K-THEORETIC POSITIVITY FOR WONDERFUL VARIETIES AND MATROIDS

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Abstract

Wonderful varieties are certain smooth projective varieties constructed from linear subspaces of a coordinated vector space. We establish a positivity property for Grothendieck rings of vector bundles of wonderful varieties. The Grothendieck ring of vector bundles of a wonderful variety depends only on the matroid represented by the linear subspace. We define a combinatorial analogue of the Grothendieck ring of vector bundles for any matroid, and we show that it has properties resembling the Grothendieck ring of a smooth projective variety. We prove the positivity property for any, not necessarily realizable, matroid.

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Introduction

1.1 Wonderful compactifications

Let $E = \{1, \ldots, n\}$. Let $\mathbf{k}$ be a field, and let $L \subseteq \mathbf{k}^E$ be a linear subspace of dimension r which is not contained in any coordinate hyperplane. The coordinate hyperplanes then give a hyperplane arrangement in L. For $S \subseteq E$, let $L_S = L \cap \mathbf{k}^{E \setminus S}$, and let $L^S = L/L_S$. For each non-empty subset S of E, we have a rational map $\mathbb{P}L \dashrightarrow \mathbb{P}L^S$. We define the *wonderful compactification* or *wonderful variety* W_L of L to be the closure of the image of the rational map

$$\mathbb{P}L \dashrightarrow \prod_{\emptyset \neq S \subseteq E} \mathbb{P}L^S.$$

This construction was introduced in [DCP95], where it is shown that W_L can be alternatively described as an iterated blow-up of $\mathbb{P}L$. We first blow up the points $\{\mathbb{P}L_S : \emptyset \neq S \subseteq E, \dim L_S = 1\}$. We then blow up the strict transform of the locus $\{\mathbb{P}L_S : \emptyset \neq S \subseteq E, \dim L_S = 2\}$. We continue in this way, blowing up the strict transforms of the $\mathbb{P}L_S$ in increasing order of dimension. From this description, we see that W_L is a smooth projective variety of dimension $r - 1$ which compactifies $\mathbb{P}L \setminus \cup_i \mathbb{P}L_{\{i\}}$.

From this description, one can also compute the cohomology ring of W_L, as there is a description of how the cohomology ring of a space changes when we blow up a smooth center. As the blow up procedure only depends on the incidences between the various L_S, the cohomology ring only depends on these incidences [DCP95]. A subset F of E is called a *flat* if, for each i such that $L_F \subseteq L_{\{i\}}$, we have that $i \in F$. That is, F is the maximal element of the set $\{S : S \subseteq E, L_S = L_F\}$. The flats form a lattice, and the cohomology of W_L has a presentation which only depends on the lattice of flats (Proposition 2.2.2).

At least if $\mathbf{k}$ is a subfield of $\mathbb{C}$, the fact that W_L is a smooth projective variety endows the singular cohomology $H^\bullet(W_L;\mathbb{Z})$ with powerful positivity properties. For example, the first Chern class of any ample line bundle $\mathcal{L}$ on W_L satisfies the *Hard Lefschetz* theorem: the multiplication map $c_1(\mathcal{L})^k \colon H^{r-1-k}(W_L;\mathbb{Q}) \to H^{r-1+k}(W_L;\mathbb{Q})$ is an isomorphism. We also have the Hodge–Riemann relations: the pairing $(a,b) \mapsto i^{r-1-k}\deg_{W_L}(c_1(\mathcal{L})^k \cdot a \cdot b)$ is positive definite on the kernel of the multiplication map $c_1(\mathcal{L})^{k+1} \colon H^{r-1-k}(W_L;\mathbb{Q}) \to H^{r+1+k}(W_L;\mathbb{Q})$. In [HK12], Huh and Katz used these positivity properties to prove remarkable combinatorial inequalities on the lattice of flats of L. This is essentially the only proof of these inequalities.

Matroids are combinatorial abstractions of hyperplane arrangements. Each hyperplane arrangement $L \subseteq \mathbf{k}^E$ gives rise to a matroid, which records the lattice of flats of the hyperplane arrangement. Matroids have a rich and extensively developed combinatorial theory.

Definition 1.1.1. A *matroid* M on a finite ground set E a collection of subsets of E, the *flats* of M, such that

1. E is a flat.
2. If F and G are flats, then $F \cap G$ is a flat.
3. If F is a flat, then each element of $E \setminus F$ is contained in exactly one flat which covers F.

For example, the flats of a hyperplane arrangement $L \subseteq \mathbf{k}^E$ form a matroid. Each subset S of E is contained in a unique minimal flat, the *closure* of S, which is denoted $\mathrm{cl}_{\mathrm{M}}(S)$. The flats of any matroid form a lattice, and every maximal chain in an interval $[F, G]$ has the same length. We say that the *rank* of a flat F, denoted

$\mathrm{rk}_{\mathrm{M}}(F)$, is the length of any maximal chain in the interval $[\mathrm{cl}_{\mathrm{M}}(\emptyset), F]$. The *rank* of M is $\mathrm{rk}_{\mathrm{M}}(E)$.

A matroid which arises from a hyperplane arrangement $L \subseteq \mathbf{k}^E$ is called *realizable* over $\mathbf{k}$. The rank of a flat F is the codimension of L_F. Although many matroids are not realizable over any field, experience shows that the properties enjoyed by realizable matroids are often shared by all matroids.

The presentation of $H^\bullet(W_L; \mathbb{Z})$ obtained from the iterated blow-up description of W_L depends only on the matroid M that $L \subseteq \mathbf{k}^E$ represents, and it makes sense for any matroid. This ring is called the *Chow ring* of M, denoted $A^\bullet(\mathrm{M})$, because it coincides with the Chow ring of cycles modulo rational equivalence on W_L. In [AHK18], it is shown that $A^\bullet(\mathrm{M})$ satisfies the properties that one expects of the cohomology ring of a smooth projective variety, even when M is not realizable. That is, there is a degree map $\deg\colon A^{r-1}(\mathrm{M}) \to \mathbb{Z}$ such that the pairing $(a, b) \mapsto \deg(a \cdot b)$ is unimodular (Poincaré duality), and there is a combinatorially defined "ample cone" in $A^1(\mathrm{M})$ such that $A^\bullet(\mathrm{M})$ satisfies the conclusions of the Hard Lefschetz theorem and the Hodge–Riemann relations. This generalizes the combinatorial inequalities obtained for realizable matroids in [HK12] to all matroids.

We study the Grothendieck ring of vector bundles $K(W_L)$. We give a presentation of $K(W_L)$ which depends only on the matroid M represented by $L \subseteq \mathbf{k}^E$ (Theorem 3.1.2). We show that $K(W_L)$ is the Grothendieck ring of vector bundles of a certain smooth non-compact toric variety $X_{\Sigma_{\mathrm{M}}}$ (Proposition 3.1.1). The definition of $X_{\Sigma_{\mathrm{M}}}$ makes sense for non-realizable matroids, so we define the K-ring of a matroid $K(\mathrm{M})$ to be the Grothendieck ring of vector bundles of this variety. From this definition, one sees that $K(\mathrm{M})$ has the properties that one expects from the Grothendieck ring of vector bundles of a variety: the Adams operations equip it a collection of commuting endomorphisms, $K(\mathrm{M})$ is an augmented lambda ring, and there is a Chern character isomorphism $\mathrm{ch}\colon K(\mathrm{M}) \otimes \mathbb{Q} \to A^\bullet(\mathrm{M}) \otimes \mathbb{Q}$ (Section 3.4).

When M is realized by $L \subseteq \mathbf{k}^E$, there is a functional $\chi\colon K(\mathrm{M}) \to \mathbb{Z}$ given by taking the sheaf Euler characteristic. The existence of this functional for non-realizable matroids is not obvious because $X_{\Sigma_{\mathrm{M}}}$ is *not* proper. Nevertheless, we construct such a functional $\chi\colon K(\mathrm{M}) \to \mathbb{Z}$ for all matroids which coincides with the sheaf Euler characteristic on $K(W_L)$ when L is a realization of M (Definition 3.4.3).

These results show that $K(\mathrm{M})$ behaves like the Grothendieck ring of vector bundles of a smooth projective variety. Our next goal is to prove "K-theoretic positivity results" for $K(\mathrm{M})$. We first sketch two examples of K-theoretic positivity results from the literature.

1.2 Lattice point counting in polytopes

Let Q be a d-dimensional lattice polytope in $\mathbb{R}^n$, and let kQ denote its kth dilate. Stanley [Sta80] showed that the h^*-vector $(h_0^*(Q), \ldots, h_d^*(Q))$ defined by

$$\sum_{k \geq 0} |\{\text{lattice points in } kQ\}| t^k = \frac{h_0^*(Q) + h_1^*(Q)t + \cdots + h_d^*(Q)t^d}{(1-t)^{d+1}}$$

is non-negative, and it is furthermore a Macaulay vector (Definition 4.1.1) if, for every k, all lattice points in kQ are sums of lattice points in Q. Via standard results in toric geometry [Ful93, Chapter 3.5], this result can be formulated geometrically as "K-theoretic positivity" in the following way.

We identify $\mathbb{Z}^n$ with the character lattice of $\mathbb{G}_m^n$. Each lattice point in Q gives a character of $\mathbb{G}_m^n$, so we have a map $\mathbb{G}_m^n \to \mathbb{G}_m^{|\{\text{lattice points in } Q\}|}$. Let X be the normalization of the closure of the image in $\mathbb{P}^{|\{\text{lattice points in } Q\}|}$, and let $\mathcal{L}$ be the ample line bundle on X obtained from pulling back $\mathcal{O}(1)$. Toric vanishing theorems imply that $\chi(X, \mathcal{L}^{\otimes k}) = \dim H^0(X, \mathcal{L}^{\otimes k}) = |\{\text{lattice points in } kQ\}|$ (for $k \geq 0$), and that the graded ring $R^\bullet_{\mathcal{L}} := \bigoplus_{k \geq 0} H^0(X, \mathcal{L}^{\otimes k})$ is Cohen–Macaulay. See Proposition 4.1.2 for a detailed review. Quotienting $R^\bullet_{\mathcal{L}}$ by a linear system of parameters, the vector $(h_0^*(\mathcal{L}), \ldots, h_d^*(\mathcal{L}))$ defined by

$$\sum_{k \geq 0} \chi(X, \mathcal{L}^{\otimes k}) t^k = \text{Hilbert series of } R^\bullet_{\mathcal{L}} = \frac{h_0^*(\mathcal{L}) + h_1^*(\mathcal{L})t + \cdots + h_d^*(\mathcal{L})t^d}{(1-t)^{d+1}}$$

is the Hilbert function of a graded artinian ring. In particular, this implies that the vector $(h_0^*(\mathcal{L}), \ldots, h_d^*(\mathcal{L}))$ is non-negative, and it is furthermore a Macaulay vector if

$R^{\bullet}_{\mathcal{L}}$ is generated in degree 1. This gives inequalities on the number of lattice points in dilates of Q. There are also combinatorial proofs of at least the non-negativity of the $h_i^*(\mathcal{L})$, see, e.g., [BS07].

1.3 Degenerations of torus-orbit closures

Let $L \subseteq \mathbf{k}^E$ be a linear subspace of dimension r. This defines a point $[L]$ in the Grassmannian $\mathrm{Gr}(r, n)$. The torus $T = \mathbb{G}_m^n$ acts on $\mathrm{Gr}(r, n)$, and we may consider the torus-orbit closure $\overline{T \cdot [L]} \subseteq \mathrm{Gr}(r, n)$.

For each cocharacter $\lambda \colon \mathbb{G}_m \to T$, there is a specialization of $[L]$ to $\lim_{t\to 0} \lambda(t) \cdot [L] = [L']$. Then $\overline{T \cdot [L]}$ degenerates to a union of torus-orbit closures that contains $\overline{T \cdot [L']}$. In [Spe08], Speyer studied the structure of the special fiber of this degeneration, and in particular conjectured a bound on the number of torus-orbits in the special fiber when L is generic. He constructed examples to show that his conjecture, if true, is tight [Spe08, Theorem 1.2]. This conjecture is equivalent to bounding the number of faces in a "tropicalized linear space."

Because $\overline{T \cdot [L]}$ is a projective toric variety (via the Plücker embedding of the Grassmannian), there is a moment map $\overline{T \cdot [L]} \to \mathbb{R}^n$. The image of the moment map is a polytope, which we now describe. Let M be the matroid that L represents. We say that $B \subseteq E$ is a *basis* of M if the coordinate projection $L \hookrightarrow \mathbf{k}^E \to \mathbf{k}^B$ is an isomorphism. Given $S \subseteq E$, let $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i$, where $\mathbf{e}_i$ is the ith standard basis vector in $\mathbb{R}^n$. Then the image of moment map is

$$B(\mathrm{M}) = \text{the convex hull of } \{\mathbf{e}_B : B \text{ basis of } \mathrm{M}\}.$$

This is called the *basis polytope* of M [GS87]. The dimension of $B(\mathrm{M})$ is $n - c$, where c is the dimension of stabilizer in T of $[L] \in \mathrm{Gr}(r, n)$.

Each strata of the special fiber is the torus-orbit closure of some point in $\mathrm{Gr}(r, n)$, and so the moment polytope of each strata of the special fiber is the basis polytope of a matroid. The theory of projective toric varieties over discrete valuations rings implies that there is a subdivision of the polytope $B(\mathrm{M})$ into the basis polytopes

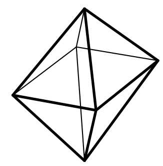

Figure 1.1: The moment polytope of the torus-orbit closure of a general point in $\mathrm{Gr}(2,4)$.

of the matroids corresponding to the strata of the special fiber. See, for instance, [Kat09, Section 4]. In order to bound the number of torus-orbits in the special fiber of a degeneration, it suffices to bound the number of polytopes appearing in any subdivision of $B(\mathrm{M})$ into basis polytopes of matroids.

In [Spe09], Speyer proved a special case of his conjecture bounding the number of torus-orbits in a degeneration of a torus-orbit closure. Speyer constructed an invariant $g_{\mathrm{M}}(t) \in \mathbb{Z}[t]$ of a matroid M with the following property: whenever there is a subdivision of a basis polytope $B(\mathrm{M})$ into basis polytopes of matroids such that $B(\mathrm{M}_1), \ldots, B(\mathrm{M}_k)$ are the interior faces of the subdivision, we have

$$g_{\mathrm{M}}(t) = g_{\mathrm{M}_1}(t) + \cdots + g_{\mathrm{M}_k}(t).$$

If one can understand the coefficients of $g_{\mathrm{M}}(t)$, and in particular show that they are non-negative for every matroid, then this gives a bound on the possible subdivisions of $B(\mathrm{M})$ into basis polytopes of matroids. Speyer computed $g_{\mathrm{M}}(t)$ when M is the matroid represented by a generic linear subspace [Spe09, Proposition 3.1], and he showed that the non-negativity of $g_{\mathrm{M}}(t)$ would prove his conjecture.

We give a definition of Speyer's invariant $g_{\mathrm{M}}(t)$ when M is realized by $L \subseteq \mathbf{k}^E$. If L is contained in a coordinate hyperplane, then $g_{\mathrm{M}}(t)$ is defined to be 0. Let X_E be the wonderful variety of the Boolean arrangement $\mathbf{k}^E \subseteq \mathbf{k}^E$; we have an inclusion $W_L \hookrightarrow X_E$. Let $\mathcal{Q}_L$ denote the normal bundle of this inclusion, and let $\mathcal{C}_L$ be the projectivization of $\mathcal{Q}_L^\vee$ over W_L; this is a variety of dimension $n-2$ equipped with a line bundle $\mathcal{O}(1)$.

One can show that the line bundle $\mathcal{O}(1)$ is globally generated. Let $H_1, \ldots, H_{n-2}$ be the vanishing loci of general sections of $\mathcal{O}(1)$. Then the constant term of $g_{\mathrm{M}}(t)$ is

0, and the t^i coefficient of $g_{\mathrm{M}}(t)$ is $(-1)^{\dim B(\mathrm{M})-n+i}\chi(H_1 \cap \cdots \cap H_{n-1-i}, \mathcal{O}(-1))$ for $i \in \{1, \ldots, n-1\}$.

In [Spe09, Proposition 3.3], Speyer showed that, if M is realizable over a field of characteristic 0, then the coefficients of $g_{\mathrm{M}}(t)$ are non-negative. He does this by reducing to the case when $\dim B(\mathrm{M}) = n-1$ and then showing that $\mathcal{O}(1)$ is a nef and big line bundle, and this remains true when $\mathcal{O}(1)$ is restricted to $H_1 \cap \cdots \cap H_{n-1-i}$ for each i. Then the Kawamata–Viehweg vanishing theorem implies that $H^j(H_1 \cap \cdots \cap H_{n-1-i}, \mathcal{O}(-1)) = 0$ for $j < i-1$, and so $(-1)^{i-1}\chi(H_1 \cap \cdots \cap H_{n-1-i}, \mathcal{O}(-1)) \geq 0$.

It has since been realized that, in order to prove that the coefficients of $g_{\mathrm{M}}(t)$ are non-negative for all matroids, it suffices to prove that the t^r coefficient of $g_{\mathrm{M}}(t)$ is non-negative for all matroid of rank r which have $\dim B(\mathrm{M}) = n-1$, see Proposition 4.5.1. Note that

$$\begin{aligned}\chi(H_1 \cap \cdots \cap H_{n-1-i}, \mathcal{O}(-1)) &= \chi(\mathcal{C}_L, [\mathcal{O}(-1)] \cdot (1 - [\mathcal{O}(-1)])^{n-1-i}) \\ &= \sum_{k=0}^{n-1-i} (-1)^k \binom{n-1-i}{k} \chi(\mathcal{C}_L, \mathcal{O}(-k-1)).\end{aligned}$$

Let $\pi \colon \mathcal{C}_L \to W_L$ be the projective bundle morphism. Note that $R\pi_*\mathcal{O}(-k) = 0$ for $1 \leq k \leq n-r-1$, and so $\chi(\mathcal{C}_L, \mathcal{O}(-k)) = 0$. In particular, the t^k coefficient of $g_{\mathrm{M}}(t)$ vanishes for $k > r$, and the coefficient of t^r is

$$\begin{aligned}(-1)^{\dim B(\mathrm{M})-1}\chi(\mathcal{C}_L, \mathcal{O}(-n+r)) &= (-1)^{\dim B(\mathrm{M})-1}\chi(W_L, R\pi_*\mathcal{O}(-n+r)) \\ &= (-1)^{\dim B(\mathrm{M})-n+r}\chi(W_L, \det \mathcal{Q}_L^\vee)\end{aligned}$$

as $R^i\pi_*\mathcal{O}(-n+r) = 0$ for $i < n-r-1$ and $R^{n-r-1}\pi_*\mathcal{O}(-n+r) = \det \mathcal{Q}_L^\vee$. The line bundle $\det \mathcal{Q}_L$ is a nef and big line bundle on W_L when $\dim B(\mathrm{M}) = n-1$, so if $\mathbf{k}$ has characteristic 0, the non-negativity follows from applying the Kawamata–Viehweg vanishing theorem to W_L. See Proposition 4.5.2 for a more general formula. One can compute all of the coefficients of $g_{\mathrm{M}}(t)$ in terms of symmetric powers of $\mathcal{Q}_L$ by pushing forward to W_L.

The definition of $g_{\mathrm{M}}(t)$ can be extended to non-realizable matroids using the following strategy. Consider the sub-Grassmannian

$$\mathrm{Gr}_e = \{[U] \in \mathrm{Gr}(r,n) : (1,\dots,1) \in U\}.$$

Kapranov's *space of visible contours* [Kap93] of $L \subseteq \mathbf{k}^E$ is $\mathcal{V}_L := \overline{T \cdot [L]} \cap \mathrm{Gr}_e$. When $\dim B(\mathrm{M}) = n-1$, there is a map $f \colon W_L \to \mathcal{V}_L$ which satisfies $f_*\mathcal{O}_{W_L} = \mathcal{O}_{\mathcal{V}_L}$ [Tev07]. The vector bundle $\mathcal{Q}_L$ on W_L is $f^*\mathcal{Q}$, where $\mathcal{Q}$ is the restriction of the tautological quotient bundle on $\mathrm{Gr}(r,n)$ to $\mathcal{V}_L$. We may compute the Euler characteristics used in the definition of $g_{\mathrm{M}}(t)$ by pushing forward from $\mathcal{C}_L$ to W_L; the resulting class in $K(W_L)$ will be pulled back from $\mathcal{V}_L$ from f. We may therefore compute the coefficients of $g_{\mathrm{M}}(t)$ as Euler characteristics on $\mathcal{V}_L$.

The intersection defining $\mathcal{V}_L$ is transverse, and so we have $[\mathcal{O}_{\mathcal{V}_L}] = [\mathcal{O}_{\overline{T\cdot L}}] \cdot [\mathcal{O}_{\mathrm{Gr}_e}]$ inside $K(\mathrm{Gr}(r,n))$. By the above discussion, we can express the t^i coefficient of $g_{\mathrm{M}}(t)$ as $\chi(\mathrm{Gr}(r,n), a_i \cdot [\mathcal{O}_{\overline{T\cdot L}}] \cdot [\mathcal{O}_{\mathrm{Gr}_e}])$ for some class $a_i \in K(\mathrm{Gr}(r,n))$ which is a linear combination of symmetric powers of $\mathcal{Q}$, and in particular is independent of L.

Therefore, to extend the definition of $\mathrm{g}_{\mathrm{M}}(t)$ to arbitrary matroids (and prove that the above definition does not depend on the choice of realization), it suffices to define a class generalizing $[\mathcal{O}_{\overline{T\cdot L}}] \in K(\mathrm{Gr}(r,n))$ for an arbitrary matroid. This was done in [FS12], using the following strategy. The torus-orbit closure $\overline{T \cdot [L]}$ is isomorphic to the toric variety of the matroid basis polytope $X_{B(\mathrm{M})}$; in particular, it is normal [Wel76, Chapter 18.6, Theorem 3]. Therefore $[\mathcal{O}_{\overline{T\cdot[L]}}] = \iota_* X_{B(\mathrm{M})}$, where ι is the inclusion. We can compute this pushforward using equivariant localization: this gives a combinatorial formula for the class $[\mathcal{O}_{\overline{T\cdot[L]}}]$ inside the T-equivariant K-theory of the Grassmannian which is described solely in terms of the combinatorics of $B(\mathrm{M})$. This formula, given in [FS12, Proposition 3.2], makes sense for any matroid. This generalizes the class $[\mathcal{O}_{\overline{T\cdot[L]}}]$. See [ELS] for a detailed discussion. For a slightly different description of the g-polynomial, see [FS12, Section 6].

1.4 K-theoretic positivity for matroids

For each line bundle $\mathcal{L}$ on the non-compact toric variety $X_{\Sigma_{\mathrm{M}}}$, the function $k \mapsto \chi(\mathrm{M}, \mathcal{L}^{\otimes k})$ is a polynomial in k called the *Snapper polynomial* of $\mathcal{L}$. We define the *h^*-vector* $(h^*_0(\mathrm{M}, \mathcal{L}), \ldots, h^*_d(\mathrm{M}, \mathcal{L}))$ of $\mathcal{L}$ by

$$\sum_{k \geq 0} \chi(\mathrm{M}, \mathcal{L}^{\otimes k}) t^k = \frac{h^*(\mathrm{M}, \mathcal{L}; t)}{(1-t)^{d+1}} \quad \text{where} \quad h^*(\mathrm{M}, \mathcal{L}; t) = \sum_{k=0}^{d} h^*_k(\mathrm{M}, \mathcal{L}) t^k,$$

and d is the degree of the Snapper polynomial of $\mathcal{L}$.

We conjecture that, for a large class of line bundles, $h^*(\mathrm{M}, \mathcal{L}; t)$ is a Macaulay vector (Conjecture 4.3.1). As in Section 1.2, if M is realizable, then this would follow from a strong cohomology vanishing statement, see, e.g., Example 4.6.3. Note that $h^*_d(\mathrm{M}, \mathcal{L}) = (-1)^d \chi(\mathrm{M}, \mathcal{L}^{-1})$, so the non-negativity of $h^*_d(\mathrm{M}, \mathcal{L})$ can in some cases be proved using the Kawamata–Viehweg vanishing theorem. Conjecture 4.3.1 implies the non-negativity of the coefficients of the Speyer's g-polynomial.

We prove Conjecture 4.3.1 for a class of line bundles (Theorem 4.0.2). This proves the non-negativity of the coefficients of Speyer's g-polynomial in new cases (Theorem 4.5.5). Our strategy is to show that we can compute $\chi(\mathrm{M}, -)$ on a certain highly reducible subvariety of a product of projective spaces (Theorem 3.6.1). This subvariety is Frobenius split, which gives strong cohomology vanishing statements.

1.5 Organization and overview

Chapter 2 consists of background information about wonderful varieties and related objects. We introduce Bergman fans, which correspond to certain non-compact toric varieties that contain wonderful varieties. We also discuss the Chow rings of wonderful varieties and matroids.

Chapter 3 computes the K-rings of wonderful varieties and defines the K-ring of a matroid. We show that the K-ring of a matroid has properties resembling the K-ring of a smooth projective varieties. The most nontrivial part is defining an analogue of the Euler characteristic, see Definition 3.4.3. This definition is justified by Theorem 3.4.1. This content of this chapter is mostly drawn from [LLPP24], although the proof of Theorem 3.4.1 is new.

Chapter 4 discusses positivity properties of K-rings of matroids. The positivity properties that we consider are best encoded in an analogue of the h^*-vector of a lattice polytope, see Definition 4.0.1. Our main result in this direction is Theorem 4.0.2. We conjecture a strengthening which would imply Speyer's conjecture. The content of this chapter is mostly drawn from [EL], although some parts are simplified using the new proof of Theorem 3.4.1.

Throughout, we will only consider linear subspaces $L \subseteq \mathbf{k}^E$ which are not contained in any coordinate hyperplane. This is because the wonderful variety W_L is not defined when L is contained in coordinate subspace. Many of the results considered here have versions for *augmented wonderful varieties*, which are varieties associated to any linear subspace $L \subseteq \mathbf{k}^E$. In some cases, this is developed in [LLPP24] and [EL].

Chapter 2

Geometry and combinatorics of wonderful varieties

Let $L \subseteq \mathbf{k}^E$ be a linear subspace of dimension r which is not contained in any coordinate hyperplane. In this chapter, we review some aspects of the geometry of the wonderful variety W_L. We describe three families of divisors on W_L which will be used in the sequel. We also introduce two key tools for understanding the geometry of wonderful varieties. The first, introduced in Section 2.2, is the toric variety of the *Bergman fan*, which is a non-compact toric variety containing W_L. The second, introduced in Section 2.5, is the realization of W_L as *multiplicity-free* subvariety of a product of projective spaces.

2.1 The geometry of wonderful varieties

We say that a matroid is *loopless* if the rank of every $i \in E$ is 1. Equivalently, a matroid is loopless if the empty set is a flat. A matroid realized by $L \subseteq \mathbf{k}^E$ is loopless if and only if L is not contained in a coordinate hyperplane.

As mentioned in the introduction, W_L can be described as an iterated blow-up of $\mathbb{P}L$. Recall that $L_S = L \cap \mathbf{k}^{E \setminus S}$ and that $L^S = L/L_S$. If F is the closure of S, i.e., the minimal flat containing S, then $L_S = L_F$ and $L^S = L^F$. Then W_L is obtained by blowing up the points $\{\mathbb{P}L_F : \mathrm{rk}(F) = r-1\}$ on $\mathbb{P}L$, the blowing up the strict transforms of the lines $\{\mathbb{P}L_F : \mathrm{rk}(F) = r-2\}$, and so on, ending by blowing up the divisors which are the strict transforms of $\{\mathbb{P}L_F : \mathrm{rk}(F) = 1\}$ [DCP95]. In particular, the dimension of W_L is $r-1$. Indeed, W_L contains $\mathbb{P}L \setminus \cup_F \mathbb{P}L_F$ as a dense open subset.

There are two distinguished families of divisors on wonderful varieties, one arising from the iterated blow-up description, and the other arising from the description as a closure in a product of projective spaces. To each proper non-empty flat F (i.e., each flat other than E and the empty set), we have a divisor D_F on W_L which is the strict transform of the exceptional divisor obtained when we blow up $\mathbb{P}L_F$.

For each non-empty flat, we have a line bundle $\mathcal{L}_F$ on W_L obtained by pulling back $\mathcal{O}(1)$ along the map $W_L \to \mathbb{P}L^F$. This line bundle is trivial if $\mathrm{rk}(F) = 1$ because $\mathbb{P}L^F$ is a point. We also obtain line bundles on W_L by pulling back $\mathcal{O}(1)$ from $\mathbb{P}L^S$ for any non-empty set S. However, if F is the smallest flat containing S, then $L^S = L^F$, so we don't obtain any new line bundles in this way. In particular, the map

$$W_L \to \prod_{F \neq \emptyset} \mathbb{P}L^F,$$

where the product is taken over all flats (and not all subsets), is an embedding.

Either of these families of divisors generate the Chow ring of W_L, which is isomorphic to the singular cohomology ring of W_L when $\mathbf{k}$ is a subfield of $\mathbb{C}$. In the next section, we will describe the Chow ring of W_L. For this, it will be convenient to introduce another variety.

2.2 Bergman fans

When $L = \mathbf{k}^E$, the torus $\mathbb{G}_m^E$ acts on W_L, and the diagonal subtorus acts trivially. The orbit of the projectivization of the line spanned by $(1, \ldots, 1)$ under $\mathbb{G}_m^E/\mathbb{G}_m$ is dense in W_L and is identified with $\mathbb{G}_m^E/\mathbb{G}_m$. As W_L is smooth and in particular normal, this gives W_L the structure of a toric variety. We call this the *permutohedral toric variety* X_E on E. It has dimension $n-1$.

We will write $\mathbb{Z}^E/\mathbb{Z}$ to denote $\mathbb{Z}^E$ module the subgroup of diagonal elements. This is the cocharacter lattice of the torus $\mathbb{G}_m^E/\mathbb{G}_m$. Set $\mathbb{R}^E/\mathbb{R} = (\mathbb{Z}^E/\mathbb{Z}) \otimes \mathbb{R}$. For $S \subseteq E$, set $\mathbf{e}_S = \sum_{i \in S} \mathbf{e}_i \in \mathbb{Z}^E/\mathbb{Z}$.

For any $L \subseteq \mathbf{k}^E$, the embedding $L \hookrightarrow \mathbf{k}^E$ induces an embedding of W_L into X_E. An important role will be played by a certain open toric subvariety of X_E which contains W_L: we will show that the restriction map on Chow and K-groups from this open subvariety to W_L is an isomorphism. These are the toric varieties associated to *Bergman fans*, which we introduced in [AK06].

Let M be a loopless matroid on ground set E. Let $\mathcal{F} = \{\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E\}$ be a flag of proper non-empty flats of M. Let $\rho_{\mathcal{F}}$ be the cone in $\mathbb{R}^E/\mathbb{R}$ generated by $\{\mathbf{e}_F : F \in \mathcal{F}\}$.

Definition 2.2.1. The *Bergman fan* of a loopless matroid M, denoted Σ_{M}, is the fan in $\mathbb{R}^E/\mathbb{R}$ whose cones are $\{\rho_{\mathcal{F}} : \mathcal{F} \text{ flag of flats of } \mathrm{M}\}$.

Let $U_{n,n}$ be the *Boolean* matroid, i.e., the matroid realized by $\mathbf{k}^E \subseteq \mathbf{k}^E$. Then $\Sigma_{U_{n,n}}$ is the fan of the permutohedral toric variety, and we denote it Σ_E. Each Σ_{M} is a subfan of Σ_E. The toric variety of Σ_{M}, denoted $X_{\Sigma_{\mathrm{M}}}$, is therefore open inside X_E. Because X_E is smooth, we see that $X_{\Sigma_{\mathrm{M}}}$ is smooth. We may therefore talk about their Chow rings of cycles modulo rational equivalence on $X_{\Sigma_{\mathrm{M}}}$. The following result motivates the definition of the Bergman fan.

Proposition 2.2.2. *Let $L \subseteq \mathbf{k}^E$ be a realization of a loopless matroid* M. *Then the inclusion $W_L \hookrightarrow X_E$ factors through $X_{\Sigma_{\mathrm{M}}}$, and the pullback map $A^\bullet(X_{\Sigma_{\mathrm{M}}}) \to A^\bullet(W_L)$ is an isomorphism.*

Proposition 2.2.2 appears in [BHM+22, Remark 2.13] and can be deduced from [DCP95, FY04].

If M is loopless, we define the *Chow ring* of a matroid M, denoted $A^\bullet(\mathrm{M})$, to be the Chow ring $A^\bullet(X_{\Sigma_{\mathrm{M}}})$. We have the following fundamental result on Chow rings of matroids.

Proposition 2.2.3. *[AHK18, Theorem 6.19] Let* M *be a loopless matroid of rank r. Then there is an isomorphism* $\deg_{\mathrm{M}}\colon A^{r-1}(\mathrm{M}) \to \mathbb{Z}$ *such that the pairing* $A^k(\mathrm{M}) \times A^{r-1-k}(\mathrm{M}) \to \mathbb{Z}$ *given by* $(a, b) \mapsto \deg_{\mathrm{M}}(ab)$ *is unimodular.*

In particular, Proposition 2.2.3 asserts that $A^\bullet(\mathrm{M})$ is a free abelian group of finite rank. The above results is a version of Poincaré duality. We call the pairing $(a, b) \mapsto \deg_{\mathrm{M}}(ab)$ the *Poincaré pairing.*

When M is realized by $L \subseteq \mathbf{k}^E$, under the identification in Proposition 2.2.2, the degree map coincides with the degree map $\deg\colon A^{r-1}(W_L) \to \mathbb{Z}$ given by pushing forward to a point. At least if $\mathbf{k}$ is a subfield of $\mathbb{C}$, Proposition 2.2.3 is a consequence of Poincaré duality in this case.

The Chow ring of a matroid, being the Chow ring of a smooth toric variety, is generated by the classes of torus-invariant irreducible divisors. This leads to a presentation of the Chow ring of a matroid, which we call the *toric presentation.* Torus-invariant irreducible divisors on a toric variety are in bijection with rays of the fan, so the torus-invariant divisors on $X_{\Sigma_{\mathrm{M}}}$ are labeled by proper non-empty flats of M. We call the class of a divisor labeled by a proper non-empty flat F x_F. Two flats of M are *incomparable* if neither contains the other.

Proposition 2.2.4. *[FY04, Theorem 3] [AHK18, Section 5.3] Let* M *be a loopless matroid on E. Then the Chow ring of* M *has the presentation*

$$A^\bullet(\mathrm{M}) = \frac{\mathbb{Z}[x_F]_{F \textit{ proper non-empty flat}}}{(\sum_{F \ni i} x_F - \sum_{G \ni j} x_G : i, j \in E) + (x_{F_1} x_{F_2} : F_1, F_2 \textit{ incomparable})}.$$

When M is realized by $L \subseteq \mathbf{k}^E$, the isomorphism $A^\bullet(\mathrm{M}) \to A^\bullet(W_L)$ sends x_F to the class of the divisor D_F, the strict transform of the exceptional divisor arising when we blow up $\mathbb{P}L_F$.

2.3 Line bundles on the permutohedral toric variety

There is a description of globally generated line bundles on a proper toric variety in terms of polytopes, see [Ful93, Section 3.4]. Each integral polytope whose normal fan coarsens the fan of the toric variety gives rise to a globally generated line bundle on that toric variety. In the case of X_E, we will give a more explicit description in terms of *polymatroids*, which are combinatorial objects introduced in [Edm70] which generalize matroids.

Definition 2.3.1. For vectors $u, v \in \mathbb{R}^E$, we say $u \geq v$ if $u - v \in \mathbb{R}^E_{\geq 0}$. A *polymatroid* on E is a non-empty lattice polytope P in the non-negative orthant $\mathbb{R}^E_{\geq 0}$ satisfying the following two properties:

1. If $v \in \mathbb{R}^E_{\geq 0}$ such that $u \geq v$ for some $u \in \mathrm{P}$, then $v \in \mathrm{P}$.
2. For any $v \in \mathbb{R}^E_{\geq 0}$, every maximal $u \in \mathrm{P}$ such that $u \leq v$ has the same coordinate sum $\langle u, \mathbf{e}_E \rangle$.

Example 2.3.2. For $S \subseteq E$, the simplex that is the convex hull

$$\Delta_S = \mathrm{Conv}(\{\mathbf{e}_i : i \in S\} \cup \{0\})$$

is a polymatroid.

Example 2.3.3. Let M be a matroid on E. Then the *independence polytope*

$$I(\mathrm{M}) = \mathrm{Conv}(\{\mathbf{e}_I : \mathrm{rk}_{\mathrm{M}}(I) = |I|\})$$

is a polymatroid.

An *independent set* of a polymatroid is a lattice point in the polymatroid. In particular, the independent sets of $I(\mathrm{M})$ are the sets $I \subseteq E$ such that $\mathrm{rk}_{\mathrm{M}}(I) = |I|$.

We will need the following equivalent description of polymatroids. A function $\mathrm{rk}\colon 2^E \to \mathbb{Z}$ with $\mathrm{rk}(\emptyset) = 0$ is said to be *non-decreasing* and *submodular* if

(non-decreasing) $\mathrm{rk}(S) \leq \mathrm{rk}(S')$ whenever $S \subseteq S' \subseteq E$, and

(submodular) $\mathrm{rk}(S \cup S') + \mathrm{rk}(S \cap S') \leq \mathrm{rk}(S) + \mathrm{rk}(S')$ for all $S, S' \subseteq E$.

Theorem 2.3.4. *[Edm70, (8)] There is a bijection between polymatroids on E and non-decreasing and submodular functions* $\mathrm{rk}\colon 2^E \to \mathbb{Z}$ *with* $\mathrm{rk}(\emptyset) = 0$. *The bijection* *is given by*

$$\begin{array}{rcl} \textit{a polytope } \mathrm{P} & \mapsto & \mathrm{rk}\colon 2^E \to \mathbb{Z} \textit{ where } \mathrm{rk}(S) = \max\{\langle u, \boldsymbol{e}_S\rangle \mid u \in \mathrm{P}\} \\ \textit{a function } \mathrm{rk}\colon 2^E \to \mathbb{Z} & \mapsto & \mathrm{P} = \{u \in \mathbb{R}^E_{\geq 0} \mid \langle \boldsymbol{e}_S, u\rangle \leq \mathrm{rk}(S) \textit{ for all } S \subseteq E\}. \end{array}$$

Under this correspondence, the independence polytope $I(\mathrm{M})$ of a matroid corresponds to the rank function of the matroid. In particular, the rank function of a matroid is submodular. Several matroid-theoretic concepts immediately generalize to polymatroids.

The *rank* of the polymatroid associated to a submodular function rk is $\mathrm{rk}(E)$. Equivalently, the rank of P is $\max\{\langle u, \mathbf{e}_E\rangle \mid u \in P\}$. A *basis* of P is a lattice point $u \in \mathrm{P}$ which has $\langle u, \mathbf{e}_E\rangle$ equal to the rank of P. The set of bases of P is denoted $B(\mathrm{P})$. The bases of a polymatroid determine the polymatroid via the formula

$$\mathrm{P} = \mathbb{R}^E_{\geq 0} \cap (\mathrm{Conv}(B(\mathrm{P})) + \mathbb{R}^n_{\leq 0}),$$

where the sum is Minkowski sum.

Later, we will use a construction of polymatroids in terms of *subspace arrangements*. Let L be a finite-dimensional vector space over $\mathbf{k}$, and let $L_1, \ldots, L_m$ be subspaces of $\mathbf{k}$. Then we have an injective map

$$L/\cap_i L_i \to \bigoplus_{i=1}^m L/L_i.$$

If $\cap_i L_i = 0$, then the subspace arrangement is determined by the inclusion $L \hookrightarrow \bigoplus_i L/L_i$: every subspace of a finite-dimensional vector space equipped with a direct sum decomposition $L \subseteq \bigoplus_i V_i$ determines a subspace arrangement, where we set $L_i = L \cap \oplus_{j \neq i} V_j$. In general, set $L_S = L \cap \oplus_{j \notin S} V_j$. Then we obtain a polymatroid P whose rank function is given by

$$\mathrm{rk}_{\mathrm{P}}(S) = \text{codimension of } L_S \text{ in } L.$$

Note that, in the case when each L_i has codimension at most 1, this specializes to the definition of a matroid associated to a hyperplane arrangement. Replacing L by $L/\cap_i L_i$ does not change the polymatroid.

We can describe the bases of P in terms of L, as follows. Choose a generic basis $x_{i,1}, \ldots, x_{i,a_i}$ for $(L/L_i)^*$ for all i. Then a vector $\mathbf{b} = (b_1, \ldots, b_m)$ is a basis for P if and only if the composition

$$L \hookrightarrow \bigoplus L/L_i \to \mathbf{k}^{b_1} \oplus \cdots \oplus \mathbf{k}^{b_m}$$

is an isomorphism, where the second map is induced by the maps $L/L_i \to \mathbf{k}^{b_i}$ using the functionals $x_{i,1}, \ldots, x_{i,b_i}$. A polymatroid arising in the above fashion is called *realizable.*

It will be useful to define a combinatorially natural class in $A^1(\mathrm{M})$. Define the class $x_\emptyset = -\sum_{S \not\ni i} x_S \in A^1(X_E)$ for some $i \in E$. The defining relations in Proposition 2.2.4 imply that $x_\emptyset$ is well-defined, i.e., independent of the choice of i. This class is often denoted $-\beta$ in the literature. By restriction, this also defines a class in $A^1(\mathrm{M})$ for each loopless matroid M.

Proposition 2.3.5. *[BEST23, Section 2.7] Each polymatroid on E determines a globally generated line bundle on X_E via the formula*

$$(\textit{polymatroid } \mathrm{P} \textit{ defined by } \mathrm{rk}\colon 2^E \to \mathbb{Z}) \mapsto \sum_{\emptyset \subseteq S \subsetneq E} \mathrm{rk}(E \setminus S) x_S \in A^1(X_E).$$

Every globally generated line bundle on X_E arises in this way.

Given a polymatroid P, the associated line bundle on X_E is denoted $\mathcal{L}_{\mathrm{P}}$. By restriction, this defines a line bundle on $X_{\Sigma_{\mathrm{M}}}$ for all loopless matroid M on E, which we also denote $\mathcal{L}_{\mathrm{P}}$. One can show that, if $\mathrm{P}_1, \mathrm{P}_2$ are polymatroids, then $\mathcal{L}_{\mathrm{P}_1} \xrightarrow{\sim} \mathcal{L}_{\mathrm{P}_2}$ on X_E if and only if $B(\mathrm{P}_1)$ is a translate of $B(\mathrm{P}_2)$ [EHL23, Appendix A].

The complete linear system of the line bundle $\mathcal{L}_{\Delta_S}$ on X_E induces the map $X_E \to \mathbb{P}(\mathbf{k}^S)$. Therefore, the restriction of $\mathcal{L}_{\Delta_S}$ to W_L induces the map $W_L \to \mathbb{P}L^S$. In particular, the line bundle $\mathcal{L}_{\Delta_S}$ on W_L is the pullback of $\mathcal{O}(1)$ from $\mathbb{P}L^S$.

2.4 Simplicial presentations of Chow rings of matroids

There is a second presentation of the Chow ring of a matroid, arising from the geometry of the embedding $W_L \hookrightarrow \prod_{F \neq \emptyset} \mathbb{P}L^F$, which will play an important role in the sequel. This presentation was introduced in [Yuz02] and extensively studied in [BES]. For each non-empty flat, we set h_F to be the first Chern class of the line bundle $\mathcal{L}_{\Delta_F}$ on $X_{\Sigma_{\mathrm{M}}}$. When M is realized by $L \subseteq \mathbf{k}^E$, this is the pullback of the hyperplane class from $\mathbb{P}L^F$. The classes $\{h_F : F \text{ non-empty flat}\}$ generate $A^1(W_L) = A^1(\mathrm{M})$. We will see this by working out the formula for h_F in terms of the toric generators.

Under the inclusion $W_L \hookrightarrow X_E$, the line bundle $\mathcal{L}_{\Delta_F}$ is the restriction of the line bundle on X_E corresponding to the simplex Δ_F. We set

$$x_E = -\sum_{S \ni i, S \neq \emptyset, E} x_S = -c_1(\mathcal{L}_{\Delta_E}) \in A^1(X_E). \tag{2.1}$$

The class x_E is usually denoted $-\alpha$ in the literature. Note that

$$x_E + x_\emptyset = -\sum_{\emptyset \subsetneq S \subsetneq E} x_S.$$

Therefore, using Proposition 2.3.5, we have that

$$c_1(\mathcal{L}_{\Delta_S}) = x_\emptyset + \sum_{S \not\subseteq T \subsetneq E} x_T = -\sum_{T \supseteq S} x_T. \tag{2.2}$$

The divisor class $x_S \in A^1(X_E)$ restricts to 0 in $A^1(\mathrm{M})$ if S is not a flat of M. Therefore, restricting this to $A^\bullet(\mathrm{M})$, we have

$$h_F = -\sum_{G \supseteq F} x_G, \tag{2.3}$$

where x_E is the restriction of x_E from $A^1(X_E)$. From this, we see that the restriction of $h_S \in A^1(X_E)$ to $A^1(\mathrm{M})$ is $h_{\mathrm{cl}_{\mathrm{M}}(S)}$.

Because this change of coordinates is upper-triangular, we see that the h_F generate $A^1(\mathrm{M})$ and therefore generate $A^\bullet(\mathrm{M})$ as a ring. We have the following presentation of $A^\bullet(\mathrm{M})$ using the h_F as generators, which we call the *simplicial presentation* of $A^\bullet(\mathrm{M})$.

Proposition 2.4.1. *[LLPP24, Appendix A] Let* M *be a loopless matroid. We have the presentation*

$$A^\bullet(\mathrm{M}) = \frac{\mathbb{Z}[h_F]_{F \text{ non-empty flat}}}{((h_F - h_{F\vee G})(h_G - h_{F\vee G}) : F, G \text{ non-empty flats}) + (h_i : i \in E)}.$$

Here $\vee$ denotes the join in the lattice of flats, i.e., $F \vee G$ is the smaller flat containing both F and G.

2.5 Multiplicity-free subvarieties

In this section, we consider certain subvarieties of products of projective spaces. An (integral) subvariety $X \subseteq \prod_{i=1}^m \mathbb{P}^{a_i}$ is said to be *multiplicity-free* if, when we express

$$[X] \in A^\bullet\left(\prod_{i=1}^m \mathbb{P}^{a_i}\right) = \bigotimes_{i=1}^m \mathbb{Z}[t_i]/(t^{a_i+1})$$

in terms of the basis given by the monomials in the hyperplane classes, the coefficients are all 0 or 1. Equivalently, X is multiplicity-free if the degree of any monomial in the first Chern classes of the $\mathcal{O}(1)$'s from the factors is either 0 or 1.

Let $\pi_j \colon \prod_{i=1}^m \mathbb{P}^{a_i} \to \mathbb{P}^{a_j}$ be the projection. The *multidegree* of X is the function from $\mathbb{Z}^m_{\geq 0}$ to $\mathbb{Z}$ which records the numbers

$$\deg_{\prod_{i=1}^m \mathbb{P}^{a_i}}(c_1(\pi_1^*\mathcal{O}(1))^{k_1} \cdots c_1(\pi_m^*\mathcal{O}(1))^{k_m} \cap [X])$$

for all possible choices of $k_1, \ldots, k_m$. The multidegree is the coefficients used in the expression of $[X]$ in terms of the monomials in the hyperplane classes, so X is multiplicity-free if and only if its multidegree only takes the values 0 and 1. Being an intersection number, the multidegree is locally constant in flat families.

A result of Brion [Bri03] shows that multiplicity-free subvarieties of a product of projective spaces have remarkable properties. To state this, we set up some notation. Let $G = \prod_{i=1}^m PGL_{a_i+1}$, and consider the Borel subgroup of lower triangular matrices. Note that G acts transitively on $\prod_{i=1}^m \mathbb{P}^{a_i}$. A *Schubert variety* in $\prod_{i=1}^m \mathbb{P}^{a_i}$ is a Borel-fixed (integral) subvariety. Concretely, a Schubert variety is a product of linear spaces of the form $\{[* : \cdots : * : 0 : \cdots : 0]\}$. Note that the class of a Schubert variety in $A^\bullet(\prod_{i=1}^m \mathbb{P}^{a_i})$ is a monomial in the hyperplane classes, and there is a unique Schubert variety representing each monomial. Therefore, given any subset B of the degree d monomials in the hyperplane classes, there is a unique reduced union of Schubert varieties Y_B whose multidegree is 1 on B and is 0 otherwise. In other words, we have

$$\deg_{\prod_{i=1}^m \mathbb{P}^{a_i}}(c_1(\pi_1^*\mathcal{O}(1))^{b_1} \cdots c_1(\pi_m^*\mathcal{O}(1))^{b_m} \cap [Y_B]) = \begin{cases} 1 & (b_1, \ldots, b_m) \in B \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.5.1. *[Bri03] Let X be a multiplicity-free subvariety of $\prod_{i=1}^m \mathbb{P}^{a_i}$ with the same multidegree as Y_B. Then X is normal and Cohen–Macaulay, and X has a flat degeneration inside of $\prod_{i=1}^m \mathbb{P}^{a_i}$ to Y_B.*

The degeneration is constructed by considering the Borel-action on the Hilbert scheme of $\prod_{i=1}^m \mathbb{P}^{a_i}$. The Borel-orbit closure of the class of X in the Hilbert scheme is a projective variety with a Borel-action, and so it has a Borel-fixed point by the Borel fixed point theorem. The corresponds to a Borel-fixed subscheme of $\prod_{i=1}^m \mathbb{P}^{a_i}$ with the same multidegree of X. Brion uses properties of multiplicity-free subvarieties to show that this Borel-fixed subscheme is Cohen–Macaulay and so has no embedded points. The only Borel-fixed subscheme with no embedded points with the right multidegree is Y_B. From this, he produces a degeneration from X to Y_B.

The possible multidegrees of multiplicity-free (integral) subvarieties of a product of projective spaces are highly constrained by the following result. The support of the multidegree of a variety is the set of monomials in the hyperplane classes whose intersection with X is non-zero.

Proposition 2.5.2. *[BH20, Corollary 4.7] Let X be a subvariety of $\prod_{i=1}^m \mathbb{P}^{a_i}$. Then the support of the multidegree of X is the set of bases of a polymatroid.*

In particular, any variety Y_B that occurs as a degeneration of a multiplicity-free subvariety must have $B = B(\mathrm{P})$ for some polymatroid P. For a polymatroid P on $[m]$, a *cage* is a sequence $(a_1, \ldots, a_m)$ such that $a_i \geq \mathrm{rk}_{\mathrm{P}}(i)$ for all i. For each cage, we may consider the reduced union of Schubert varieties in $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ whose multidegree is given by $B(\mathrm{P})$. Observe that this scheme, and restrictions of the $\mathcal{O}(1)$ from the factors, is independent of the choice of cage. We define Y_{P} to be this scheme. Note that Y_{P} is defined over $\operatorname{Spec} \mathbb{Z}$, but we will often consider it over a chosen field $\mathbf{k}$. The following result is a useful property of Y_{P}.

Proposition 2.5.3. *Let* P *be a polymatroid. Then the variety* Y_{P} *is Cohen–Macaulay.*

Proof. See [CCRC23, Proof of Theorem 5.6], which was obtained by using properties of "polymatroid ideals" in [HH11, Chapter 12.6]. □

In particular, this implies the Cohen–Macaulayness of multiplicity-free subvarieties, as the locus of Cohen–Macaulay fibers is open in flat families [SP, 045U]. However, Brion's construction of the degeneration requires one to first prove that multiplicity-free subvarieties are Cohen–Macaulay, so this does not give a new proof. The Cohen–Macaulayness of Y_B does not hold for arbitrary B.

Example 2.5.4. Let $B = \{(2,0),(0,2)\}$. Inside of $\mathbb{P}^2 \times \mathbb{P}^2$, Y_B is the surface $\mathbb{P}^2 \times [1,0,0] \cup [1,0,0] \times \mathbb{P}^2$. Then Y_B is not S_2 and therefore not Cohen–Macaulay.

Finally, we will need the following result, which gives a formula for $[\mathcal{O}_{Y_{\mathrm{P}}}] \in K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m})$. This formula originates in the work of Knutson, who studied the more general problem of calculating the K-class of a reduced union of Schubert varieties inside a homogeneous space. He showed that one can compute the K-class in terms of Möbius inversion on the poset of Schubert varieties. The special case of products of projective spaces was also proven in [CCRMMn, Theorem 7.12]. For each tuple $\mathbf{b} = (b_1, \ldots, b_m)$ with $b_i \leq a_i$, let $Y_{\mathbf{b}}$ be a $\mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_m}$ embedded linearly into $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$; the class $[\mathcal{O}_{Y_{\mathbf{b}}}]$ does not depend on the choice of an embedding. The classes $\{[\mathcal{O}_{Y_{\mathbf{b}}}]\}$ form a basis for $K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m})$.

Proposition 2.5.5. *[Knu] Let* P *be a polymatroid with cage* $(a_1, \ldots, a_m)$. *Write* $[\mathcal{O}_{Y_{\mathrm{P}}}] = \sum_{\boldsymbol{b}} c_{\boldsymbol{b}}[\mathcal{O}_{Y_{\boldsymbol{b}}}] \in K(\prod_{i=1}^m \mathbb{P}^{a_i})$. *If* $\sum_{i=1}^m b_i > \mathrm{rk}(\mathrm{P})$, *then* $c_{\boldsymbol{b}} = 0$. *If* $\sum_{i=1}^m b_i = \mathrm{rk}(\mathrm{P})$, *then*

$$c_{\boldsymbol{b}} = \begin{cases} 1 & \text{if } \boldsymbol{b} \in B(\mathrm{P}) \\ 0 & \text{otherwise.} \end{cases}$$

If $\sum_{i=1}^m b_i < \mathrm{rk}(\mathrm{P})$, *then* $c_{\boldsymbol{b}} = 1 - \sum_{\boldsymbol{b}' > \boldsymbol{b}} c_{\boldsymbol{b}'}$.

Note that the function $\mathbf{b} \mapsto c_{\mathbf{b}}$ is zero unless $\mathbf{b}$ is an independent set of P.

Because of Proposition 2.5.1, Proposition 2.5.5 also computes the K-class of the structure sheaf of any multiplicity-free subvariety in a product of projective spaces. Indeed, the class in $K(\prod_{i=1}^m \mathbb{P}^{a_i})$ of the structure sheaf of a subvariety X of $\prod_{i=1}^m \mathbb{P}^{a_i}$ is determined by $\chi(X, \mathcal{O}(k_1, \ldots, k_m))$ for all $(k_1, \ldots, k_m)$. As Euler characteristics are locally constant in proper flat families, the class of the structure sheaf of a fiber in $K(\prod_{i=1}^m \mathbb{P}^{a_i})$ is locally constant for a flat family of closed subschemes of $\prod_{i=1}^m \mathbb{P}^{a_i}$.

2.6 Examples of multiplicity-free subvarieties

In this section, we record several examples of multiplicity-free subvarieties of products of projective spaces. The most important examples will be certain varieties arising from subspace arrangements which generalize wonderful varieties.

Proposition 2.6.1. *Let* $L_1, \ldots, L_m$ *be a subspace arrangement in* L, *and assume that* $L_i \neq L$ *for all* i. *Then the closure of the image of*

$$\mathbb{P}L \dashrightarrow \prod_{i=1}^m \mathbb{P}(L/L_i)$$

is a multiplicity-free subvariety.

We allow the L_i to contain each other. When $\{L_i\} = \{L_F : F \text{ non-empty flat}\}$ are the non-empty flats of a hyperplane arrangement, this implies that the wonderful variety is a multiplicity-free subvariety.

Proof of Proposition 2.6.1. By replacing L by $L/\cap L_i$, we may assume that $\cap L_i = 0$. Let X denote the closure of the image, and let U be the image of $\mathbb{P}L \setminus \cup_i \mathbb{P}L_i$. Note that U is open and dense in X.

Let $\mathcal{L}_i$ be the pullback of $\mathcal{O}(1)$ from $\mathbb{P}(L/L_i)$. The pullback of $H^0(\mathbb{P}(L/L_i), \mathcal{O}(1))$ is a linear system which globally generates $H^0(X, \mathcal{L}_i)$. We need to show that

$$\int_X c_1(\mathcal{L}_1)^{k_1} \cdots c_1(\mathcal{L}_n)^{k_m} \in \{0,1\}$$

for each $k_1, \ldots, k_m$ with $k_1 + \cdots + k_m = \dim X$. By extending scalars, we may assume that $\mathbf{k}$ is infinite. Choose k_i generic elements of the linear system $H^0(\mathbb{P}(L/L_i), \mathcal{O}(1))$ for each i. Let their respective vanishing loci be $V_1, \ldots, V_{\dim X}$. By Bertini's theorem, we can assume that $V_1 \cap \cdots \cap V_{\dim X} \subset U$ and that the intersection is 0-dimensional. Note that, for each j, $V_j \cap U \subset \mathbb{P}L$ is a hyperplane. These hyperplanes either intersect transversely in 0 or 1 points in U. $\square$

The polymatroid corresponding to the subvariety in Proposition 2.6.1 is somewhat complicated to describe; a special case (which can be used to understand the general one, see [EL24]) is given in Proposition 2.7.4. One case, however, is simple.

Proposition 2.6.2. *Let $L_1, \ldots, L_m \subseteq L$ be a subspace arrangement, and consider the subspace arrangement $\tilde{L}_i = L_i \oplus 0 \subseteq L \oplus \mathbf{k}$. Then the multidegree of the closure of the image of*

$$\mathbb{P}(L \oplus \mathbf{k}) \dashrightarrow \prod_{i=1}^m \mathbb{P}((L \oplus \mathbf{k})/\tilde{L}_i) = \prod_{i=1}^m \mathbb{P}(L/L_i \oplus \mathbf{k})$$

is the polymatroid corresponding to the subspace arrangement.

Proof. We may replace L by $L/\cap_i L_i$. Let X denote the closure of the image, and let P be the polymatroid corresponding to the multidegree of X. Note X contains a dense open subset which is identified with L, the image of $\{(v,1) : v \in L\} \subseteq \mathbb{P}(L \oplus \mathbf{k})$. Similarly, L/L_i is naturally identified with a dense open subset of $\mathbb{P}(L/L_i \oplus \mathbf{k})$. Choose a generic basis for $x_{i,1}, \ldots, x_{i,\dim L/L_i}$ for each each $(L/L_i)^*$. Then $\mathbf{b} = (b_1, \ldots, b_m)$ is a basis for P if and only if the rational map

$$X \hookrightarrow \prod_{i=1}^{m} \mathbb{P}((L \oplus \mathbf{k})/\tilde{L}_i) = \prod_{i=1}^{m} \mathbb{P}(L/L_i \oplus \mathbf{k}) \dashrightarrow \prod_{i=1}^{m} \mathbb{A}^{b_i}$$

is dominant, where the right rational map is induced by the map $L/L_i \to \mathbf{k}^{b_i}$ given by the functions $x_{i,1}, \ldots, x_{i,b_i}$ for each i. If this map is dominant, then the fact that X is multiplicity-free implies that its degree is 1, and so the map is birational and in particular is generically étale. Note that this map is generically étale if and only if the pullback map on differentials at the generic point is an isomorphism.

Consider the module of differentials $\Omega_{K(X)/\mathbf{k}}$ of the function field $K(X)$. Because L is a dense open subset of X, this is identified with $\Omega_{K(L)/\mathbf{k}}$, which can be canonically identified with $L^* \otimes_{\mathbf{k}} K(L)$, via the map which sends a linear function ℓ to $d\ell \otimes 1$. Each $x_{i,j}$ defines a function on L, so we may consider its differential $dx_{i,j} \in \Omega_{K(X)/\mathbf{k}} = L^* \otimes_{\mathbf{k}} K(L)$. For each i, the subspace $\{v \in L \otimes_{\mathbf{k}} K(L) : dx_{ij}(v) = 0 \text{ for all } j\}$ is $L_i \otimes_{\mathbf{k}} K(L)$. The description of the polymatroid associated to a subspace arrangement in Section 2.3 then implies the result. $\square$

Example 2.6.3. Let $X \subseteq \mathbf{k}^{mn}$ be the locus of $m \times n$ matrices of rank at most r. If $r = 1$ or $r = m - 1$, then the closure of X in $(\mathbb{P}^1)^{mn}$ is a multiplicity-free subvariety.

Example 2.6.4. The closure of the locus of 4×4 matrices of rank at most 2 in $(\mathbb{P}^1)^{16}$ is not multiplicity-free. Indeed, if one fills in the off-diagonal entries of a 4×4 matrix with generic elements of $\mathbf{k}$, there are exactly two ways to fill in the diagonal so that the resulting matrix has rank at most 2.

Example 2.6.5. Let $X = \overline{M}_{0,n}$ be the Deligne–Mumford–Knudsen moduli space of genus 0 stable curves with n marked points. For each $S \subseteq \{1, \ldots, n\}$ of size at least 3, there is a forgetful map $f_S \colon \overline{M}_{0,n} \to \overline{M}_{0,S}$, where $\overline{M}_{0,S}$ is the moduli space with points marked by S. For $i \in \{1, \ldots, n\}$, let $\mathbb{L}_i$ be the line bundle whose fiber at a point of $\overline{M}_{0,n}$ is the ith cotangent line of the corresponding curve. Then each $\mathbb{L}_i$ is base-point-free, and its complete linear system induces a birational map $\overline{M}_{0,n} \to \mathbb{P}^{n-3}$ [Kap93]. For each subset S of $\{1, \ldots, n\}$ which contains n, we have a map $\overline{M}_{0,n} \to \mathbb{P}^{|S|-3}$ by composing f_S with the map $\overline{M}_{0,S} \to \mathbb{P}^{|S|-3}$ given by $\mathbb{L}_n$. We therefore obtain a map

$$\overline{M}_{0,n} \to \prod_{S \subseteq \{1,\dots,n\},\, n \in S,\, |S| \geq 3} \mathbb{P}^{|S|-3}.$$

This map is a closed embedding, and its image is a multiplicity-free subvariety. See [DCP95, BELL].

We now show that, for a given characteristic $p \geq 0$, there is a multiplicity-free subvariety $X \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ whose multidegree is the bases of a polymatroid P if and only if P is realizable over a field of characteristic p. By Proposition 2.6.2, this implies that there is a wonderful variety $W_L \subseteq \prod_{F \neq \emptyset} \mathbb{P}L^F$ whose projection to the some of the factors has the same multidegree.

Proposition 2.6.6. *Let $X \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ be a multiplicity-free subvariety whose multidegree is given by a polymatroid* P*. Then there is a realization of* P *over the field* $K(X)$*.*

Proof. For each i, we have a subspace of $\Omega_{K(X)/\mathbf{k}}$ obtained by pulling back the differentials of rational functions on the ith factor. The sum of these subspaces is $\Omega_{K(X)/\mathbf{k}}$, so, dualizing, we obtain an embedding $\Omega^*_{K(X)/\mathbf{k}} \subseteq \bigoplus_i V_i$, where V_i is a $K(X)$ vector space of dimension a_i. As in the proof of Proposition 2.6.2, the bases of the polymatroid that this subspace arrangement represents record whether the restriction of the rational map $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m} \dashrightarrow \mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_m}$, given by projecting away from a generic linear space in each factor, to X is generically étale. Because X is multiplicity-free, such a projection is birational if and only if it is dominant, and so asking whether a projection is generically étale is the same as asking whether it is dominant. □

In particular, there is a subvariety of $(\mathbb{P}^1)^n$ whose multidegree is given by the bases of a matroid M over a field of characteristic $p \geq 0$ if and only if M is realizable over a field of that characteristic.

2.7 Multidegree of wonderful varieties

In this section, we state a formula for the multidegree of the defining embedding of a wonderful variety into a product of projective spaces and its generalization to arbitrary matroids. This formula was proven in [BES]. See [Lar] for an elementary proof. With the exception of Section 3.5, we will only need that the support of the multidegree is the bases of a polymatroid. When the matroid is realizable, this is an immediate consequence of Proposition 2.5.2. In general, it can be deduced from [AHK18, Theorem 8.9]. First we make a combinatorial definition.

Definition 2.7.1. We say that a sequence $(S_1, \ldots, S_m)$ of non-empty subsets of E satisfies the *dragon-Hall–Rado* condition (with respect to M) if

$$\operatorname{rk}_{\mathrm{M}}\left(\bigcup_{i \in I} S_i\right) \geq 1 + |I| \quad \text{for every } \emptyset \neq I \subseteq [m].$$

Moreover, we say that $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{Z}^m_{\geq 0}$ satisfies the *dragon-Hall–Rado* condition if the sequence $(S_1^{t_1}, \ldots, S_m^{t_m})$, where $S_i^{t_i}$ denotes S_i repeated t_i times, satisfies the dragon-Hall–Rado condition, or, equivalently if

$$\operatorname{rk}_{\mathrm{M}}\left(\bigcup_{i \in I} S_i\right) \geq 1 + \sum_{i \in I} t_i \quad \text{for every } \emptyset \neq I \subseteq [m].$$

We will later need the following combinatorial result, which implies that the multisets of maximal cardinality which satisfy the dragon-Hall–Rado condition are the bases of a polymatroid.

Proposition 2.7.2. *The vectors $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{Z}^m_{\geq 0}$ such that $(S_1^{t_1}, \ldots, S_m^{t_m})$ satisfies the dragon-Hall–Rado condition form the independent sets of a polymatroid on $[m]$.*

To prove Proposition 2.7.2, we will need the following result of Edmonds.

Proposition 2.7.3. *[Edm70, (8)] Let $f \colon 2^E \to \mathbb{Z}$ be a submodular and increasing* *function. Then the multisets I whose elements are contained in E which satisfy*

$$|I'| \leq f(I') \textit{ for all } \emptyset \neq I' \subseteq I$$

are the independent sets of a polymatroid on E.

Here we are extending f to a function on multisets whose elements all belong to E by ignoring repetitions, i.e., the value of f on a multiset S is the value of f on subset of E which consists of elements in S.

Proof of Proposition 2.7.2. Let $f\colon 2^{[m]} \to \mathbb{Z}$ be the function

$$f(S) = \mathrm{rk}_{\mathrm{M}}\left(\bigcup_{i \in S} S_i\right) - 1.$$

Clearly f is increasing; it is submodular because the rank function of M is. Applying Proposition 2.7.3 gives a polymatroid whose independent sets are the multisets which satisfy the dragon-Hall–Rado condition. □

The relevance of the dragon-Hall–Rado condition for us comes from the following result.

Proposition 2.7.4. *[BES, Theorem 5.2.4] Let* M *be a loopless matroid of rank r, and let $F_1, \ldots, F_m$ be flats of* M*. If $\mathbf{t} \in \mathbb{Z}^m_{\geq 0}$ has $\sum t_i = r - 1$, then*

$$\deg_{\mathrm{M}}(h_{F_1}^{t_1} \cdots h_{F_m}^{t_m}) = \begin{cases} 1 & \mathbf{t} \textit{ satisfies dragon-Hall–Rado} \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, this describes the multidegree of the defining embedding of any wonderful variety and gives a strengthening of Proposition 2.6.1.

Chapter 3

K-theory of wonderful varieties

In this chapter, we study the K-theory of wonderful varieties and Bergman fans. We will define the K-ring of a matroid, and we will show that it has the structures that one expects of the K-ring of a smooth projective variety. All the varieties that we consider are smooth, so the map from the Grothendieck ring of vector bundles to the Grothendieck group of coherent sheaves is an isomorphism [SP, 0FDJ, 0F8A]. We may therefore unambiguously talk about the K-ring of a variety.

3.1 K-rings of matroids

We begin by analyzing the K-ring of wonderful varieties and toric varieties of Bergman fans.

Proposition 3.1.1. *Let* M *be a loopless matroid realized by* $L \subseteq \mathbf{k}^E$. *Then the restriction map* $K(X_{\Sigma_{\mathrm{M}}}) \to K(W_L)$ *is an isomorphism.*

Proof. By Proposition 2.2.3, $A^\bullet(X_{\Sigma_{\mathrm{M}}})$ is torsion-free, so it is isomorphic to the associated graded of $K(X_{\Sigma_{\mathrm{M}}})$ with respect to the coniveau filtration [Ful93, Example 15.2.16]. Since the associated graded map $A^\bullet(\mathrm{M}) \to A^\bullet(W_L)$ is an isomorphism by Proposition 2.2.2, the filtered map $K(X_{\Sigma_{\mathrm{M}}}) \to K(W_L)$ is also an isomorphism [Wei94, Theorem 5.2.12]. □

Motivated by this, if M is loopless, then we define the K-*ring* of M to be $K(X_{\Sigma_{\mathrm{M}}})$. We now compute a presentation of K-ring of a matroid, which we call the *toric presentation* of $K(\mathrm{M})$.

Theorem 3.1.2. *Let* M *be a loopless matroid on E. Then the K-ring of* M *has the presentation*

$$K(\mathrm{M}) = \frac{\mathbb{Z}[\xi_F]_{F \text{ proper non-empty flat}}}{(\prod_{F \ni i}(1-\xi_F) - \prod_{G \ni j}(1-\xi_G) : i, j \in E) + (\xi_{F_1}\xi_{F_2} : F_1, F_2 \text{ incomparable})}.$$

Theorem 3.1.2 should be compared with Proposition 2.2.4. The additive relation in the presentation of $A^\bullet(\mathrm{M})$ is replaced by a multiplicative relation whose lowest order term is the additive relation. As the proof will show, ξ_F is the class of the structure sheaf of the torus-fixed divisor corresponding to F on $X_{\Sigma_{\mathrm{M}}}$.

Computations of a presentation of the K-ring of a smooth projective toric variety have appeared in [SU03, San08]. We use a similar strategy: we identify some geometrically obvious relations, and then prove that they generate all relations by using the connection between $K(X_{\Sigma_{\mathrm{M}}})$ and $A^\bullet(X_{\Sigma_{\mathrm{M}}})$.

We now prepare for the proof of Theorem 3.1.2. Recalling that $A^\bullet(\mathrm{M}) = A^\bullet(X_{\Sigma_{\mathrm{M}}})$ is a free abelian group, the following lemma will be useful.

Lemma 3.1.3. *Let X be a smooth variety. If $A^\bullet(X)$ is a free abelian group of finite rank p, then so is $K(X)$, and the Chern character* $\mathrm{ch}\colon K(X) \to A^\bullet(X) \otimes \mathbb{Q}$ *is injective.*

Proof. The Chern character becomes an isomorphism after tensoring with $\mathbb{Q}$ [Ful98, Example 15.2.16], so $K(X)$ has rank p. There is a surjective map from $A^\bullet(X)$ to the associated graded of $K(X)$ with respect to the coniveau filtration [Ful98, Example 15.1.5]. Since $A^\bullet(X)$ is free of rank p and $K(X)$ also has rank p, this implies that that $K(X)$ is free. Finally, the Chern character factors as $K(X) \to K(X) \otimes \mathbb{Q} \to A^\bullet(X) \otimes \mathbb{Q}$, with the first map being injective by freeness of $K(X)$ and the second map being an isomorphism, so the Chern character is injective. □

Lemma 3.1.4. *Let X be a smooth variety, and suppose that $A^\bullet(X)$ as a ring by divisor classes. Let $D_1, \ldots, D_k$ be divisors on X. If $A^1(X)$ is generated as an abelian group by $[D_1], \ldots, [D_k]$, then $K(X)$ is generated as a ring by $[\mathcal{O}_{D_1}], \ldots, [\mathcal{O}_{D_k}]$.*

Proof. First we claim that $K(X)$ is generated as a ring by classes of line bundles. Let K_{line} be the subring of $K(X)$ generated by classes of line bundles. Note that K_{line} is equipped with a filtration, obtained by intersecting the coniveau filtration on $K(X)$ with K_{line}. Then the image of $\operatorname{gr} K_{\text{line}}$ in $\operatorname{gr} K(X)$ is the subring generated by classes of divisors. As there is a surjective ring homomorphism $A^\bullet(X) \to \operatorname{gr} K(X)$ and $A^\bullet(X)$ is generated by classes of divisors, we see that $\operatorname{gr} K_{\text{line}} = \operatorname{gr} K(X)$. Therefore $K_{\text{line}} = K(X)$.

Let R be the subring of $K(X)$ generated by $[\mathcal{O}_{D_1}], \ldots, [\mathcal{O}_{D_k}]$. We need to show that the class of every line bundle is contained in R. Since $[D_1], \ldots, [D_k]$ generate $A^1(X)$ as an abelian group, the line bundles $\mathcal{O}(\pm D_1), \ldots, \mathcal{O}(\pm D_k)$ generate the Picard group of X under multiplication, so it suffices to show that $[\mathcal{O}(\pm D_i)] \in R$ for all i.

For any divisor D, we have an exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0,$$

which implies that

$$[\mathcal{O}(-D)] = [\mathcal{O}] - [\mathcal{O}_D] = 1 - [\mathcal{O}_D].$$

We also have

$$[\mathcal{O}(D)] = [\mathcal{O}(-D)]^{-1} = \frac{1}{1 - [\mathcal{O}_D]} = 1 + [\mathcal{O}_D] + [\mathcal{O}_D]^2 + \cdots .$$

Since $[\mathcal{O}_D]$ lives in the first piece of the coniveau filtration on $K(X)$, it is nilpotent, so the sum terminates. This allows us to conclude that both $[\mathcal{O}(-D_i)]$ and $[\mathcal{O}(D_i)]$ live in the ring R. □

Proof of Theorem 3.1.2. By Lemma 3.1.4, Lemma 3.1.3, and Proposition 2.2.4, the map from $\mathbb{Z}[\xi_F]_{F \text{ proper non-empty flat}}$ to $K(\mathrm{M})$ sending ξ_F to the class of the structure sheaf of the torus-fixed divisor corresponding to a proper non-empty flat F is surjective.

Next, we show that the relations described in Theorem 3.1.2 are indeed satisfied in $K(\mathrm{M})$. For F_1, F_2 incomparable flats, the corresponding divisors are disjoint, so the product of their structure sheaves is 0 in $K(\mathrm{M})$. To prove that

$$\prod_{F \ni i}(1-\xi_F) - \prod_{G \ni j}(1-\xi_G) \tag{3.1}$$

for all $i, j \in E$, we apply the Chern character ch, which is injective by Lemma 3.1.3. If D_F is the divisor on X_{Σ_M} corresponding to F, then we have that $\xi_F = 1 - [\mathcal{O}(-D_F)]$. We see that

$$\mathrm{ch}(\xi_F) = 1 - \mathrm{ch}([\mathcal{O}(-D_F)]) = 1 - \exp(-x_F) = x_F - x_F^2/2! + x_F^3/3! - \cdots .$$

To prove that (3.1) holds is then equivalent to the statement that

$$\exp\left(-\sum_{F \ni i} x_F\right) = \exp\left(-\sum_{G \ni j} x_G\right).$$

This follows from Proposition 2.2.4.

Let R be the quotient $\mathbb{Z}[\xi_F]_{F \text{ proper non-empty flat}}$ by the ideals in Theorem 3.1.2. We have shown that R surjects on $K(\mathrm{M})$, and we need to prove that the map is injective. Let p be the rank of the free abelian group $A^\bullet(\mathrm{M})$. Consider the decreasing filtration

$$R = F_0 \supset F_1 \supset \cdots ,$$

where F_i is the span of all monomials of total degree $\geq i$. Since the leading terms of the generators of the relations in R are the relations in $A^\bullet(\mathrm{M})$, we have a surjection $A^\bullet(\mathrm{M}) \to \mathrm{gr}\, R$. In particular, this implies that the abelian group $\mathrm{gr}\, R$ can be generated by p elements, and so the same is true of R. Lemma 3.1.3 tells us that $K(\mathrm{M})$ is also free abelian of rank p, so any surjection from R to $K(\mathrm{M})$ must be an isomorphism. □

3.2 Simplicial generators of $K(\mathrm{M})$

In this section, we construct an isomorphism $\zeta_\mathrm{M} \colon K(\mathrm{M}) \to A^\bullet(\mathrm{M})$, which we call the *exceptional isomorphism*. Its exceptional nature is that it unrelated to the Chern character $\mathrm{ch}\colon K(\mathrm{M}) \otimes \mathbb{Q} \to A^\bullet(\mathrm{M}) \otimes \mathbb{Q}$. The existence of such an isomorphism was first observed in [BEST23] in the case of the permutohedral toric variety X_E, where it was constructed using equivariant techniques. We will later use ζ_M to define the Euler characteristic map $\chi(\mathrm{M}, -)\colon K(\mathrm{M}) \to \mathbb{Z}$.

For each flat F of M, we have defined a line bundle $\mathcal{L}_{\Delta_F}$ on X_{Σ_M}. Define $\eta_F = 1 - [\mathcal{L}_{\Delta_F}^{-1}] \in K(\mathrm{M})$. Note that $\mathcal{L}_{\Delta_F}$ is globally generated, so η_F represents the structure sheaf of the vanishing locus of a generic section of $\mathcal{L}_{\Delta_F}$. These classes will play the role of "K-theoretic simplicial generators."

As $A^\bullet(\mathrm{M})$ is spanned by $\{h_F\}_{F \text{ non-empty flat}}$, Lemma 3.1.4 implies that $K(\mathrm{M})$ is generated by the η_F. We will prove the following theorem.

Theorem 3.2.1. *Let* M *be a loopless matroid. Then there is a ring isomorphism* $\zeta_\mathrm{M}\colon K(\mathrm{M}) \to A^\bullet(\mathrm{M})$ *defined by* $\zeta_\mathrm{M}(\eta_F) = h_F$.

In particular, this gives another presentation of $K(\mathrm{M})$: the relations are the same in Proposition 2.4.1.

To prepare the proof of Theorem 3.2.1, we first obtain a description of the classes η_F in terms of the toric generators of $K(\mathrm{M})$ so that we can use Theorem 3.1.2. We set

$$\xi_E := 1 - \prod_{F \ni i} (1 - \xi_F)^{-1}$$

for any $i \in E$. This is well-defined by Theorem 3.1.2 and the observation that each ξ_F is nilpotent (which can be seen, for example, by applying the Chern character).

Proposition 3.2.2. *Let F be a non-empty flat of* M. *Then we have*

$$\eta_F = 1 - \prod_{G \supseteq F} (1 - \xi_G)^{-1} = 1 - \prod_{G \supseteq F} (1 + \xi_G + \xi_G^2 + \cdots).$$

Proof. First we do the case $F = E$, i.e., we show that

$$\eta_E := 1 - [\mathcal{L}_{\Delta_E}^{-1}] = 1 - (1 - \xi_E)^{-1}.$$

Using the definition of ξ_E and ξ_F, we get

$$(1-\xi_E)^{-1} = \prod_{E \neq F \ni i} (1-\xi_F) = [\bigotimes_{E \neq F \ni i} \mathcal{O}(-D_F)],$$

where D_F is the divisor on $X_{\Sigma_{\mathrm{M}}}$ corresponding to F. Then the claims follows from (2.1).

The computation for general F is similar: we have $1-\eta_F = [\mathcal{L}_{\Delta_F}^{-1}]$. By (2.3), this is equal to $[\mathcal{L}_{\Delta_E}^{-1} \otimes \bigotimes_{F \subseteq G \subsetneq E} \mathcal{O}(D_G)]$, and then we use that $\mathcal{L}_{\Delta_E}^{-1} = (1-\xi_E)^{-1}$ by the special case above and $[\mathcal{O}(D_G)] = (1-\xi_G)^{-1}$ when G is a proper non-empty flat. □

Proof of Theorem 3.2.1. We define a map $\kappa_{\mathrm{M}} \colon \mathbb{Z}[h_F]_{F \text{ non-empty flat}} \to K(\mathrm{M})$ by setting $\kappa_{\mathrm{M}}(h_F) = \eta_F$. We will show that κ_{M} descends to an isomorphism $A^\bullet(\mathrm{M}) \to K(\mathrm{M})$, and ζ_{M} will be the inverse of this map.

We need to check that κ_{M} vanishes on the generators of the ideal defining $A^\bullet(\mathrm{M})$. Recall that $\xi_F \xi_G = 0$ if F and G are incomparable. We first check that κ_{M} vanishes on an element of the form $(h_F - h_{F \vee G})(h_G - h_{F \vee G})$. We have

$$\begin{aligned} & \kappa_M\Big((h_F - h_{F\vee G})(h_G - h_{F\vee G})\Big) \\ = & \left(\prod_{H \supseteq F} (1-\xi_H)^{-1} - \prod_{I \supseteq F \vee G} (1-\xi_I)^{-1}\right)\left(\prod_{J \supseteq G} (1-\xi_J)^{-1} - \prod_{I \supseteq F \vee G} (1-\xi_I)^{-1}\right) \\ = & \prod_{I \supseteq F \vee G} (1-\xi_I)^{-2} \left(\prod_{F \subseteq H \subsetneq F \vee G} (1-\xi_H)^{-1} - 1\right)\left(\prod_{G \subseteq J \subsetneq F \vee G} (1-\xi_J)^{-1} - 1\right), \end{aligned}$$

which vanishes because H and J are incomparable for any H appearing in the second product and J appearing in the third product. We next check that $\kappa_{\mathrm{M}}(h_i) = 0$ for each $i \in E$, i.e., we need to check that $\eta_i = 0$ in $K(\mathrm{M})$. This holds because $\mathcal{L}_{\Delta_i}$ is the trivial line bundle on $X_{\Sigma_{\mathrm{M}}}$. Indeed, $\mathcal{L}_{\Delta_i}$ is trivial on X_E by Proposition 2.3.5. We therefore obtain a map $A^\bullet(\mathrm{M}) \to K(\mathrm{M})$, which is surjective by Lemma 3.1.4. By Lemma 3.1.3, $A^\bullet(\mathrm{M})$ and $K(\mathrm{M})$ are free abelian groups of the same finite rank, so this map is an isomorphism. We define ζ_{M} to be the inverse of this map. □

If M is a loopless matroid on E, then Σ_{M} is a subfan of $\Sigma_{U_{n,n}} = \Sigma_E$, so there is an inclusion $i\colon X_{\Sigma_{\mathrm{M}}} \hookrightarrow X_E$. Therefore, there are pullback maps $i^*\colon K(X_E) \to K(X_{\Sigma_{\mathrm{M}}}) = K(\mathrm{M})$ and $i^*\colon A^\bullet(X_E) \to A^\bullet(X_{\Sigma_{\mathrm{M}}}) = A^\bullet(\mathrm{M})$. As described in Section 2.4, we have $i^*(h_S) = h_{\mathrm{cl}_{\mathrm{M}}(S)}$, and similarly $i^*\eta_S = \eta_{\mathrm{cl}_{\mathrm{M}}(S)}$. The formula $\zeta_{\mathrm{M}}(\eta_F) = h_F$ gives the following compatibility between ζ_{M} and $\zeta_E := \zeta_{U_{n,n}}\colon K(X_E) \to A^\bullet(X_E)$.

Proposition 3.2.3. *Let* M *be a loopless matroid on* $E = \{1, \ldots, n\}$. *Then the following diagram commutes:*

$$\begin{tikzcd} K(X_E) \arrow[r, "\zeta_E"] \arrow[d, "i^*"] & A^\bullet(X_E) \arrow[d, "i^*"] \\ K(\mathrm{M}) \arrow[r, "\zeta_{\mathrm{M}}"] & A^\bullet(\mathrm{M}). \end{tikzcd}$$

3.3 Euler characteristics on multiplicity-free subvarieties

We wish to show that $K(\mathrm{M})$ behaves like the K-ring of a smooth projective variety. Many properties of $K(\mathrm{M})$ can be easily deduced from the description of $K(\mathrm{M})$ as the Grothendieck ring of vector bundles on the (non-proper) toric variety of the Bergman fan (see Section 3.4). The K-ring of a smooth projective variety X has an *Euler characteristic map* $\chi\colon K(X) \to \mathbb{Z}$, given by pushing forward to a point. The existence of this map requires X to be proper, and so we cannot easily deduce its existence using $X_{\Sigma_{\mathrm{M}}}$. It is this map that is required to formulate the K-theoretic positivity results mentioned previously.

When M is realized by $L \subseteq \mathbf{k}^E$, we will compute the Euler characteristic map on $K(W_L)$. Our strategy is based on the realization of W_L as a multiplicity-free subvariety of a product of projective spaces and Knutson's formula Proposition 2.5.5. Our approach works for a large class of multiplicity-free subvarieties.

We begin with the observation that $A^\bullet(\mathbb{P}^a) \cong \mathbb{Z}[t]/(t^{a+1})$, where $t = c_1(\mathcal{O}(1))$ is the class of a hyperplane. Also, $K(\mathbb{P}^a) \cong \mathbb{Z}[t]/(t^{a+1})$, where $t = [\mathcal{O}_H]$ is the class of the structure sheaf of a hyperplane. Therefore there is an isomorphism $\zeta\colon K(\mathbb{P}^a) \to$

$A^\bullet(\mathbb{P}^a)$, given by sending $[\mathcal{O}_H]$ to $c_1(\mathcal{O}(1))$. We also have isomorphisms

$$A^\bullet\left(\prod_{i=0}^m \mathbb{P}^{a_i}\right) = \bigotimes_{i=0}^m A^\bullet(\mathbb{P}^{a_i}) \text{ and } K\left(\prod_{i=0}^m \mathbb{P}^{a_i}\right) = \bigotimes_{i=0}^m K(\mathbb{P}^{a_i}).$$

In particular, there is an isomorphism $\otimes\zeta\colon K(\prod_{i=0}^m \mathbb{P}^{a_i}) \to A^\bullet(\prod_{i=0}^m \mathbb{P}^{a_i})$.

For each $j \in \{0, 1, \ldots, m\}$, let $\pi_j\colon \prod_{i=0}^m \mathbb{P}^{a_i} \to \mathbb{P}^{a_j}$ be the projection. We will study the classes $[\mathcal{O}_{Y_{\mathrm{P}}}] \in K(\prod_{i=0}^m \mathbb{P}^{a_i})$ for certain polymatroids P. By Proposition 2.5.1, this is the same as the K-class of any multiplicity-free subvariety whose multidegree is given by P.

Theorem 3.3.1. *Let* P *be a polymatroid with cage* $(a_0, \ldots, a_m)$ *such that* $\mathrm{rk}_{\mathrm{P}}(0) = \mathrm{rk}(\mathrm{P})$. *Then*

$$\otimes\zeta([\mathcal{O}_{Y_{\mathrm{P}}}]) = [Y_{\mathrm{P}}] \cdot \prod_{i=1}^m (1 - c_1(\pi_i^*\mathcal{O}(1))).$$

Proof. Let $r = \mathrm{rk}(\mathrm{P})$. For $i \in \{0, 1, \ldots, m\}$, let $x_i = c_1(\pi_i^*\mathcal{O}(1)) \in A^\bullet(\prod_{i=0}^m \mathbb{P}^{a_i})$. Then we have

$$[Y_{\mathrm{P}}] \in A^\bullet\left(\prod_{i=0}^m \mathbb{P}^{a_i}\right) = \sum_{\mathbf{b} \in B(\mathrm{P})} x_0^{a_0 - b_0} x_1^{a_1 - b_1} \cdots x_m^{a_m - b_m},$$

where $\mathbf{b} = (b_0, \ldots, b_m)$. Then the assumption that $\mathrm{rk}_{\mathrm{P}}(0) = \mathrm{rk}(\mathrm{P})$ means that $(r, 0, \ldots, 0) \in B(\mathrm{P})$, so $x_1^{a_1} \cdots x_k^{a_m}$ appears as a monomial in the above expression.

For a vector $p = (p_0, \ldots, p_m) \in \mathbb{Z}_{\geq 0}^{m+1}$, set $d(p) = (a_0 - p_0, \ldots, a_m - p_m)$, so the monomial $x_0^{p_0} \cdots x_m^{p_m}$ is equal to $[Y_{d(p)}]$ in $A^\bullet(\prod_{i=0}^m \mathbb{P}^{a_i})$ if $p_i \leq a_i$ for each i. Then we define $c_{\mathbf{b}}$ via the formula

$$[X] \cdot \prod_{i=1}^m (1 - x_i) = \sum_{\mathbf{b}} (-1)^{r - |\mathbf{b}|} c_{\mathbf{b}} [Y_{\mathbf{b}}].$$

Observe that $c_{\mathbf{b}}$ is the number of subsets S of $\{1, \ldots, m\}$ such that $\mathbf{b} + \mathbf{1}_S \in B(\mathrm{P})$, where $\mathbf{1}_S$ is the indicator vector of S. In particular, $c_{\mathbf{b}}$ is zero unless $\mathbf{b}$ is an independent set of P.

Recall that Proposition 2.5.5 gives a recursive formula for $[\mathcal{O}_{Y_{\mathrm{P}}}]$. In order to prove the theorem, we need to show that, for each $\mathbf{b}'$, we have

$$\sum_{\mathbf{b}\geq\mathbf{b}'}(-1)^{r-|\mathbf{b}|}c_{\mathbf{b}} = \sum_{\mathbf{b}\geq\mathbf{b}'}(-1)^{r-|\mathbf{b}|}\sum_{S\subset\{1,\dots m\},\,\mathbf{b}+\mathbf{1}_S\in B(\mathrm{P})} 1 = 1.$$

Each term in the above sum, which is labeled by a pair $(\mathbf{b}, S)$, is associated to a basis $\mathbf{b}+\mathbf{1}_S = \mathbf{d} \in B(\mathrm{P})$. Group the terms by their associated basis $\mathbf{d} = (d_0, \dots, d_m)$. Let $T = \{i \in \{1, \dots, m\} : d_i \neq 0\}$. The sum of the terms associated to $\mathbf{d}$ is $\sum_{S\subseteq T}(-1)^{r-|T|}$. If T is non-empty, this sum vanishes. If $T = \emptyset$, which occurs exactly for the basis $\mathbf{d} = (r, 0, \dots, 0)$, this sum is 1. $\square$

Let P be a polymatroid with cage $(a_0, \dots, a_m)$ such that $\mathrm{rk}_\mathrm{P}(0) = \mathrm{rk}(\mathrm{P})$. We will relate the Euler characteristics of certain classes in $K^\circ(Y_\mathrm{P})$ to the degrees of certain Chow classes on Y_P. Set

$$\overline{K}_\mathrm{P} = K\left(\prod_{i=0}^m \mathbb{P}^{a_i}\right)/\operatorname{ann}([\mathcal{O}_{Y_\mathrm{P}}]) \text{ and } \overline{A}^\bullet_\mathrm{P} = A^\bullet\left(\prod_{i=0}^m \mathbb{P}^{a_i}\right)/\operatorname{ann}([Y_\mathrm{P}]).$$

Note that $\overline{K}_\mathrm{P}$ and $\overline{A}^\bullet_\mathrm{P}$ are equipped with an Euler characteristic and a degree map, respectively. Indeed, let $\iota\colon Y_\mathrm{P} \hookrightarrow \prod_{i=0}^m \mathbb{P}^{a_i}$ be the inclusion. Then we have a map $\chi(Y_\mathrm{P}, -)\colon \overline{K}_\mathrm{P} \to \mathbb{Z}$ given by

$$\chi(Y_\mathrm{P}, \iota^*\mathcal{E}) = \chi\left(\prod_{i=0}^m \mathbb{P}^{a_i}, [\mathcal{O}_{Y_\mathrm{P}}]\cdot\mathcal{E}\right).$$

This is well-defined by construction. Similarly, we have a map $\deg\colon \overline{A}^\bullet_\mathrm{P} \to \mathbb{Z}$. A more geometric proof of the following result was given in [LLPP24, Section 3].

Corollary 3.3.2. *Let* P *be a polymatroid with cage* $(a_0, \dots, a_m)$ *such that* $\mathrm{rk}_\mathrm{P}(0) = \mathrm{rk}(\mathrm{P})$*. Then the isomorphism* $\otimes\zeta\colon K(\prod_{i=0}^m \mathbb{P}^{a_i}) \to A^\bullet(\prod_{i=0}^m \mathbb{P}^{a_i})$ *descends to an isomorphism* $\zeta\colon \overline{K}_\mathrm{P} \to \overline{A}^\bullet_\mathrm{P}$ *which satisfies*

$$\chi(Y_\mathrm{P}, \mathcal{E}) = \deg_{Y_\mathrm{P}}(\zeta(\mathcal{E})\cdot(1 + c_1(\pi_0^*\mathcal{O}(1)) + c_1(\pi_0^*\mathcal{O}(1))^2 + \cdots)).$$

Proof. Theorem 3.3.1 implies that the isomorphism $\otimes\zeta$ sends $[\mathcal{O}_{Y_\mathrm{P}}]$ to a unit times

$[Y_{\mathrm{P}}]$. This implies that $\otimes\zeta$ maps $\mathrm{ann}([\mathcal{O}_{Y_{\mathrm{P}}}])$ isomorphically onto $\mathrm{ann}([Y_{\mathrm{P}}])$.

By direct computation, one sees that the isomorphism $\zeta\colon K(\mathbb{P}^{a_i}) \to A^\bullet(\mathbb{P}^{a_i})$ satisfies

$$\chi(\mathbb{P}^{a_i}, \mathcal{E}) = \deg_{\mathbb{P}^{a_i}}(\zeta(\mathcal{E}) \cdot (1 + c_1(\mathcal{O}(1)) + c_1(\mathcal{O}(1))^2 + \cdots))$$

for all $\mathcal{E} \in K(\mathbb{P}^{a_i})$. Indeed, it suffices to check this on a basis for $K(\mathbb{P}^{a_i})$, and it is evident for the structure sheaf of a linear subspace. We then see that the isomorphism $\otimes\zeta$ satisfies

$$\chi\left(\prod_{i=0}^m \mathbb{P}^{a_i}, \mathcal{E}\right) = \deg_{\prod_{i=0}^m \mathbb{P}^{a_i}}\left(\zeta(\mathcal{E}) \cdot \prod_{i=0}^m (1 + c_1(\pi_i^*\mathcal{O}(1)) + c_1(\pi_i^*\mathcal{O}(1))^2 + \cdots)\right).$$

Let $\iota\colon Y_{\mathrm{P}} \to \prod_{i=0}^m \mathbb{P}^{a_i}$ be the inclusion. Using the projection formula, we compute that, for any $\mathcal{E} \in K(\prod_{i=0}^m \mathbb{P}^{a_i})$, we have

$$\begin{aligned}
\chi(Y_{\mathrm{P}}, \iota^*\mathcal{E}) &= \chi\left(\prod_{i=0}^m \mathbb{P}^{a_i}, [\mathcal{O}_{Y_{\mathrm{P}}}] \cdot \mathcal{E}\right) \\
&= \deg_{\prod_{i=0}^m \mathbb{P}^{a_i}}\left(\zeta([\mathcal{O}_{Y_{\mathrm{P}}}]) \cdot \zeta(\mathcal{E}) \cdot \prod_{i=0}^m (1 + c_1(\pi_i^*\mathcal{O}(1)) + c_1(\pi_i^*\mathcal{O}(1))^2 + \cdots)\right) \\
&= \deg_{\prod_{i=0}^m \mathbb{P}^{a_i}}\left([Y_{\mathrm{P}}] \cdot \zeta(\mathcal{E}) \cdot (1 + c_1(\pi_0^*\mathcal{O}(1)) + c_1(\pi_0^*\mathcal{O}(1))^2 + \cdots)\right) \\
&= \deg_{Y_{\mathrm{P}}}\left(\iota^*\zeta(\mathcal{E}) \cdot (1 + c_1(\pi_0^*\mathcal{O}(1)) + c_1(\pi_0^*\mathcal{O}(1))^2 + \cdots)\right),
\end{aligned}$$

where we use Theorem 3.3.1 in the third equality. $\square$

We discuss a consequence of Corollary 3.3.2. Let X be a smooth multiplicity-free subvariety of $\mathbb{P}^{a_0} \times \cdots \times \mathbb{P}^{a_m}$ such that $\pi_0\colon X \to \mathbb{P}^{a_0}$ is birational. Let P be the polymatroid such that the multidegree of X is given by the bases of P. The assumption that π_0 is birational implies that $\mathrm{rk}_{\mathrm{P}}(0) = \mathrm{rk}(\mathrm{P}) = \dim X$. In general, the rings $\overline{K}_{\mathrm{P}}$ and $\overline{A}^\bullet_{\mathrm{P}}$ are subquotients of $K(X)$ and $A^\bullet(X)$.

Suppose that the restriction maps on K and Chow from $\prod_{i=0}^m \mathbb{P}^{a_i}$ to X are surjective. Suppose also that the pairing $(x, y) \mapsto \deg_X(xy)$ on $A^\bullet(X)$ is nondegenerate. Then the Hirzebruch–Riemann–Roch theorem implies that the pairing $(x, y) \mapsto \chi(X, xy)$ on $K(X)$ is nondegenerate. This implies that $K(X) = \overline{K}_{\mathrm{P}}$ and

$A^\bullet(X) = \overline{A}^\bullet(\mathrm{P})$, and so Corollary 3.3.2 gives an isomorphism $K(X) \to A^\bullet(X)$. As we will see, these hypotheses are satisfied by wonderful varieties in their defining embeddings.

Example 3.3.3. Recalling the embedding of $\overline{M}_{0,n}$ as a multiplicity-free subvariety of a product of projective spaces from Example 2.6.5. The restriction map on Chow is surjective: this is equivalent to showing that $A^\bullet(\overline{M}_{0,n})$ is generated by pullbacks of ψ_n along forgetful maps $\overline{M}_{0,n} \to \overline{M}_{0,S}$ where $n \in S$, which is shown in [EHKR10, Theorem 5.5]. Kapranov's description of $\overline{M}_{0,n}$ as an iterated blow-up [Kap93] implies that the cycle class map $A^\bullet(\overline{M}_{0,n}) \to H^{2\bullet}(\overline{M}_{0,n})$ is an isomorphism, so the Poincaré pairing on $A^\bullet(\overline{M}_{0,n})$ is nondegenerate. Corollary 3.3.2 then constructs an isomorphism $K(\overline{M}_{0,n}) \to A^\bullet(\overline{M}_{0,n})$. See [LLPP24, Section 4] for an alternative description of this isomorphism.

3.4 The structure of K-rings of matroids

We first motivate the definition of the Euler characteristic map $\chi(\mathrm{M}, -)\colon K(\mathrm{M}) \to \mathbb{Z}$ in the case where M has a realization $L \subseteq \mathbf{k}^E$. Recall that, by Proposition 2.2.2 and Proposition 3.1.1, we can identify $A^\bullet(W_L)$ with $A^\bullet(\mathrm{M})$ and can identify $K(W_L)$ with $K(\mathrm{M})$.

Theorem 3.4.1. *Let* M *be a loopless matroid realized $L \subseteq \mathbf{k}^E$. Then, for any $a \in K(W_L)$,*

$$\chi(W_L, a) = \deg_{\mathrm{M}}(\zeta_{\mathrm{M}}(a) \cdot (1 + h_E + h_E^2 + \cdots)).$$

Remark 3.4.2. Theorem 3.4.1 can be stated as saying that the diagram

$$\begin{tikzcd} K(W_L) \arrow[r, "\zeta_{\mathrm{M}}"] \arrow[d, "\chi(W_L,-)"] & A^\bullet(W_L) \arrow[d, "\deg_{W_L}(-\cdot(1+h_E+\cdots))"] \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

commutes. This should be compared with the classical Hirzebruch–Riemann–Roch theorem, which says that for any smooth projective variety X, the diagram

$$\begin{tikzcd} K(X) \otimes \mathbb{Q} \arrow[r, "\mathrm{ch}"] \arrow[d, "\chi(X,-)"] & A^\bullet(X) \otimes \mathbb{Q} \arrow[d, "\deg_X(- \cdot \mathrm{Todd}_X)"] \\ \mathbb{Q} \arrow[r] & \mathbb{Q} \end{tikzcd}$$

commutes. The map ζ_{M} is unrelated to the Chern character, and Theorem 3.4.1 cannot be deduced from Hirzebruch–Riemann–Roch. We will later give a version of Hirzebruch–Riemann–Roch for K-rings of matroids in Proposition 3.4.5.

Proof of Theorem 3.4.1. By Proposition 2.6.1, the embedding $W_L \hookrightarrow \prod_{F \neq \emptyset} \mathbb{P}L^F$ realizes W_L as a multiplicity-free subvariety. The restriction maps $K(\prod_{F \neq \emptyset} \mathbb{P}L^F) \to K(W_L) = K(\mathrm{M})$ and $A^\bullet(\prod_F \mathbb{P}L^F) \to A^\bullet(W_L) = A^\bullet(\mathrm{M})$ are surjective by Proposition 2.4.1 and Lemma 3.1.4. By Proposition 2.2.3, the Poincaré pairing on $A^\bullet(W_L)$ is nondegenerate. By the discussion following Corollary 3.3.2, we have $\overline{K}_{\mathrm{P}} = K(W_L)$ and $\overline{A}^\bullet_{\mathrm{P}} = A^\bullet(W_L)$, where P is the dragon-Hall–Rado polymatroid (see Proposition 2.7.2).

The projection $W_L \to \mathbb{P}L$ is birational. By Corollary 3.3.2, we have an isomorphism $\zeta \colon K(W_L) \to A^\bullet(W_L)$ satisfying the desired formula. This isomorphism satisfies $\zeta(\eta_F) = h_F$ by construction, so it agrees with ζ_{M}. □

Motivated by Theorem 3.4.1, we make the following definition of $\chi(\mathrm{M}, -)$.

Definition 3.4.3. Let M be a loopless matroid. Define $\chi(\mathrm{M}, -) \colon K(\mathrm{M}) \to \mathbb{Z}$ by $\chi(\mathrm{M}, a) = \deg_{\mathrm{M}}(\zeta_{\mathrm{M}}(a) \cdot (1 + h_E + h_E^2 + \cdots))$ for any $a \in K(\mathrm{M})$.

The following property of $K(\mathrm{M})$ holds for realizable matroids by [AP15, Theorem 1.3].

Proposition 3.4.4. *Let* M *be a loopless matroid. Then the pairing* $K(\mathrm{M}) \times K(\mathrm{M}) \to \mathbb{Z}$ *given by* $(a, b) \mapsto \chi(\mathrm{M}, ab)$ *is unimodular.*

Proof. It suffices to show that the induced pairing on $K(\mathrm{M})_{\mathbb{F}_p}$ is nondegenerate for every prime p. If $x \in K(\mathrm{M})$ has the property that $\chi(\mathrm{M}, xy) \equiv 0 \pmod p$ for all $y \in K(\mathrm{M})$, then the unimodularity of the Poincaré pairing on the Chow ring of a matroid (Proposition 2.2.3) and the definition of the Euler characteristic implies that $\zeta_{\mathrm{M}}(x) \cdot (1 + h_E + h_E^2 + \cdots) = 0$ in $A^\bullet(\mathrm{M})_{\mathbb{F}_p}$. But this implies that $\zeta_{\mathrm{M}}(x) = 0$ in

$A^\bullet(\mathrm{M})_{\mathbb{F}_p}$, so the fact that ζ_{M} is an integral isomorphism implies that x vanishes in $K(\mathrm{M})_{\mathbb{F}_p}$. $\square$

A version of the usual Hirzebruch–Riemann–Roch theorem for K-rings of matroids will be useful in the sequel.

Proposition 3.4.5. *Let* M *be a loopless matroid. There is a ring homomorphism* $\mathrm{ch}\colon K(\mathrm{M}) \to A^\bullet(\mathrm{M})_{\mathbb{Q}}$ *which induces an isomorphism* $K(\mathrm{M})_{\mathbb{Q}} \to A^\bullet(\mathrm{M})_{\mathbb{Q}}$ *defined by*

$$\mathrm{ch}([\mathcal{L}]) = \exp(c_1(\mathcal{L})) = 1 + c_1(\mathcal{L}) + c_1(\mathcal{L})^2/2! + \cdots .$$

There is a class $\mathrm{Todd}_{\mathrm{M}} \in A^\bullet(\mathrm{M})_{\mathbb{Q}}$ *such that, for any* $\xi \in K(\mathrm{M})_{\mathbb{Q}}$,

$$\chi(\mathrm{M}, \xi) = \deg_{\mathrm{M}}\big(\mathrm{ch}(\xi) \cdot \mathrm{Todd}_{\mathrm{M}}\big).$$

Moreover, the degree 0 part of $\mathrm{Todd}_{\mathrm{M}}$ *is 1.*

Proof. We first recall $K(\mathrm{M}) = K(X_{\Sigma_{\mathrm{M}}})$ and $A^\bullet(\mathrm{M}) = A^\bullet(X_{\Sigma_{\mathrm{M}}})$, i.e., the K and Chow rings of the toric variety $X_{\Sigma_{\mathrm{M}}}$ (respectively). Hence, that the Chern character map ch is well-defined and is an isomorphism after tensoring with $\mathbb{Q}$ is a general fact about algebraic varieties [Ful98, Example 15.2.16]. Because $K(\mathrm{M})$ is generated as a ring by classes of line bundles by Lemma 3.1.4 and Proposition 2.2.4, the formula $\mathrm{ch}([\mathcal{L}]) = \exp(c_1(\mathcal{L}))$ determines ch. By Proposition 2.2.3, the pairing $A^\bullet(\mathrm{M})_{\mathbb{Q}} \otimes A^\bullet(\mathrm{M})_{\mathbb{Q}} \to \mathbb{Q}$ given by $(x, y) \mapsto \deg_{\mathrm{M}}(x \cdot y)$ is a perfect pairing. Therefore there is some class $\mathrm{Todd}_{\mathrm{M}} \in A^\bullet(\mathrm{M})_{\mathbb{Q}}$ such that the linear functional $x \mapsto \chi(\mathrm{M}, \mathrm{ch}^{-1}(x))$ on $A^\bullet(\mathrm{M})_{\mathbb{Q}}$ is given by $x \mapsto \deg_{\mathrm{M}}(x \cdot \mathrm{Todd}_{\mathrm{M}})$.

Lastly, we check that the degree 0 part of $\mathrm{Todd}_{\mathrm{M}}$, which is some number in $\mathbb{Q}$, must be 1. We have that

$$\zeta_{\mathrm{M}}(\eta_E) = \zeta_{\mathrm{M}}(1 - [\mathcal{L}_{\Delta_E}]^{-1}) = h_E.$$

In particular, $\zeta_{\mathrm{M}}([\mathcal{L}_{\Delta_E}]) = (1 + h_E + h_E^2 + \cdots)$. Using Proposition 2.7.4 and the definition of $\chi(\mathrm{M}, -)$, we have

$$\chi(\mathrm{M}, \mathcal{L}_{\Delta_E}^{\otimes t}) = \deg_{\mathrm{M}}((1 + h_E + h_E^2 + \cdots)^{t+1}) = \frac{t^{r-1}}{(r-1)!} + O(t^{r-2})$$

as $\deg_{\mathrm{M}}(c_1(\mathcal{L}_{\Delta_E})^{r-1}) = \deg_{\mathrm{M}}(h_E^{r-1}) = 1$. On the other hand, Hirzebruch–Riemann–Roch gives that

$$\chi(\mathrm{M}, \mathcal{L}_{\Delta_E}^{\otimes t}) = \frac{\deg_{\mathrm{M}}(c_1(\mathcal{L})^{r-1} \cdot \mathrm{Todd}_{\mathrm{M}})}{(r-1)!} t^{r-1} + O(t^{r-2}).$$

Comparing these implies that the degree 0 part of $\mathrm{Todd}_{\mathrm{M}}$ is 1. □

Additionally, the fact that $K(\mathrm{M}) = K(X_{\Sigma_{\mathrm{M}}})$ is the K-ring of a smooth variety endow it with the structure of an *augmented λ-ring* [SGA71, Exposé V, Exemple 3.9.1]. This means that we have a rank function ϵ that takes values in $\mathbb{Z}$, and for each natural number k, we have operations λ^k and Ψ^k (the latter called *Adams operations*) characterized by the property that

$$\lambda^k([\mathcal{E}]) = [\wedge^k \mathcal{E}] \text{ and } \Psi^k([\mathcal{L}]) = [\mathcal{L}^{\otimes k}]$$

for any vector bundle $\mathcal{E}$ and any line bundle $\mathcal{L}$. Since our simplicial generators η_F are all of the form $1 - [\mathcal{L}]$ for some line bundle $\mathcal{L}$, we have $\epsilon(\eta_F) = 0$, and

$$\Psi^k(\eta_F) = \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} \eta_F^i.$$

Note that the Adams operations are ring homomorphisms, which is not at all combinatorially obvious from the above formula. The Adams operations become simultaneously diagonalizable after tensoring with $\mathbb{Q}$, and their eigenspaces (which are independent of $k > 1$) map isomorphically to the graded pieces of the Chow ring via the Chern character. We also have a *duality automorphism* D, characterized by the property that $D([\mathcal{E}]) = [\mathcal{E}^\vee]$ for any vector bundle $\mathcal{E}$. In terms of the simplicial generators, this takes the form

$$D(\eta_F) = \frac{-\eta_F}{1-\eta_F} = -\eta_F - \eta_F^2 - \cdots .$$

We now establish a combinatorial version of Serre duality for K-rings of matroids. The proof uses some results from [BEST23] to reduce to Serre duality on X_E. A different proof is given in [LLPP24, Section 6], which uses the theory of *valuative* invariants of matroids to reduce to the case of realizable matroids. The following result will not be used in the sequel.

Theorem 3.4.6. *Let* M *be a loopless matroid of rank* r. *Then there is a class* $\omega_{\mathrm{M}} \in K(\mathrm{M})$ *such that*

$$\chi(\mathrm{M}, \mathcal{E}) = (-1)^{r-1}\chi(\mathrm{M}, \omega_{\mathrm{M}} \cdot D(\mathcal{E}))$$

for all $\mathcal{E} \in K(\mathrm{M})$.

We first indicate the geometry of this result and its proof. Suppose M is realized by $L \subseteq \mathbf{k}^E$. Then Serre duality implies the identity

$$\chi(W_L, \mathcal{E}) = (-1)^{r-1}\chi(W_L, [\omega_{W_L}] \cdot D(\mathcal{E})) \tag{3.2}$$

for all $\mathcal{E} \in K(W_L)$, where ω_{W_L} is the canonical bundle of W_L. Note that W_L is embedded in X_E. In [BEST23, Theorem 7.10], it is shown that there is a vector bundle $\mathcal{Q}_L$ of rank $n-r$ on X_E which has a regular section that cuts out W_L. This implies that there is a Koszul resolution

$$0 \to \wedge^{n-r}\mathcal{Q}_L^\vee \to \cdots \to \mathcal{Q}_L^\vee \to \mathcal{O}_{X_E} \to \mathcal{O}_{W_L} \to 0.$$

In particular, this exact sequence implies that

$$[\mathcal{O}_{W_L}] = \sum_{i=0}^{n-r}(-1)^i[\wedge^i\mathcal{Q}_L^\vee].$$

We have that

$$(\wedge^i\mathcal{Q}_L^\vee)^\vee = \wedge^i\mathcal{Q}_L \xrightarrow{\sim} \det \mathcal{Q}_L \otimes \wedge^{n-r-i}\mathcal{Q}_L^\vee.$$

Therefore

$$\begin{aligned} D([\mathcal{O}_{W_L}]) &= \sum_{i=0}^{n-r} (-1)^i D([\mathcal{Q}_L^\vee]) \\ &= [\det \mathcal{Q}_L] \cdot \sum_{i=0}^{n-r} (-1)^i [\wedge^{n-r-i} \mathcal{Q}_L^\vee] \\ &= (-1)^{n-r} [\det \mathcal{Q}_L] \cdot [\mathcal{O}_{W_L}]. \end{aligned}$$

The restriction map $i^*\colon K(X_E) \to K(W_L)$ is surjective. Using the projection formula, the fact that duality commutes with restriction, and Serre duality on X_E, we have

$$\begin{aligned} \chi(W_L, i^*[\mathcal{E}]) &= \chi(X_E, [\mathcal{O}_{W_L}] \cdot [\mathcal{E}]) \\ &= (-1)^{n-1} \chi(X_E, D([\mathcal{O}_{W_L}]) \cdot D([\mathcal{E}]) \cdot [\omega_{X_E}]) \\ &= (-1)^{r-1} \chi(X_E, [\mathcal{O}_{W_L}] \cdot D([\mathcal{E}]) \cdot [\det \mathcal{Q}_L] \cdot [\omega_{X_E}]) \\ &= (-1)^{r-1} \chi(W_L, D(i^*[\mathcal{E}]) \cdot i^*[\det \mathcal{Q}_L] \cdot i^*[\omega_{X_E}]). \end{aligned}$$

The adjunction formula implies that $\omega_{W_L} = i^* \det \mathcal{Q}_L \otimes \omega_{X_E}$ as $i^*\mathcal{Q}_L$ is the normal bundle of W_L in X_E, so this gives a second proof of (3.2) which only uses Serre duality on X_E. This proof can be adapted to prove Theorem 3.4.6. Recall that, for each $i \geq 0$, there is an operation $\lambda^i\colon K(X_E) \to K(X_E)$ which satisfies $\lambda^i([\mathcal{E}]) = [\wedge^i \mathcal{E}]$ if $\mathcal{E}$ is a vector bundle on X_E. We will need the following three facts:

1. For each loopless matroid M, there is a class $[\Delta_{\mathrm{M}}] \in A^\bullet(X_E)$ such that

   $$\deg_{\mathrm{M}}(i^* a) = \deg_{X_E}([\Delta_{\mathrm{M}}] \cdot a)$$

   for all $a \in X_E$, where $i^*\colon A^\bullet(X_E) \to A^\bullet(\mathrm{M})$ is the restriction map.

2. For each loopless matroid M of rank r, there is a class $[\mathcal{Q}_{\mathrm{M}}] \in K(X_E)$ such that

   $$\zeta_E \left(\sum_{i=0}^{n-r} (-1)^i \lambda^i (D([\mathcal{Q}_{\mathrm{M}}])) \right) = [\Delta_{\mathrm{M}}].$$

3. We have that

$$D(\lambda^i(D([\mathcal{Q}_{\mathrm{M}}]))) = \lambda^{n-r}([\mathcal{Q}_{\mathrm{M}}]) \cdot \lambda^{n-r-i}(D([\mathcal{Q}_{\mathrm{M}}])).$$

The first fact follows the description of the Chow cohomology of a toric variety in terms of Minkowski weights, see [FS97, Theorem 3.1]. The class $[\Delta_{\mathrm{M}}]$ is called the *Bergman class* of M. The class $[\mathcal{Q}_{\mathrm{M}}]$ is constructed in [BEST23] using equivariant localization on X_E. In [BEST23], an isomorphism $\zeta_{X_E} \colon K(X_E) \to A^\bullet(X_E)$ is constructed using equivariant techniques. That this isomorphism agrees with $\zeta_E \colon K(X_E) \to A^\bullet(X_E)$ follows from [BEST23, Corollary 10.6]; the proof of this corollary also gives the second fact. The third fact follows from the description of $[\mathcal{Q}_{\mathrm{M}}]$ in terms of equivariant localization, see [BEST23, Section 2].

We first prove a version of the projection formula for $K(\mathrm{M})$. Let $i^* \colon K(X_E) \to K(\mathrm{M})$ be the (surjective) restriction map. Define

$$[\mathcal{O}_{\mathrm{M}}] = \sum_{i=0}^{n-r} (-1)^i \lambda^i(D([\mathcal{Q}_{\mathrm{M}}])).$$

Proposition 3.4.7. *Let* M *be a loopless matroid. Then for any* $[\mathcal{E}] \in K(X_E)$*, we have*

$$\chi(\mathrm{M}, i^*[\mathcal{E}]) = \chi(X_E, [\mathcal{O}_{\mathrm{M}}] \cdot [\mathcal{E}]).$$

Proof. Using the first and second fact above, Theorem 3.4.1, and Proposition 3.2.3, we compute

$$\begin{aligned}
\chi(X_E, [\mathcal{O}_{\mathrm{M}}] \cdot [\mathcal{E}]) &= \deg_{X_E}(\zeta_E([\mathcal{O}_{\mathrm{M}}]) \cdot \zeta_E([\mathcal{E}]) \cdot (1 + h_E + \cdots)) \\
&= \deg_{X_E}([\Delta_{\mathrm{M}}] \cdot \zeta_E([\mathcal{E}]) \cdot (1 + h_E + h_E^2 + \cdots)) \\
&= \deg_{\mathrm{M}}(i^*\zeta_E([\mathcal{E}]) \cdot i^*(1 + h_E + h_E^2 + \cdots)) \\
&= \deg_{\mathrm{M}}(\zeta_{\mathrm{M}}(i^*[\mathcal{E}]) \cdot (1 + h_E + h_E^2 + \cdots)).
\end{aligned}$$

This is equal to $\chi(\mathrm{M}, i^*[\mathcal{E}])$ by Theorem 3.4.1. $\square$

Proof of Theorem 3.4.6. The third fact above implies that, for any loopless matroid

M of rank r,

$$D([\mathcal{O}_{\mathrm{M}}]) = (-1)^{n-r}\lambda^{n-r}([\mathcal{Q}_{\mathrm{M}}]) \cdot [\mathcal{O}_{\mathrm{M}}].$$

As the restriction map $i^*\colon K(X_E) \to K(\mathrm{M})$ is surjective, it suffices to prove the equality in Theorem 3.4.6 for a class of the form $i^*[\mathcal{E}]$, where $[\mathcal{E}] \in K(X_E)$. Using Serre duality on X_E, we have

$$\begin{aligned}
\chi(\mathrm{M}, i^*[\mathcal{E}]) &= \chi(X_E, [\mathcal{O}_{\mathrm{M}}] \cdot [\mathcal{E}]) \\
&= (-1)^{n-1}\chi(X_E, D([\mathcal{O}_{\mathrm{M}}]) \cdot D([\mathcal{E}]) \cdot [\omega_{X_E}]) \\
&= (-1)^{r-1}\chi(X_E, [\mathcal{O}_{\mathrm{M}}] \cdot D([\mathcal{E}]) \cdot [\det \mathcal{Q}_{\mathrm{M}}] \cdot [\omega_{X_E}]) \\
&= (-1)^{r-1}\chi(\mathrm{M}, D(i^*[\mathcal{E}]) \cdot i^*[\det \mathcal{Q}_{\mathrm{M}}] \cdot i^*[\omega_{X_E}]).
\end{aligned}$$

Setting $\omega_{\mathrm{M}} = i^*[\det \mathcal{Q}_{\mathrm{M}}] \cdot i^*[\omega_{X_E}]$ gives the result. □

Remark 3.4.8. One can check that ω_{M} is the class of the line bundle whose first Chern class is $x_\emptyset + x_E + c_1(\mathcal{L}_{\mathrm{M}^\perp})$, where $\mathrm{M}^\perp$ is the dual matroid to M.

3.5 A formula for the Euler characteristic

We now give a formula for the Euler characteristic map on the K-ring of a matroid. As $K(\mathrm{M})$ is generated by the classes η_F, for F a non-empty flat, it suffices to compute $\chi(\mathrm{M}, \eta_{F_1}^{t_1} \cdots \eta_{F_m}^{t_m})$ for any non-empty flats $F_1, \ldots, F_m$ and a vector $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{Z}_{\geq 0}^m$. We have the following result.

Theorem 3.5.1. *Let* M *be a loopless matroid, let* $F_1, \ldots, F_m$ *be flats of* M*, and let* $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{Z}_{\geq 0}^m$. *Then we have*

$$\chi(\mathrm{M}, \eta_{F_1}^{t_1} \cdots \eta_{F_m}^{t_m}) = \begin{cases} 1 & \mathbf{t} \text{ satisfies dragon-Hall–Rado} \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Proof. Using the definition of $\chi(\mathrm{M}, -)$, we have

$$\chi(\mathrm{M}, \eta_{F_1}^{t_1} \cdots \eta_{F_m}^{t_m}) = \deg_{\mathrm{M}}(h_{F_1}^{t_1} \cdots h_{F_m}^{t_m} \cdot (1 + h_E + h_E^2 + \cdots)). \tag{3.4}$$

If $\sum t_i > r-1$, then both sides of (3.3) vanish, so we may assume that $\sum t_i \leq r-1$. The unique term on the right-hand side of (3.4) which is in degree $r-1$ is $h_{F_1}^{t_1} \cdots h_{F_m}^{t_m} h_E^{r-1-\sum t_i}$. The result follows from the observation that $t_1F_1, \ldots, t_mF_m$ satisfies the dragon-Hall–Rado condition if and only if $t_1F_1, \ldots, t_mF_m, (r-1-\sum t_i)E$ satisfies the dragon-Hall–Rado condition and has sum $r-1$. The result then follows from Proposition 2.7.4. □

Remark 3.5.2. A computation of the Euler characteristics of monomials in the toric generators of $K(\mathrm{M})$ is given in [LLPP24, Section 8].

3.6 Projection formulas

In this section, we show that $\chi(\mathrm{M}, -)$ can be computed as a sheaf Euler characteristic on a projective scheme, the scheme Y_{P} considered previously for a certain polymatroid P. This will enable us to use vanishing theorems on Y_{P} to control the sign of $\chi(\mathrm{M}, -)$. In Section 4, we will use this to prove inequalities about $\chi(\mathrm{M}, -)$ which are not clear combinatorially. It also allows one to compute $\chi(\mathrm{M}, -)$ using, for example, Čech cohomology.

We first sketch the geometry of the formula below. Let $L \subseteq \mathbf{k}^E$ be a linear subspace of dimension r which is not contained in any coordinate hyperplane, and let M be the matroid represented by L. Assume that $\mathbf{k}$ has characteristic 0. Let $F_1, \ldots, F_m$ be flats of M. Let X denote the image of the map $W_L \to \prod_{i=1}^m \mathbb{P}L^{F_i}$. Assume that $L_{F_1} \cap \cdots \cap L_{F_m} = 0$. Then X has dimension $r-1$, and the map $p \colon W_L \to X$ is birational. By [BF22, Theorem 4.3], which is based on [Bri01, Theorem 5], X has rational singularities because it is a multiplicity-free subvariety. As W_L is smooth, this implies that $Rp_*\mathcal{O}_{W_L} = \mathcal{O}_X$. The projection formula then gives that, for any line bundle $\mathcal{L}$ on X, we have

$$H^i(X, \mathcal{L}) = H^i(W_L, p^*\mathcal{L}) \text{ for all } i.$$

In particular, $\chi(X, \mathcal{L}) = \chi(W_L, p^*\mathcal{L})$. Suppose $\mathcal{L}$ is the restriction of a line bundle on $\prod_{i=1}^m \mathbb{P}L^F$. Then $\mathcal{L}$ extends over the degeneration of X to some Y_{P} given by

Proposition 2.5.1. As Euler characteristics are locally constant in proper flat families, this implies that we may compute the Euler characteristic of $\mathcal{L}$ on Y_{P}.

Theorem 3.6.1. *Let $F_1, \dots, F_m$ be non-empty flats of a loopless matroid* M*. Let* P *be the polymatroid on $[m]$ whose bases are the $\mathbf{t} \in \mathbb{Z}^m_{\geq 0}$ which satisfy the dragon-Hall–Rado formula. Then*

$$\chi(\mathrm{M}, \mathcal{L}_{\Delta_{F_1}}^{\otimes k_1} \otimes \cdots \otimes \mathcal{L}_{\Delta_{F_m}}^{k_m}) = \chi(Y_{\mathrm{P}}, \mathcal{O}(k_1, \dots, k_m)).$$

Note that, by Proposition 2.7.2, P is indeed a polymatroid. We prepare for the proof of Theorem 3.6.1 with the following proposition.

Proposition 3.6.2. *Let* P *be a polymatroid with cage $(a_1, \dots, a_m)$ on $[m]$, and let $Y_{\mathrm{P}} \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ be the corresponding reduced union of Schubert varieties. Let $p \colon \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m} \to \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_{m-1}}$ be the projection that forgets the last factor. Let P' be the polymatroid on $[m-1]$ whose rank function is obtained by restricting the rank function of* P *to $[m-1]$. Then $p_* \colon K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}) \to K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_{m-1}})$ satisfies $p_*[\mathcal{O}_{Y_{\mathrm{P}}}] = [\mathcal{O}_{Y'_{\mathrm{P}}}]$.*

Proof. First note that if $Y_{\mathbf{b}}$ is a product of projective spaces embedded linearly in $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ whose multidegree is $\mathbf{b} = (b_1, \dots, b_m)$, then $p(Y_{\mathbf{b}})$ is the product of projective spaces embedded linearly in $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_{m-1}}$ whose multidegree is $\mathbf{b}' = (b_1, \dots, b_{m-1})$. Also, $p_*[\mathcal{O}_{Y_{\mathbf{b}}}] = [\mathcal{O}_{Y_{\mathbf{b}'}}]$ because $Y_{\mathbf{b}} \to Y_{\mathbf{b}'}$ is projective bundle.

We write $p_*[\mathcal{O}_{Y_{\mathrm{P}}}]$ in terms of the basis for $K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m})$ given by the structure sheaves of products of projective spaces $[\mathcal{O}_{Y_{\mathbf{b}}}]$, and then we check that the claimed formula for $p_*[\mathcal{O}_{Y_{\mathrm{P}}}]$ satisfies the recursion of Proposition 2.5.5. First we do the case $\mathbf{b} = (a_1, \dots, a_{m-1})$, i.e., we compute the coefficient of $[\mathcal{O}_{\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_{m-1}}}]$. The classes in $K(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m})$ which contribute to this term are $[\mathcal{O}_{Y_{\mathbf{e}}}]$, for $\mathbf{e} = (a_1, \dots, a_{m-1}, i)$ for some $i \in \{0, \dots, a_m\}$. We see that the coefficient of $[\mathcal{O}_{\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_{m-1}}}]$ is the sum of the coefficients of these classes in the expression of $[\mathcal{O}_{Y_{\mathrm{P}}}]$ as a sum of structure sheaves of products of linear spaces. By Proposition 2.5.5, this sum is 1 if $\mathbf{b}$ is independent in P' and is 0 otherwise.

We may now inductively assume that the formula holds for all $\mathbf{b}' > \mathbf{b}$ and prove it for $\mathbf{b} = (b_1, \dots, b_{m-1})$. We say that $\mathbf{e}$ *extends* $\mathbf{b}$ if it is of the form $(b_1, \dots, b_{m-1}, i)$

for some i. Let $c_{\mathbf{b}'}$ be the coefficient of $[\mathcal{O}_{Y_{\mathbf{b}'}}]$, and let $\tilde{c}_{\mathbf{e}}$ be the coefficient of $[\mathcal{O}_{Y_{\mathbf{e}}}]$ in the expression of $[\mathcal{O}_{Y_{\mathrm{P}}}]$ as a sum of structure sheaves of products of linear spaces. Then we have

$$\begin{aligned}\sum_{\mathbf{b}'\geq\mathbf{b}} c_{\mathbf{b}} &= \sum_{\mathbf{b}'\geq\mathbf{b}} \sum_{\mathbf{e}' \text{ extending } \mathbf{b}'} \tilde{c}_{\mathbf{e}'} \\ &= \sum_{\mathbf{e}'\geq(b_1,\dots,b_{m-1},0)} \tilde{c}_{\mathbf{e}} \\ &= 1,\end{aligned}$$

where in the last step we use Proposition 2.5.5. □

For any projection p from a product of projective spaces to some of the factors and a polymatroid P, then $p(Y_{\mathrm{P}}) = Y_{\mathrm{P}'}$, where P$'$ is the polymatroid obtained by restricting the rank function of P. By iterating Proposition 3.6.2, we see that $p_*[\mathcal{O}_{Y_{\mathrm{P}}}] = [\mathcal{O}_{Y_{\mathrm{P}'}}]$.

Proof of Theorem 3.6.1. In the case when $F_1, \dots, F_m$ are all of the non-empty flats of M, then the result follows Corollary 3.3.2 and definition of the Euler characteristic map on $K(\mathrm{M})$. It suffices to show that if the theorem holds for $F_1, \dots, F_{m+1}$, then it holds for $F_1, \dots, F_m$. Let P be the polymatroid associated to $F_1, \dots, F_{m+1}$, and let P$'$ be the polymatroid associated to $F_1, \dots, F_m$. Let $p \colon \mathbb{P}^{\mathrm{rk}(F_1)-1} \times \cdots \times \mathbb{P}^{\mathrm{rk}(F_{m+1})-1} \to \mathbb{P}^{\mathrm{rk}(F_1)-1} \times \cdots \times \mathbb{P}^{\mathrm{rk}(F_m)-1}$ be the projection, and note that $p(Y_{\mathrm{P}}) = Y_{\mathrm{P}'}$. We have

$$\begin{aligned}\chi(\mathrm{M}, \mathcal{L}_{\Delta_{F_1}}^{\otimes a_1} \otimes \cdots \otimes \mathcal{L}_{\Delta_{F_m}}^{\otimes a_m}) &= \chi(\mathrm{M}, \mathcal{L}_{\Delta_{F_1}}^{\otimes a_1} \otimes \cdots \otimes \mathcal{L}_{\Delta_{F_m}}^{\otimes a_m} \otimes \mathcal{L}_{\Delta_{F_{m+1}}}^{\otimes 0}) \\ &= \chi(Y_{\mathrm{P}}, \mathcal{O}(a_1, \dots, a_m, 0)) \\ &= \chi\left(\prod_{i=1}^{m+1} \mathbb{P}^{\mathrm{rk}(F_i)-1}, [\mathcal{O}_{Y_{\mathrm{P}}}] \otimes \mathcal{O}(a_1, \dots, a_m, 0)\right) \\ &= \chi\left(\prod_{i=1}^{m} \mathbb{P}^{\mathrm{rk}(F_i)-1}, p_*[\mathcal{O}_{Y_{\mathrm{P}}}] \otimes \mathcal{O}(a_1, \dots, a_m)\right) \\ &= \chi(Y_{\mathrm{P}'}, \mathcal{O}(a_1, \dots, a_m)).\end{aligned}$$

□

Chapter 4

K-theoretic positivity

In this chapter, we study positivity properties of K-rings of matroids. Our guiding principle is that wonderful varieties resemble smooth projective toric varieties, and so we expect analogues of the positivity properties for K-rings of toric varieties described in Section 1.2. These positivity properties are best expressed in terms of analogues of the h^*-vector of a lattice polytope. We also expect those positivity properties to hold for matroids which are not necessarily realizable. See Conjecture 4.3.1. Our strongest result in this direction is Theorem 4.0.2

Let M be a loopless matroid. For a line bundle $\mathcal{L}$ on $X_{\Sigma_{\mathrm{M}}}$, it follows from the definition of the Euler characteristic map (and more explicitly, from Theorem 3.5.1) that the function $t \mapsto \chi(\mathrm{M}, \mathcal{L}^{\otimes t})$ is a polynomial in t, which we call the *Snapper polynomial* of $\mathcal{L}$ on M.

Definition 4.0.1. For a loopless matroid M on a ground set E and a line bundle $\mathcal{L}$ in $K(\mathrm{M})$, we define its *h^*-vector* $(h^*_0(\mathrm{M}, \mathcal{L}), \ldots, h^*_d(\mathrm{M}, \mathcal{L}))$ by

$$\sum_{k\geq 0} \chi(\mathrm{M}, \mathcal{L}^{\otimes k}) t^k = \frac{h^*(\mathrm{M}, \mathcal{L}; t)}{(1-t)^{d+1}} \quad \text{where} \quad h^*(\mathrm{M}, \mathcal{L}; t) = \sum_{k=0}^{d} h^*_k(\mathrm{M}, \mathcal{L}) t^k,$$

and d is the degree of the Snapper polynomial of $\mathcal{L}$.

We will show that, for certain $\mathcal{L}$, the h^*-vector is a *Macaulay vector*, i.e., it is the Hilbert function of a graded $\mathbf{k}$-algebra $R^\bullet$ with $R^0 = \mathbf{k}$ which is generated in degree

1. In particular, the h^*-vector is non-negative. See Section 4.1 for more on Macaulay vectors. Our main result is the following theorem.

We say that a line bundle $\mathcal{L}$ on X_{Σ_M} is *simplicially positive* if $c_1(\mathcal{L})$ is a non-negative linear combination of the h_F's. Equivalently, $\mathcal{L}$ is simplicially positive if, in the unique expression

$$\mathcal{L} = \otimes_{F \neq \emptyset} \mathcal{L}_{\Delta_F}^{\otimes k_F},$$

we have $k_F \geq 0$ for all F.

A line bundle $\mathcal{L}$ is simplicially positive on X_{Σ_M} if and only if it is of the form $\mathcal{L}_\mathrm{P}$ for a polymatroid P which is a Minkowski sum of simplices. Note that P is not determined by $\mathcal{L}$, i.e., we may have $\mathcal{L}_\mathrm{P} \xrightarrow{\sim} \mathcal{L}_\mathrm{Q}$ where P is a Minkowski sum of simplices but Q is not.

Theorem 4.0.2. *Let* M *be a loopless matroid, and let* $\mathcal{L}$ *be a simplicially positive line bundle on* X_{Σ_M}. *Then* $h^*(\mathrm{M}, \mathcal{L})$ *is a Macaulay vector.*

In Section 4.1, we review Macaulay vectors and their relationship with Cohen–Macaulayness and cohomology vanishing. In Section 4.2, we use properties of Y_P to prove Theorem 4.0.2. A generalization of Theorem 4.0.2 is conjectured in Section 4.3. Results on the degree of Snapper polynomials, necessary for studying h^*-vectors, are given in Section 4.4. We discuss applications and examples in Section 4.5 and 4.6.

4.1 Macaulay vectors

Recall that the Hilbert function of a graded algebra over a field $\mathbf{k}$ is the sequence of the $\mathbf{k}$-dimensions of the graded pieces. For the numerical properties we consider, we may extend scalars to an extension of $\mathbf{k}$, so we may assume $\mathbf{k}$ is infinite as needed.

Definition 4.1.1. A sequence $(h_0, h_1, \ldots, h_d)$ of integers is a *Macaulay vector* if $(h_0, h_1, \ldots, h_d, 0, 0, \ldots)$ is the Hilbert function of a graded artinian $\mathbf{k}$-algebra $A^\bullet$ which is generated in degree 1 and has $A^0 = \mathbf{k}$.

Macaulay vectors are also called M-vectors and O-sequences. Macaulay gave an explicit description of these vectors as follows [BH93, Theorem 4.2.10]. Given positive integers n and d, there is a unique expression

$$n = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_\delta}{\delta}, \quad k_d > k_{d-1} > \cdots > k_\delta \geq 1.$$

Set $n^{\langle d \rangle} = \binom{k_d+1}{d+1} + \cdots + \binom{k_\delta+1}{\delta+1}$. Then $(1, a_1, \ldots, a_d)$ is a Macaulay vector if and only if $0 \leq a_{t+1} \leq a_t^{\langle t \rangle}$ for all $t \geq 1$.

Macaulay vectors often appear in the following way. Suppose $R^\bullet$ is a graded Cohen–Macaulay algebra of Krull dimension $d+1$ with $R^0 = \mathbf{k}$. If the quotient of $R^\bullet$ by the ideal generated by R^1 is artinian, then $R^\bullet$ admits a linear system of parameters [BH93, Propositions 1.5.11 and 1.5.12]. In this case, the quotient by a linear system of parameters is a graded artinian algebra $A^\bullet$ with the property that

$$\sum_{k \geq 0} (\dim_{\mathbf{k}} R^k) t^k = \frac{\dim_{\mathbf{k}} A^0 + (\dim_{\mathbf{k}} A^1) t^1 + \cdots + (\dim_{\mathbf{k}} A^d) t^d}{(1-t)^{d+1}}.$$

See, for instance, [BH93, Remark 4.1.11]. In particular, if $R^\bullet$ is generated in degree 1, then the numerator of its Hilbert series $\sum_{k \geq 0} (\dim_{\mathbf{k}} R^k) t^k$ is a polynomial whose coefficients form a Macaulay vector. For the proof of Theorem 4.0.2, we record the following cohomological criterion for a section ring to be Cohen–Macaulay.

Proposition 4.1.2. *Let $\mathcal{L}$ be an ample line bundle on a geometrically connected and geometrically reduced projective variety X over $\mathbf{k}$ of dimension d. Suppose that $H^i(X, \mathcal{L}^{\otimes k}) = 0$ for all $i > 0$ when $k \geq 0$, and $H^i(X, \mathcal{L}^{\otimes k}) = 0$ for all $i < d$ when $k < 0$. Then, the section ring*

$$R^\bullet_{\mathcal{L}} := \bigoplus_{k \geq 0} H^0(X, \mathcal{L}^{\otimes k})$$

is a graded Cohen–Macaulay $\mathbf{k}$-algebra with $R^0_{\mathcal{L}} = \mathbf{k}$. If furthermore $R^\bullet_{\mathcal{L}}$ is generated in degree 1, then the sequence $(h_0, \ldots, h_d)$ defined by

$$\sum_{k \geq 0} \chi(X, \mathcal{L}^{\otimes k}) t^k = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^{d+1}} \tag{4.1}$$

is a Macaulay vector.

Proof. The sequence $(h_0, \ldots, h_d)$ is well-defined via (4.1) because $\chi(X, \mathcal{L}^{\otimes k})$ is a polynomial in k (see [Sta12, Section 4.3]). Because X is geometrically connected, geometrically reduced, and proper over $\operatorname{Spec} \mathbf{k}$, we have $R^0_{\mathcal{L}} = \mathbf{k}$. Because all of the higher cohomology vanishes, we have $\chi(X, \mathcal{L}^{\otimes k}) = \dim H^0(X, \mathcal{L}^{\otimes k})$ for $k \geq 0$. Therefore the second statement follows from the first by our discussion above about Macaulay vectors.

It remains to show that $R^\bullet_{\mathcal{L}}$ is a Cohen–Macaulay graded ring. That is, we show that the local cohomology $H^i_{\mathfrak{m}}(R^\bullet_{\mathcal{L}}; R^\bullet_{\mathcal{L}})$ with respect to the irrelevant ideal $\mathfrak{m}$ of $R^\bullet_{\mathcal{L}}$ vanishes for $i < d+1$. The vanishing when $i = 0, 1$ is automatic since $R^\bullet_{\mathcal{L}}$ is the section ring of $\mathcal{O}(1)$ on $X = \operatorname{Proj} R^\bullet_{\mathcal{L}}$. For $i \geq 2$, we have $H^i_{\mathfrak{m}}(R^\bullet_{\mathcal{L}}; R^\bullet_{\mathcal{L}}) = \bigoplus_{k \in \mathbb{Z}} H^{i-1}(\operatorname{Proj} R^\bullet_{\mathcal{L}}, \mathcal{L}^{\otimes k})$ by [BS98, Theorem 20.4.4]. As $X = \operatorname{Proj} R^\bullet_{\mathcal{L}}$, the sheaf cohomology vanishing hypothesis gives desired vanishing of local cohomology. □

4.2 Properties of Y_{P} and Theorem 4.0.2

Let P be a polymatroid with cage $(a_1, \ldots, a_m)$, and let $Y_{\mathrm{P}} \subseteq \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_m}$ be the subscheme defined in Section 2.5. We note that Y_{P} is Cohen–Macaulay (Proposition 2.5.3) and compatibly Frobenius split, and we use these properties of prove Theorem 4.0.2.

Note that Y_{P} is defined over $\operatorname{Spec} \mathbb{Z}$, with an embedding in a product of projective spaces over $\operatorname{Spec} \mathbb{Z}$. Viewing the product of projective spaces as a homogeneous space, Y_{P} is a reduced union of Schubert varieties, and hence it is a compatibly Frobenius split subscheme of the product of projective spaces when base changed to any field $\mathbf{k}$ of positive characteristic [BK05, Proposition 1.2.1, Theorem 2.3.10]. Together with Proposition 2.5.3, this gives the following strong cohomology vanishing results on Y_{P}.

Proposition 4.2.1. *Let $\mathcal{L}$ be the restriction of a very ample line bundle from the product of projective spaces to Y_{P}. Then, we have $H^i(Y_{\mathrm{P}}, \mathcal{L}^{\otimes k}) = 0$ for all $i > 0$ when $k \geq 0$, and $H^i(Y_{\mathrm{P}}, \mathcal{L}^{\otimes k}) = 0$ for all $i < \operatorname{rk}(P)$ when $k < 0$. Moreover, Y_{P} is geometrically reduced and geometrically connected, and the section ring $R^\bullet_{\mathcal{L}} = \bigoplus_{k \geq 0} H^0(Y_{\mathrm{P}}, \mathcal{L}^{\otimes k})$ is* *generated in degree* 1.

Proof. The cohomology vanishing follows from [BK05, Theorem 1.2.8(ii), Theorem 1.2.9] because Y_{P} is Cohen–Macaulay. By [BK05, Theorem 1.2.8(ii)], Y_{P} is projectively normal in the embedding given by $\mathcal{L}$, so $R^{\bullet}_{\mathcal{L}}$ is generated in degree 1. It remains to check that Y_{P} is geometrically reduced and geometrically connected. That it is geometrically reduced is obvious; it is geometrically connected because each component of Y_{P} contains the point $[1, 0, \ldots, 0] \times [1, 0, \ldots, 0] \times \cdots \times [1, 0, \ldots, 0]$. $\square$

Proof of Theorem 4.0.2. Let $\mathcal{L} = \bigotimes_{i=1}^{m} \mathcal{L}_{\Delta_{F_i}}^{\otimes k_i}$ for some flats $F_1, \ldots, F_m$ of the matroid M and integers $k_i > 0$. Let P be the restriction of the dragon-Hall–Rado polymatroid to the flats $F_1, \ldots, F_m$. By Theorem 3.6.1, we have that $\chi(\mathrm{M}, \mathcal{L}^{\otimes \ell}) = \chi(Y_{\mathrm{P}}, \mathcal{O}(k_1, \ldots, k_m)^{\otimes \ell})$. Note that $\mathcal{O}(k_1, \ldots, k_m)$ is the restriction of a very ample divisor from the product of projective spaces to Y_{P}. By Proposition 4.2.1, we have that Y_{P} and $\mathcal{O}(k_1, \ldots, k_m)$ satisfy the conditions of Proposition 4.1.2, including the generation of $\bigoplus_{k \geq 0} H^0(Y_{\mathrm{P}}, \mathcal{O}(k_1, \ldots, k_m)^{\otimes \ell})$ in degree 1. Hence, we conclude that $h^*(\mathrm{M}, \mathcal{L})$ is a Macaulay vector. $\square$

4.3 Positivity for more general line bundles

We conjecture a generalization of Theorem 4.0.2. In Section 4.5, we explain how the conjecture contains a question of Speyer [Spe09] as a special case, and how Theorem 4.0.2 answers this question for a new family of cases.

Conjecture 4.3.1. *Let* M *be a loopless matroid on* E*, and let* P *be a polymatroid on* E*. Then, the* h^**-vector* $h^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}})$ *is a Macaulay vector and is in particular nonnegative.*

It is reasonable to expect positivity properties from $\mathcal{L}_{\mathrm{P}}$ because $\mathcal{L}_{\mathrm{P}}$ is globally generated as it is the restriction of the line bundle corresponding to P on X_E, see Proposition 2.3.5. Several other cases in which Conjecture 4.3.1 holds are discussed in Section 4.6.

4.4 Degree of Snapper polynomials and numerical dimension

To study the h^*-vectors of the line bundles $\mathcal{L}_{\mathrm{P}}$ in Conjecture 4.3.1, one needs some tools to understand the degrees of Snapper polynomials, since the degree is essential in the definition of $h^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}})$. We give one such tool.

Definition 4.4.1. The *numerical dimension* of a line bundle $\mathcal{L}$ in $K(\mathrm{M})$ is the largest non-negative integer k such that $c_1(\mathcal{L})^k \neq 0$ in $A^\bullet(\mathrm{M})$.

Our main result on numerical dimensions is the following.

Theorem 4.4.2. *Let* M *be a loopless matroid of rank r on a ground set E.*

1. *For $\mathcal{L}$ a line bundle in $K(\mathrm{M})$, the degree of the Snapper polynomial $\chi(\mathrm{M}, \mathcal{L}^{\otimes t})$ is at most the numerical dimension of $c_1(\mathcal{L})$. Moreover, the degree equals $r-1$ if and only if the numerical dimension is $r-1$.*

2. *For* P *a polymatroid on E such that the base polytope $B(\mathrm{P})$ is full dimensional (i.e., $(n-1)$-dimensional), then the numerical dimension of $\mathcal{L}_{\mathrm{P}}$ is $r-1$, so the degree of the Snapper polynomial of $\mathcal{L}_{\mathrm{P}}$ is $r-1$.*

Proof. Let $\mathcal{L}$ be a line bundle of numerical dimension d. Because $c_1(\mathcal{L}^{\otimes t}) = tc_1(\mathcal{L})$, we have, by Proposition 3.4.5, that

$$\chi(\mathrm{M}, \mathcal{L}^{\otimes t}) = \deg_{\mathrm{M}}((1 + tc_1(\mathcal{L}) + t^2 c_1(\mathcal{L})^2/2! + \cdots) \cdot \mathrm{Todd}_{\mathrm{M}}).$$

Since $c_1(\mathcal{L})^{d+1} = 0$, we see that the right-hand side is a polynomial in t whose leading term is $t^\ell \deg_{\mathrm{M}}(c_1(\mathcal{L})^\ell \cdot \mathrm{Todd}_{\mathrm{M}})/\ell!$ for the largest $0 \leq \ell \leq d$ such that $\deg_{\mathrm{M}}(c_1(\mathcal{L})^\ell \cdot \mathrm{Todd}_{\mathrm{M}}) \neq 0$. Moreover, because the degree 0 part of $\mathrm{Todd}_{\mathrm{M}}$ is 1, we have

$$\chi(\mathrm{M}, \mathcal{L}^{\otimes t}) = \deg_{\mathrm{M}}(c_1(\mathcal{L})^{r-1}) \frac{t^{r-1}}{(r-1)!} + O(t^{r-2}).$$

Thus, $\mathcal{L}$ has numerical dimension $r-1$ if and only if the Snapper polynomial has degree $r-1$. We have proven the first statement (1).

For second statement (2), we only need show that the numerical dimension of $\mathcal{L}_{\mathrm{P}}$ is $r-1$ if $B(\mathrm{P})$ is full dimensional. When $B(\mathrm{P})$ is full dimensional, the line bundle $\mathcal{L}_{\mathrm{P}}$ on X_E is big and nef by Proposition 2.3.5. By [Laz04, Corollary 2.2.7], we can write the first Chern class as the sum of an ample class and an effective divisor class (inside $A^\bullet(X_E) \otimes \mathbb{Q}$). Restricting this to $A^\bullet(\mathrm{M})$, we get that $c_1(\mathcal{L}_{\mathrm{P}}) = A + E$, where A is the restriction of an ample class from X_E and E is the restriction of an effective class.

We now prove by induction on k that $\deg_{\mathrm{M}}(c_1(\mathcal{L}_{\mathrm{P}})^k A^{r-1-k}) > 0$, using Proposition 4.4.3 stated below. The case $k=0$ is Proposition 4.4.3(1). For $k>0$, Proposition 4.4.3(2) gives that

$$\begin{aligned}\deg_{\mathrm{M}}(c_1(\mathcal{L}_{\mathrm{P}})^k A^{r-1-k}) &= \deg_{\mathrm{M}}(c_1(\mathcal{L}_{\mathrm{P}})^{k-1} A^{r-k}) + \deg_{\mathrm{M}}(c_1(\mathcal{L}_{\mathrm{P}})^{k-1} E A^{r-1-k}) \\ &\geq \deg_{\mathrm{M}}(c_1(\mathcal{L}_{\mathrm{P}})^{k-1} A^{r-k}),\end{aligned}$$

which is positive by induction. $\square$

Proposition 4.4.3. *Let* M *be a loopless matroid of rank* r*.*

1. *Let* $A \in A^1(\mathrm{M})$ *be the restriction of an ample class from* X_E*. Then we have* $\deg_{\mathrm{M}}(A^{r-1}) > 0$*.*

2. *Let* $\mathrm{P}_1, \dots, \mathrm{P}_{r-2}$ *be polymatroids. Then, for any class* $E \in A^1(\mathrm{M})$ *which is the restriction of an effective divisor class on* X_E*,* $\deg_{\mathrm{M}}(c_1(\mathcal{L}_{\mathrm{P}_1}) \cdots c_1(\mathcal{L}_{\mathrm{P}_{r-2}}) \cdot E) \geq 0$*.*

This result, which is well-known to experts, can be deduced from the mixed Hodge–Riemann relations in degree 0 [AHK18, Theorem 8.9]. We briefly indicate the proof.

Proof of 4.4.3. The polytopal description of nef and ample line bundles on a toric variety (see [Ful93, Section 3.4]) implies the following:

- If A is an ample divisor class on X_E, then for any flag $\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$, we may write $A = \sum_{\emptyset \subsetneq S \subsetneq E} c_S x_S \in A^1(X_E)$, where $c_S > 0$ if $S \neq S_i$ for some i, and $c_{S_i} = 0$ for $i = 1, \dots, k$.

- If N is a nef divisor class on X_E, then for any flag $\emptyset \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq E$, we may write $N = \sum_{\emptyset \subsetneq S \subsetneq E} c_S x_S \in A^1(X_E)$, where $c_S \geq 0$ for all S, and $c_{S_i} = 0$ for $i = 1, \ldots, k$.

To prove Proposition 4.4.3(1), we show by induction that if $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E$ is a chain of flats of M, then $\deg_{\mathrm{M}}(x_{F_1} \cdots x_{F_k} \cdot A^{r-1-k}) > 0$. The case $k = r-1$ follows from the construction of the degree map (see [AHK18, Definition 5.9]). For the inductive step, we choose an expression $A = \sum_{\emptyset \subsetneq S \subsetneq E} c_S x_S$ with $c_S > 0$ for all $S \notin \{F_1, \ldots, F_k\}$ and $c_{F_i} = 0$. Restricting this to $A^1(\mathrm{M})$, we have $A = \sum_{\emptyset \subsetneq F \subsetneq E} c_F x_F$, where the sum is over non-empty proper flats of M. Using the linearity of $\deg_{\mathrm{M}}$, we then get

$$\deg_{\mathrm{M}}(x_{F_1} \cdots x_{F_k} \cdot A^{r-1-k}) = \sum_{\emptyset \subsetneq F \subsetneq E} c_F \deg_{\mathrm{M}}(x_F \cdot x_{F_1} \cdots x_{F_k} \cdot A^{r-2-k}).$$

If $F \in \{F_1, \ldots, F_k\}$, then the corresponding term in the above sum is 0 because c_F vanishes. If F is not comparable with $\{F_1, \ldots, F_k\}$, then the corresponding term is 0 by Proposition 2.2.4. If F is comparable with $\{F_1, \ldots, F_k\}$, then the corresponding term is positive by induction. There is at least one term like this because every maximal chain of proper non-empty flats has length $r - 1$.

The proof of Proposition 4.4.3(2) is similar, using that $c_1(\mathcal{L}_{\mathrm{P}_i}) \in A^1(X_E)$ is a nef divisor class. □

4.5 Application to Speyer's g-polynomial

In this section, we apply Theorem 4.0.2 to study Speyer's g-polynomial, which is described and motivated in Section 1.3. An outstanding problem about the g-polynomial is to show that it has non-negative coefficients for all matroids. In [Spe09], Speyer used the Kawamata–Viehweg vanishing theorem to show the non-negativity for matroids realizable over a field of characteristic 0. The non-negativity was proved for all sparse paving matroids in [FS, Theorem 13.16]. Using Theorem 4.0.2 we give a new infinite family of matroids for which the non-negativity holds.

We begin by explaining how the non-negativity of the coefficients of Speyer's g-polynomial is a special case of Conjecture 4.3.1. For a loopless matroid M of rank r, let $\omega(\mathrm{M})$ be the t^r coefficient of $g_{\mathrm{M}}(t)$. We say that M is *connected* if $B(\mathrm{M})$ has dimension $n-1$, i.e., it is full-dimensional. If $\dim B(\mathrm{M})$ has dimension $n-c$, then there is a partition $E = E_1 \sqcup \cdots \sqcup E_c$ and connected matroids $\mathrm{M}_1, \ldots, \mathrm{M}_c$ on ground sets $E_1, \ldots, E_c$ such that $B(\mathrm{M}) = B(\mathrm{M}_1) \times \cdots \times B(\mathrm{M}_c)$. In forthcoming work, Alex Fink, Kris Shaw, and David Speyer show the following result.

Proposition 4.5.1. *Suppose that* $\omega(\mathrm{M}) \geq 0$ *for all connected matroids. Then all coefficients of* $g_{\mathrm{M}}(t)$ *are non-negative for all loopless matroids.*

The following result was communicated to the author by Alex Fink, Kris Shaw, and David Speyer. We sketch a proof using a result from [BEST23]. The *dual* $\mathrm{M}^\perp$ of a matroid M is the matroid whose rank function is given by $\mathrm{rk}_{\mathrm{M}^\perp}(S) = |S| - \mathrm{rk}_{\mathrm{M}}(E) + \mathrm{rk}_{\mathrm{M}}(E \setminus S)$.

Proposition 4.5.2. *Let* M *be a loopless matroid of rank* r *with* c *connected components. Then, we have*

$$\omega(\mathrm{M}) = (-1)^{r-c}\chi(\mathrm{M}, \mathcal{L}_{\mathrm{M}^\perp}^{-1}).$$

A geometric proof for realizable matroids of the above result was sketched in Section 1.3, as the line bundle on W_L denoted $\det \mathcal{Q}_L$ is $\mathcal{L}_{\mathrm{M}^\perp}$.

Proof of Proposition 4.5.2. By Definition 3.4.3, we have

$$\chi(\mathrm{M}, \mathcal{L}_{\mathrm{M}^\perp}^{-1}) = \deg_{\mathrm{M}}(\zeta_{\mathrm{M}}([\mathcal{L}_{\mathrm{M}^\perp}^{-1}]) \cdot (1 + h_E + h_E^2 + \cdots)).$$

We compute $\zeta_{\mathrm{M}}([\mathcal{L}_{\mathrm{M}^\perp}^{-1}])$ by computing on X_E. By Proposition 3.2.3, we have that

$$\zeta_{\mathrm{M}}([\mathcal{L}_{\mathrm{M}^\perp}^{-1}]) = \iota^*\zeta_E([\mathcal{L}_{\mathrm{M}^\perp}^{-1}]).$$

In [BEST23, Theorem 10.1], there is a description of ζ_E using torus localization on X_E; that the map denoted ζ there agrees with ζ_E follows from [BEST23, Corollary 10.6] (see [EHL23, Proof of Theorem 1.8]). Computing in the equivariant Chow groups of the permutohedral variety X_E using [BEST23, Theorem 10.1] (see [EHL23, Corollary 6.5]), we have that $\zeta_E([\mathcal{L}_{M^\perp}^{-1}])$ is the class denoted $c(\mathcal{Q}_M^\vee)$ in [BEST23]. Therefore $\zeta_M([\mathcal{L}_{M^\perp}^{-1}])$ is the restriction to $A^\bullet(M)$ of this class, and so the result follows from [BEST23, Theorem 10.12]. $\square$

Now, recall the formal identity satisfied by the h^*-vector

$$h_d^*(M, \mathcal{L}) = (-1)^d \chi(M, \mathcal{L}^{-1}), \tag{4.2}$$

where d is the degree of the Snapper polynomial. See, for instance, [Sta12, Section 4.3]. Moreover, when M is connected, the polytope $B(M^\perp)$ is full dimensional, so Theorem 4.4.2 implies that the degree d of the Snapper polynomial is $r-1$. Therefore, the two preceding propositions show that the non-negativity of the coefficients of $g_M(t)$ is a special case of Conjecture 4.3.1 with $P = M^\perp$.

We now make explicit how Theorem 4.0.2 proves the positivity of $\omega(M)$ in some special cases. For this, we will need some matroid terminology, see [Oxl11]. The first step is to express $[\mathcal{L}_{M^\perp}]$ as a Laurent monomial in the $[\mathcal{L}_{\Delta_F}]$, or, equivalently, write $c_1(\mathcal{L}_{M^\perp})$ as a linear combination of the h_F. To do so, for a matroid M, we recall its *β-invariant* [Cra67], defined by two properties:

- $\beta(U_{0,1}) = 0$, $\beta(U_{1,1}) = 1$, and $\beta(M) = 0$ if M is disconnected, and
- the recursive relation: for any i which is not a loop or coloop of M,

$$\beta(M) = \beta(M/i) + \beta(M \setminus i).$$

Equivalently, the β-invariant is the coefficient of x in the Tutte polynomial of M.

Proposition 4.5.3. *Let* M *be a loopless matroid on* $[n]$*. Then, the polytope* $B(M^\perp)$ *satisfies*

$$c_1(\mathcal{L}_{M^\perp}) = \sum_{\substack{F \text{ connected flat} \\ \text{of } \mathrm{rk}_M(F) \geq 2}} \sum_{\mathrm{cl}_M(S) = F} (-1)^{|S| - \mathrm{rk}_M(S) + 1} \beta(M|_S) h_F \in A^\bullet(M).$$

Proof. Let $B(\Delta_S)$ be the simplex $\mathrm{Conv}(\{\mathbf{e}_i : i \in S\})$. Then [ABD10, Theorem 2.6] expressed $B(\mathrm{M}^\perp)$ as a signed Minkowski sum of these simplices as follows:

$$B(\mathrm{M}^\perp) = \sum_{S \subseteq [n], |S| \geq 2} (-1)^{|S| - \mathrm{rk}_{\mathrm{M}}(S) + 1} \beta(\mathrm{M}|_S) B(\Delta_S) + \sum_{i \text{ loop of } \mathrm{M}} B(\Delta_i).$$

This gives an expression for $c_1(\mathcal{L}_{\mathrm{M}^\perp}) \in A^1(X_E)$ as a sum of the simplicial generators h_S. As the restriction of $h_S \in A^1(X_E)$ to $X_{\Sigma_{\mathrm{M}}}$ is equal to $h_{\mathrm{cl}_{\mathrm{M}}(S)} \in A^1(\mathrm{M})$ and $h_i = 0$, we obtain the desired expression. $\square$

Example 4.5.4. Let M be the graphical matroid associated to the complete graph K_4; this is a matroid on $E = \{1, 2, 3, 4, 5, 6\}$. There are 5 connected flats of rank at least two: the four triangles F_1, F_2, F_3, F_4 and E. Then Proposition 4.5.3 implies that

$$c_1(\mathcal{L}_{\mathrm{M}^\perp}) = -h_E + h_{F_1} + h_{F_2} + h_{F_3} + h_{F_4} \in A^1(\mathrm{M}).$$

Theorem 4.5.5. *Let* M *be a loopless matroid of rank* r *such that, for all connected flats* F *of* M *of rank at least* 2, *we have* $\sum_{\mathrm{cl}_{\mathrm{M}}(S) = F} (-1)^{|S| - \mathrm{rk}_{\mathrm{M}}(S) + 1} \beta(\mathrm{M}|_S) \geq 0$. *Then* $\omega(\mathrm{M}) \geq 0$.

Proof. First suppose that M is connected, so the polytope $B(\mathrm{M}^\perp)$ is full dimensional. By Theorem 4.4.2(2), the degree of the Snapper polynomial of $\mathcal{L}_{\mathrm{M}^\perp}$ is $r - 1$ in this case. By Theorem 4.0.2 along with (4.2), we thus have $\omega(\mathrm{M}) = (-1)^{r-1} \chi(\mathrm{M}, \mathcal{L}_{\mathrm{M}^\perp}^{-1}) = h^*_{r-1}(\mathrm{M}, \mathcal{L}_{\mathrm{M}^\perp}) \geq 0$.

Now suppose that $\mathrm{M} = \mathrm{M}_1 \oplus \cdots \oplus \mathrm{M}_c$, with each M_i connected. The hypothesis implies that each $\mathcal{L}_{\mathrm{M}_i^\perp}$ is simplicially positive for each i, and so $\omega(\mathrm{M}_i) \geq 0$. By [FS12, Proposition 7.2], $g_{\mathrm{M}}(t) = g_{\mathrm{M}_1}(t) \cdots g_{\mathrm{M}_c}(t)$. Because the t^i coefficient of $g_{\mathrm{M}}(t)$ vanishes for $i > \mathrm{rk}(\mathrm{M})$, we have

$$\omega(\mathrm{M}) = \omega(\mathrm{M}_1) \cdots \omega(\mathrm{M}_c) \geq 0. \qquad \square$$

Equivalently, Theorem 4.5.5 states that $\omega(\mathrm{M}) \geq 0$ if $\mathcal{L}_{\mathrm{M}^\perp}$ is simplicially positive. While it appears that this is not often satisfied, Theorem 4.5.5 does show that $\omega(\mathrm{M}) \geq 0$ for many matroids. We give two examples.

Example 4.5.6. For a non-empty subset $S \subseteq E$, let H_S be the corank 1 matroid on E with S as its unique circuit. A *co-transversal matroid* is a matroid M that arises as the matroid intersection $\mathrm{M} = \mathrm{H}_{S_1} \wedge \cdots \wedge \mathrm{H}_{S_c}$ for some (not necessarily distinct) subsets $S_1, \ldots, S_c$. In this case, one verifies that $c_1(\mathcal{L}_{\mathrm{M}^\perp}) = \sum_{i=1}^c h_{S_i} \in A^\bullet(\mathrm{M})$. Thus, Theorem 4.5.5 applies to all co-transversal matroids.

Co-transversal matroids are realizable over an infinite field of arbitrary characteristic, so we could have used [Spe09, Proposition 3.3] or Remark 4.6.3 below to deduce that $\omega(\mathrm{M}) \geq 0$. We now construct an infinite family of matroids to which Theorem 4.5.5 applies but which are not realizable over a field of characteristic 0, as follows. We will use the notion of *principal extensions*, whose definition and properties can be found in [Oxl11, §7.2].

Lemma 4.5.7. *Let* M *be a loopless matroid on* E*, and fix a non-empty flat* G*. Denote by* $\mathrm{M}' = \mathrm{M} +_G \star$ *the principal extension of* M *by* G*. Then, writing*

$$c_1(\mathcal{L}_{\mathrm{M}^\perp}) = \sum_{F \text{ a flat of } \mathrm{M}} c_F h_F \in A^\bullet(\mathrm{M}),$$

then the expression for $c_1(\mathcal{L}_{(\mathrm{M}')^\perp}) \in A^\bullet(\mathrm{M}')$ *is*

$$c_1(\mathcal{L}_{(\mathrm{M}')^\perp}) = h_{G\cup\star} + \sum_{F \supseteq G} c_F h_{F\cup\star} + \sum_{F \not\supseteq G} c_F h_F.$$

Proof. A computation using Proposition 2.3.5 gives that $c_1(\mathcal{L}_{\mathrm{M}^\perp}) = \sum_F \mathrm{rk}_\mathrm{M}(F) x_F$, so (2.3) implies that the coefficients c_F are defined by the property that $\sum_{F' \subseteq F} c_{F'} = \mathrm{rk}_\mathrm{M}(F)$ for all flats F of M.

Now, we recall that the set of flats of M′ is partitioned into three categories [Oxl11, Corollary 7.2.5]:

1. $\{F : F \text{ flat of M with } F \not\supseteq G\}$, in which case $\mathrm{rk}_{\mathrm{M}'}(F) = \mathrm{rk}_\mathrm{M}(F)$,
2. $\{F \cup \star : F \text{ flat of M with } F \supseteq G\}$, in which case $\mathrm{rk}_{\mathrm{M}'}(F \cup \star) = \mathrm{rk}_\mathrm{M}(F)$, and
3. $\{F \cup \star : F \text{ flat of M with } F \not\supseteq G,\ F \text{ is not covered by an element in } [G, E]\}$, in which case $\mathrm{rk}_{\mathrm{M}'}(F \cup \star) = \mathrm{rk}_\mathrm{M}(F) + 1$.

Thus, in $A^\bullet(\mathrm{M}')$, since $h_\star = 0$ so that $-x_{E\cup\star} = \sum_{\emptyset\subseteq F\subsetneq E} x_{F\cup\star}$, we have

$$h_{G\cup\star} = \sum_{\emptyset\subseteq F\subsetneq E} x_{F\cup\star} + \sum_{G\subseteq F\subsetneq E} -x_{F\cup\star} = \sum_{F\not\supseteq G} x_{F\cup\star}.$$

The claimed expression for $c_1(\mathcal{L}_{(\mathrm{M}')^\perp})$ in all three cases of flats now follows, as the above expression for $h_{G\cup\star}$ contributes only to the case (iii) and not to cases (i) or (ii). Explicitly, we have:

1. In this case, the coefficient of x_F is $\sum_{F'\subseteq F} c_{F'} = \mathrm{rk}_\mathrm{M}(F) = \mathrm{rk}_{\mathrm{M}'}(F)$.
2. In this case, the coefficient of $x_{F\cup\star}$ is again $\sum_{F'\subseteq F} c_{F'} = \mathrm{rk}_\mathrm{M}(F) = \mathrm{rk}_{\mathrm{M}'}(F\cup\star)$.
3. In this case, the coefficient of $x_{F\cup\star}$ is

$$1 + \sum_{F'\subseteq F} c_{F'} = 1 + \mathrm{rk}_\mathrm{M}(F) = \mathrm{rk}_{\mathrm{M}'}(F\cup\star). \qquad \square$$

Given any loopless matroid M, repeatedly applying the lemma provides a method for constructing a matroid $\widetilde{\mathrm{M}}$ for which Theorem 4.5.5 applies. Moreover, a matroid is realizable over an infinite field $\mathbf{k}$ if and only if its principal extensions are realizable over the same field $\mathbf{k}$. Thus, the matroid $\widetilde{\mathrm{M}}$ has the same realizability property as M over infinite fields. In particular, the lemma produces infinite families of matroids that are not realizable or are realizable only in certain positive characteristics for which Theorem 4.5.5 applies.

4.6 Examples and problems

We present a few cases in which Conjecture 4.3.1 holds.

Example 4.6.1. When M is the Boolean matroid, then $h^*(\mathrm{M}, \mathcal{L}_\mathrm{P})$ is the usual h^*-vector of the base polytope $B(\mathrm{P})$, and hence is non-negative. Moreover, because base polytopes of polymatroids have the property that every lattice point in $kB(\mathrm{P})$ is a sum of k lattice points in $B(\mathrm{P})$ (see [Wel76, Chapter 18.6, Theorem 3]), $h^*(\mathrm{M}, \mathcal{L}_\mathrm{P})$ is a Macaulay vector.

Example 4.6.2. Let ∇ be the uniform matroid of corank 1, i.e., we have $B(\nabla) = \mathrm{Conv}(\{(0,1,\ldots,1),(1,0,1,\ldots,1),\ldots,(1,1,\ldots,1,0)\})$ and $c_1(\mathcal{L}_\nabla) \in A^1(\mathrm{M})$ is the class usually denoted $-x_\emptyset = \beta$. Then [LLPP24, Lemma 8.5] implies that

$$\chi(\mathrm{M}, \mathcal{L}_\nabla^{\otimes t}) = \sum_i f_{r-1-i}(BC_>(\mathrm{M}))\binom{t}{r-1-i},$$

where $f_j(BC_>(\mathrm{M}))$ is the number of j-dimensional faces of the reduced broken circuit complex of M under any ordering $>$. As $\binom{t}{r-1-i} = \sum_{j=0}^{i}(-1)^j\binom{i}{j}\binom{t+i}{r-1}$, we may express the Snapper polynomial in terms of the h-vector of the reduced broken circuit complex:

$$\chi(\mathrm{M}, \mathcal{L}_\nabla^{\otimes t}) = \sum_i h_{r-1-i}(BC_>(\mathrm{M}))\binom{t+i}{r-1}.$$

Comparing this with the definition of $h^*(\mathrm{M}, \mathcal{L}_\nabla)$, we have $h_i(BC_>(\mathrm{M})) = h_i^*(\mathrm{M}, \mathcal{L}_\nabla)$. By [Bjö92], the reduced broken circuit complex is shellable and therefore Cohen–Macaulay, so its h-vector is a Macaulay vector. This argument is closely related to [PS06].

Example 4.6.3. Let M be a connected matroid which has a realization $L \subseteq \mathbf{k}^E$ over a field of characteristic 0. It follows from [BEST23, Theorem 7.10] that there is a vector bundle $\mathcal{Q}_L$ of rank $n-r$ on X_E which has a section which cuts out $W_L \subseteq X_E$, so there is a Koszul resolution

$$0 \to \wedge^{n-r}\mathcal{Q}_L^\vee \to \wedge^{n-r-1}\mathcal{Q}_L^\vee \to \cdots \to \mathcal{Q}_L^\vee \to \mathcal{O}_{X_E} \to \mathcal{O}_{W_L} \to 0.$$

By [BF22, Theorem 5.1], we have that

$$H^j(X_E, \wedge^{n-r-i}\mathcal{Q}_L \otimes (\det \mathcal{Q}_L)^{\otimes(k-1)}) = 0 \text{ for } j > 0, k \geq 1.$$

Recall that $\wedge^{n-r-i}\mathcal{Q}_L \otimes (\det \mathcal{Q}_L)^{\otimes(k-1)} \cong \wedge^i\mathcal{Q}_L^\vee \otimes (\det \mathcal{Q}_L)^{\otimes k}$. Using an equivariant description of the class $[\mathcal{Q}_L] \in K_T(X_E)$ [BEST23, Proposition 3.7], one can check that $\det \mathcal{Q}_L = \mathcal{L}_{\mathrm{M}^\perp}$; see [EHL23, Proposition 4.6] for a similar computation. By [Laz04, Proposition B.1.2], for $k \geq 1$, the above cohomology vanishing result implies that

$H^i(W_L, \mathcal{L}_{\mathrm{M}^\perp}^{\otimes k}) = 0$ for $i > 0$, and the restriction map $H^0(X_E, \mathcal{L}_{\mathrm{M}^\perp}^{\otimes k}) \to H^0(W_L, \mathcal{L}_{\mathrm{M}^\perp}^{\otimes k})$ is surjective. By [Wel76, Chapter 18.6, Theorem 3], the ring $\oplus_{k\geq 0} H^0(X_E, \mathcal{L}_{\mathrm{M}^\perp}^{\otimes k})$ is generated in degree 1. Therefore the ring

$$R^\bullet := \bigoplus_{k\geq 0} H^0(W_L, \mathcal{L}_{\mathrm{M}^\perp}^{\otimes k})$$

is generated in degree 1. This implies that $\operatorname{Proj} R^\bullet$ is the image of W_L under the complete linear system of $\mathcal{L}_{\mathrm{M}^\perp}$ which is the *space of visible contours* of L. It is proved in [Tev07, Theorem 1.4 and 1.5] that $\operatorname{Proj} R^\bullet$ has rational singularities. In particular,

$$H^i(W_L, \mathcal{L}_{\mathrm{M}^\perp}^{\otimes k}) = H^i(\operatorname{Proj} R^\bullet, \mathcal{O}(k))$$

for all i and k. Because $B(\mathrm{M}^\perp)$ is full dimensional, the line bundle $\mathcal{L}_{\mathrm{M}^\perp}$ is nef and big. By the Kawamata–Viehweg vanishing theorem, $H^i(W_L, \mathcal{L}_{\mathrm{M}^\perp}^{\otimes k}) = H^i(\operatorname{Proj} R^\bullet, \mathcal{O}(k)) = 0$ for $k < 0$ and $i < \dim W_L$. As W_L is rational, $H^i(W_L, \mathcal{O}_{W_L}) = 0$ for $i > 0$. Then Proposition 4.1.2 implies that $R^\bullet$ is Cohen–Macaulay, and so $h^*(\mathrm{M}, \mathcal{L}_{\mathrm{M}^\perp})$ is a Macaulay vector.

We conjecture a monotonicity property for h^*-vectors of matroids, inspired by Stanley's monotonicity result for h^*-vectors of polytopes [Sta93], which implies the following conjecture when M is the Boolean matroid.

Conjecture 4.6.4. *Let* $\mathrm{P}_1, \mathrm{P}_2$ *be polymatroids with* $B(\mathrm{P}_1) \subseteq B(\mathrm{P}_2)$*. Then for any loopless matroid* M*,* $h_i^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}_1}) \leq h_i^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}_2})$ *for all* i*.*

If the degree of the Snapper polynomial of $\mathcal{L}$ is $\mathrm{rk}(\mathrm{M}) - 1$, then $\sum h_i^*(\mathrm{M}, \mathcal{L}) = \deg_{\mathrm{M}}(c_1(\mathcal{L})^{r-1})$, so the following result gives evidence for Conjecture 4.6.4.

Proposition 4.6.5. *Let* $\mathrm{P}_1, \mathrm{P}_2$ *be polymatroids with* $B(\mathrm{P}_1) \subseteq B(\mathrm{P}_2)$*. Then*

$$\deg_{\mathrm{M}}(c_1(\mathcal{L}_{\mathrm{P}_1})^{r-1}) \leq \deg_{\mathrm{M}}(c_1(\mathcal{L}_{\mathrm{P}_2})^{r-1}).$$

Proof. Because $B(\mathrm{P}_1) \subseteq B(\mathrm{P}_2)$, the difference of the divisor class in $A^1(X_E)$ corresponding to $B(\mathrm{P}_2)$ with the divisor class corresponding to $B(\mathrm{P}_1)$ is an effective divisor class, see Section 2.3. Then,

$$\begin{aligned}&c_1(\mathcal{L}_{\mathrm{P}_2})^{r-1} - c_1(\mathcal{L}_{\mathrm{P}_1})^{r-1}\\ &= (c_1(\mathcal{L}_{\mathrm{P}_2}) - c_1(\mathcal{L}_{\mathrm{P}_1})) \cdot (c_1(\mathcal{L}_{\mathrm{P}_2})^{r-2} + c_1(\mathcal{L}_{\mathrm{P}_2})^{r-3}c_1(\mathcal{L}_{\mathrm{P}_1}) + \cdots + c_1(\mathcal{L}_{\mathrm{P}_1})^{r-2}).\end{aligned}$$

By Proposition 4.4.3, the degree of this class is non-negative. □

In [Sta91], Stanley proved restrictions on the Hilbert function of a graded Cohen–Macaulay integral domain. When M is realized by $L \subseteq \mathbf{k}^E$, then the ring $R^\bullet = \oplus_{k \geq 0} H^0(W_L, \mathcal{L}_{\mathrm{P}}^{\otimes k})$ is an integral domain. If $R^\bullet$ is Cohen–Macaulay, then [Sta91, Theorem 2.1] implies Conjecture 4.6.6. If $R^\bullet$ is Cohen–Macaulay and $\mathbf{k}$ has characteristic 0, then [Sta91, Proposition 3.4] implies Conjecture 4.6.7.

Conjecture 4.6.6. *Let* M *be a loopless matroid, and let* P *be a polymatroid. Let s be the largest integer such that $h_s^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}}) \neq 0$. Then, for all $0 \leq i \leq s$,*

$$h_0^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}}) + \cdots + h_i^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}}) \leq h_s^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}}) + \cdots + h_{s-i}^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}}).$$

Conjecture 4.6.7. *Let* M *be a loopless matroid, and let* P *be a polymatroid. Let s be the largest integer such that $h_s^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}}) \neq 0$. Then, for all $m \geq 0$ and $n \geq 1$ with $m + n < s$,*

$$h_1^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}}) + \cdots + h_n^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}}) \leq h_{m+1}^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}}) + \cdots + h_{m+n}^*(\mathrm{M}, \mathcal{L}_{\mathrm{P}}).$$

If $\mathcal{L}$ is a simplicially positive line bundle on $X_{\Sigma_{\mathrm{M}}}$, then Proposition 2.5.1 and a Frobenius splitting argument implies that, for any realization of M, the ring section ring $R^\bullet$ is Cohen–Macaulay. In particular, in this case Conjecture 4.6.6 holds if M is realizable, and Conjecture 4.6.7 holds if M is realizable over a field of characteristic 0.

By Example 4.6.3, Conjecture 4.6.6 and Conjecture 4.6.7 hold for $h^*(\mathrm{M}, \mathcal{L}_{\mathrm{M}^\perp})$ when M is realizable over a field of characteristic 0. In the case of Example 4.6.2, the inequalities in Conjecture 4.6.6 and Conjecture 4.6.7 hold by [ADH23, Theorem 1.4].

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