independent sets of size 2 with no active elements, and every feasible set has an active element. The complex defined in Proposition 3.9 has f-vector $(1,6,6)$, so $U_D(u,-1)=6u+6u^2+u^3$. This complex is not pure because 1 is not contained in any facet.

3.3. **Enveloping matroids.** We now recall the definition of an enveloping matroid of a delta-matroid, which was introduced for algebro-geometric reasons in [EFLS, Section 6]. A closely related notion was considered in [Bou97].

For $S \subseteq [n,\bar{n}]$, let u_S denote the corresponding indicator vector in $\mathbb{R}^{[n,\bar{n}]}$. For a matroid M on $[n,\bar{n}]$, let $P(M)=\text{Conv}\{u_B : B \text{ basis of } M\}$, and let $IP(M)=\text{Conv}\{u_S : S \text{ independent in M}\}$.

Definition 3.11. Let $\text{env}\colon \mathbb{R}^{[n,\overline{n}]} \to \mathbb{R}^n$ be the map given by $(x_1,\ldots,x_n,x_{\overline{1}},\ldots,x_{\overline{n}}) \mapsto (x_1-x_{\overline{1}},\ldots,x_n-x_{\overline{n}})$. Let D be a delta-matroid on $[n,\overline{n}]$, and let M be a matroid on $[n,\overline{n}]$. We say that M is an *enveloping matroid* for D if $\text{env}(P(M))=P(D)$.

Note that enveloping matroids necessarily have rank n. In [EFLS, Section 6.3], it is shown that many different types of delta-matroids have enveloping matroids, such as realizable delta-matroids, delta-matroids arising from the independent sets or bases of a matroid, and delta-matroids associated to graphs or embedded graphs. We will need the following property of enveloping matroids.

Proposition 3.12. *Let M be an enveloping matroid for a delta-matroid D on $[n,\overline{n}]$. Let $S \in \text{AdS}_n$ be an admissible set. Then S is independent in M if and only if it is independent in D.*

Proof. If $S \in \text{AdS}_n$, then $\text{env}(u_S)=e_S$, and S is the only admissible set with this property. Furthermore, if $S \in \text{AdS}_n$ has size n, then u_S is the only indicator vector of a subset of $[n,\bar{n}]$ of size n which is a preimage of e_S under env. Because $\text{env}(P(M))=P(D)$, we see that if B is a feasible set of D, then B is a basis for M. This implies that the independent sets in D are independent in M.

By [EFLS, Lemma 7.6], $\text{env}(IP(M))=\frac{1}{2}(P(D)+\square)$. If S is admissible and independent in M, then $\text{env}(u_S)=e_S \in \frac{1}{2}(P(D)+\square)$, so by Proposition 3.6, S is independent in D. □

3.4. **Lorentzian polynomials.** For a multi-index $\mathbf{m}=(m_0,m_1,\ldots)$, let $w^{\mathbf{m}}=w_0^{m_0}w_1^{m_1}\cdots$. A homogeneous polynomial $f(w_0,w_1,\ldots)$ of degree d with real coefficients is said to be *strictly Lorentzian* if all its coefficients are positive, and the quadratic form obtained by taking $d-2$ partial derivatives is nondegenerate with exactly one positive eigenvalue. We say that f is *Lorentzian* if it is a coefficient-wise limit of strictly Lorentzian polynomials. Lorentzian polynomials enjoy strong log-concavity properties, and the class of Lorentzian polynomials is preserved under many natural operations.

The following lemma is a special case of [RSW, Proposition 3.3]. Alternatively, it can be deduced from the proof of [BH20, Corollary 3.5]. We thank Nima Anari for discussing this lemma with us.

Lemma 3.13. *For a polynomial* $f(w_0, w_1, \dots) = \sum_{\boldsymbol{m}} c_{\boldsymbol{m}} w^{\boldsymbol{m}}$, *let*

$$\overline{f}(w_0, w_1, \dots) = \sum_{\boldsymbol{m}: m_i \leq 1 \text{ for } i \neq 0} c_{\boldsymbol{m}} w^{\boldsymbol{m}}.$$

If f *is Lorentzian, then* $\overline{f}$ *is Lorentzian.*

For $S \in \mathrm{AdS}_n$, recall that $\underline{S} \subset [n]$ denotes the unsigned version of S. For a set T, let $w^T = \prod_{a \in T} w_a$. We now state a strengthening of Theorem 1.6.

Theorem 3.14. *Let* D *be a delta-matroid on* $[n, \overline{n}]$ *which has an enveloping matroid. Then the polynomial*

$$\sum_{S \text{ independent in } D} w_0^{2n-|S|} w^{\underline{S}} \in \mathbb{R}[w_0, w_1, \dots, w_n]$$

is Lorentzian.

Remark 3.15. In [EFLS, Theorem 8.1], it is proven that if D has an enveloping matroid, then the polynomial

$$\sum_{S \text{ independent in } D} \frac{w_0^{|S|}}{|S|!} w^{[n] \setminus \underline{S}} \in \mathbb{R}[w_0, w_1, \dots, w_n]$$

is Lorentzian.

Proof of Theorem 1.6. By [BH20, Theorem 2.10], the specialization

$$\sum_{S \text{ independent in } D} w_0^{2n-|S|} y^{|S|} = \sum_{i=0}^{n} f_{i-1}(D) w_0^{2n-i} y^i$$

is Lorentzian. By [BH20, Example 2.26], the coefficients of a Lorentzian polynomial in two variables of degree $2n$ are log-concave after dividing the coefficient of $w_0^{2n-i} y^i$ by $\binom{2n}{i}$, which implies the result. ☐

Proof of Theorem 3.14. Let M be an enveloping matroid of D. By [BH20, Proof of Theorem 4.14], the polynomial

$$\sum_{S \text{ independent in } M} w_0^{2n-|S|} w^S \in \mathbb{R}[w_0, w_1, \dots, w_n, w_{\overline{1}}, \dots, w_{\overline{n}}]$$

is Lorentzian. Setting $w_{\overline{i}} = w_i$, by [BH20, Theorem 2.10] the polynomial

$$\sum_{S \text{ independent in } M} w_0^{2n-|S|} w^{S \cap [n]} w^{\overline{S \cap [\overline{n}]}} \in \mathbb{R}[w_0, w_1, \dots, w_n]$$

is Lorentzian. A term $w_0^{2n-|S|} w^{S \cap [n]} w^{\overline{S \cap [\overline{n}]}}$ has degree at most 1 in each of the variables $w_1, \dots, w_n$ if and only if S is admissible, in which case it is equal to $w^{\underline{S}}$. Therefore, by Lemma 3.13, the polynomial

$$\sum_{S \in \mathrm{AdS}_n \text{ independent in } M} w_0^{2n-|S|} w^{\underline{S}} \in \mathbb{R}[w_0, w_1, \dots, w_n]$$

is Lorentzian. By Proposition 3.12, this polynomial is equal to the polynomial in Theorem 3.14. ☐

Remark 3.16. Let (U, Ω, r) be a multimatroid [Bou97], i.e., U is a finite set, Ω is a partition of U, and r is a function on partial transversals of Ω satisfying certain conditions. An *independent set* is a partial transversal S of Ω with $r(S) = |S|$. A multimatroid is called *shelterable* if r can be extended to the rank function of a matroid on U. Then the argument used to prove Theorem 1.6 shows that, if a_k is the number of independent sets of a shelterable multimatroid of size k, then

$$a_k^2 \geq \frac{|U| - k + 1}{|U| - k} \frac{k+1}{k} a_{k+1} a_{k-1}.$$

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