DETERMINANTS OF HODGE–RIEMANN FORMS AND SIMPLICIAL MANIFOLDS

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Abstract. We calculate the determinant of the bilinear form in middle degree of the generic artinian reduction of the Stanley–Reisner ring of an odd-dimensional simplicial sphere. This proves the odd multiplicity conjecture of Papadakis and Petrotou and implies that this determinant is a complete invariant of the simplicial sphere. We extend this result to odd-dimensional connected oriented simplicial homology manifolds, and we conjecture a generalization to the Hodge–Riemann forms of any connected oriented simplicial homology manifold. We show that our conjecture follows from the strong Lefschetz property for certain quotients of the Stanley–Reisner rings.

1. Introduction

Let Δ be a simplicial complex with vertex set $V = \{1, \dots, n\}$ of dimension $d - 1 > 0$. Let k be a field, and set $K = k(a_{i,j})_{1\leq i\leq d,\, 1\leq j\leq n}$. We assume that Δ is a connected homology manifold over k, i.e., Δ is connected, and the link of every nonempty face G of Δ has the same homology as a sphere of dimension $d - |G| - 1$ over k. Let $K[\Delta]$ be the Stanley–Reisner ring of Δ, and set $\theta_i = a_{i,1}x_1 + \cdots + a_{i,n}x_n \in K[\Delta]$ for $i \in \{1, \dots, d\}$, so $\theta_1, \dots, \theta_d$ is a linear system of parameters for $K[\Delta]$. Let $H(\Delta) = K[\Delta]/(\theta_1, \dots, \theta_d)$ be the generic artinian reduction of $K[\Delta]$.

Assume that Δ is oriented. Then there is a distinguished isomorphism $\deg\colon H^d(\Delta) \to K$ [Bri97], see Section 2. Let $\overline{H}(\Delta)$ be the Gorenstein quotient of $H(\Delta)$, i.e., the quotient by the ideal $(y \in H(\Delta) : (y\cdot z)_d = 0$ for all $z \in H(\Delta))$, where y_d denotes the degree d component of y in $H(\Delta)$. One has $\overline{H}^q(\Delta) = H^q(\Delta)$ for $q \in \{0, 1, d\}$; see, for example, Proposition 4.4. Also, if Δ is a homology sphere over k, i.e., a homology manifold with the same homology over k as a sphere of dimension $d-1$, then $\overline{H}(\Delta) = H(\Delta)$. By construction, $\overline{H}(\Delta)$ is an artinian Gorenstein ring: for each q, the bilinear form $\overline{H}^q(\Delta) \times \overline{H}^{d-q}(\Delta) \to K$ given by $(y, z) \mapsto \deg(y \cdot z)$ is nondegenerate.

Suppose that d is even. Let $D_{d/2} \in K^\times/(K^\times)^2$ be the determinant of the nondegenerate bilinear form on $\overline{H}^{d/2}(\Delta)$. That is, choose a basis $y_1, \dots, y_p$ for $\overline{H}^{d/2}(\Delta)$, and let M be the symmetric matrix whose (i, j)th entry is $\deg(y_i \cdot y_j)$. Then $D_{d/2}$ is the image of $\det M$ in $K^\times/(K^\times)^2$; choosing a different basis for $\overline{H}^{d/2}(\Delta)$ only changes $\det M$ by a square, so $D_{d/2}$ is well-defined. For a subset $F = \{j_1 < \cdots < j_d\}$ of V of size d, set $[F]$ to be the determinant of the $d \times d$ matrix whose (i, m)th entry is a_{i,j_m}.

Theorem 1.1. *Let d be even, and let Δ be a connected oriented simplicial k-homology manifold of dimension $d - 1$. Then*

$$D_{d/2} = \lambda \prod_{F \text{ facet of } \Delta} [F] \in K^\times/(K^\times)^2$$

for some $\lambda \in k^\times/(k^\times)^2$.

Papadakis and Petrotou proved Theorem 1.1 for 1-dimensional simplicial spheres [PP23, Proposition 5.1].

Let F be a subset of V of size d. As $[F]$ is an irreducible polynomial (see Lemma 4.1), it defines a valuation $\mathrm{ord}_{[F]}\colon K^\times \to \mathbb{Z}$ given by the order of vanishing along the hypersurface defined by $[F]$. This descends to a homomorphism $\mathrm{ord}_{[F]}\colon K^\times/(K^\times)^2 \to \mathbb{Z}/2\mathbb{Z}$. We immediately deduce the following corollary to Theorem 1.1. It implies that the determinant of the bilinear form on $\overline{H}^{d/2}(\Delta)$ is a complete invariant of the connected oriented simplicial k-homology manifold Δ.

Corollary 1.2. *Let d be even, and let Δ be a connected oriented simplicial k-homology manifold of dimension $d-1$ with vertex set V. Let F be a subset of V of size d. Then*

$$\operatorname{ord}_{[F]}(D_{d/2}) = \begin{cases} 1 & \text{if } F \text{ is a facet of } \Delta \\ 0 & \text{if otherwise.} \end{cases}$$

When Δ is a simplicial sphere, Corollary 1.2 was conjectured by Papadakis and Petrotou [PP23, Conjecture 5.4], who called it the *odd multiplicity conjecture*. This conjecture has motivated our work.

We conjecture a generalization of the odd multiplicity conjecture. Let $\ell = \sum_{j=1}^{n} x_j \in \overline{H}^1(\Delta)$. For $0 \leq q \leq d/2$, define the Hodge–Riemann form $\overline{H}^q(\Delta) \times \overline{H}^q(\Delta) \to K$ via $(y, z) \mapsto \deg(\ell^{d-2q} \cdot y \cdot z)$. When d is even and $q = d/2$, the Hodge–Riemann form is the bilinear form on $\overline{H}^{d/2}(\Delta)$ considered above, and Conjecture 1.3 below is Corollary 1.2. Let D_q be the determinant of the Hodge–Riemann form on $\overline{H}^q(\Delta)$.

Conjecture 1.3. *Let Δ be a connected oriented simplicial k-homology manifold of dimension $d-1$ with vertex set V, and let $0 \leq q \leq d/2$. Let F be a subset of V of size d. Then*

$$\operatorname{ord}_{[F]}(D_q) = \begin{cases} 1 & \text{if } F \text{ is a facet of } \Delta \\ 0 & \text{if otherwise.} \end{cases}$$

The nondegeneracy of the Hodge–Riemann form, which is part of Conjecture 1.3, is equivalent to the map $\overline{H}^q(\Delta) \to \overline{H}^{d-q}(\Delta)$ given by multiplication by ℓ^{d-2q} being an isomorphism. By Lemma 4.5, this is equivalent to $\overline{H}(\Delta)$ having the strong Lefschetz property in degree q, i.e., that there is some $y \in \overline{H}^1(\Delta)$ such that the map $\overline{H}^q(\Delta) \to \overline{H}^{d-q}(\Delta)$ given by multiplication by y^{d-2q} is an isomorphism.

In particular, Conjecture 1.3 is a generalization of the algebraic g-conjecture for Δ (that $\overline{H}(\Delta)$ has the strong Lefschetz property), and it implies that the Hodge–Riemann form in any degree is a complete invariant of Δ. A proof of the algebraic g-conjecture for connected oriented simplicial k-homology manifolds was announced in [APP21], see also [Adi18, KX23, PP20]. While our work does not directly rely on it, we have been heavily inspired by the recent progress on the algebraic g-conjecture, and, in particular, the key insight that one should study the generic artinian reduction of $K[\Delta]$ and the corresponding degree map. See also [APP24].

In Example 4.11, we verify Conjecture 1.3 for simplicial spheres obtained from the boundary of the d-dimensional simplex by successively doing stellar subdivisions in the interiors of facets. In Conjecture 5.1, we give an alternative statement that we show is equivalent to Conjecture 1.3 holding for all $0 \leq q \leq d/2$.

We can verify Conjecture 1.3 when $q = 0$. As $\overline{H}^0(\Delta)$ is 1-dimensional, the determinant of the Hodge–Riemann form is equal to the image of $\deg(\ell^d)$ in $K^\times/(K^\times)^2$. Then we can prove a more precise result.

Theorem 1.4. *Let Δ be a connected oriented simplicial k-homology manifold of dimension $d-1$ with vertex set V. Let F be a subset of V of size d. Then*

$$\operatorname{ord}_{[F]}(\deg(\ell^d)) = \begin{cases} -1 & \text{if } F \text{ is a facet of } \Delta \\ 0 & \text{if otherwise.} \end{cases}$$

We show that Conjecture 1.3 follows from a strengthening of the algebraic g-conjecture for less generic artinian reductions of $K[\Delta]$. Let F be a subset of V of size d which is not a facet of Δ, and set $\theta_1^F = \sum_{j \notin F} a_{1,j}$. Then $\theta_1^F, \theta_2, \ldots, \theta_d$ is still a linear system of parameters for $K[\Delta]$ by Stanley's criterion (see Proposition 2.2).

Let $H_F(\Delta) = K[\Delta]/(\theta_1^F, \theta_2, \dots, \theta_d)$. Then there is a distinguished isomorphism $\deg\colon H_F^d(\Delta) \to K$ (see Section 2). Set $\overline{H}_F(\Delta)$ to be the Gorenstein quotient of $H_F(\Delta)$. For example, if Δ is a homology sphere over k, then $H_F(\Delta) = \overline{H}_F(\Delta)$.

Conjecture 1.5. *Let Δ be a connected oriented simplicial k-homology manifold of dimension $d-1$, and let $0 \le q \le d/2$. Then for every non-face F of size d, $\overline{H}_F(\Delta)$ has the strong Lefschetz property in degree q.*

When d is even and $q = d/2$, Conjecture 1.5 is vacuously true. The following result is then a generalization of Corollary 1.2.

Theorem 1.6. *If Conjecture 1.5 holds for all Δ of dimension $d-1$ in degree q, then Conjecture 1.3 holds for all Δ of dimension $d-1$ in degree q.*

In order to deduce Conjecture 1.3 for Δ, we use Conjecture 1.5 for Δ and for the stellar subdivisions of Δ in the interiors of facets.

We also show that, when $q = 0$, Conjecture 1.5 is a consequence of Theorem 1.4. See Remark 4.3.

Our paper is organized as follows. In Section 2, we recall the construction and properties of the degree map. In Section 3, we compute some special cases which will be used in the proofs of the main theorems. In Section 4, we prove the main theorems. In Section 5, we give some examples and discuss possible extensions.

Throughout, we fix a connected oriented simplicial k-homology manifold Δ of dimension $d-1$ with vertex set V. If G is a face of Δ with vertices $\{j_1, \dots, j_r\}$, we write $x_G \coloneqq x_{j_1} \cdots x_{j_r}$ for the corresponding monomial in $K[\Delta]$. We will sometimes abuse notation and use x_G to denote its image in $H(\Delta)$ or $\overline{H}(\Delta)$. See [Sta84] for any undefined terminology.

We will assume throughout that $d > 1$. If $d = 1$, the (not connected) case of a simplicial sphere of dimension 0, i.e., Δ consists of two points, is discussed in Example 3.3.

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2. Degree maps

We now discuss degree maps on artinian reductions of Stanley–Reisner rings of connected oriented simplicial k-homology manifolds. The normalization of the degree map will be crucial in what follows, as the results of the introduction can fail if we use an arbitrary isomorphism $H^d(\Delta) \to K$. Explicitly, two such isomorphisms vary by multiplication by a nonzero element $\omega \in K$, and if $p = \dim \overline{H}^q(\Delta)$ is odd, then the determinant of a nondegenerate bilinear form on $\overline{H}^q(\Delta)$ will vary by multiplication by $\omega^p = \omega \in K^\times/(K^\times)^2$.

We first discuss orientations in the case when the characteristic of k is not 2. If $d > 1$, then an *orientation* on a $(d-1)$-dimensional simplex is a choice of ordering of the vertices, up to changing the ordering by an even permutation. If $d = 1$, then an orientation on a $(d-1)$-dimensional simplex is a choice of $\epsilon \in \{1, -1\}$. An orientation on a $(d-1)$-dimensional simplex induces an orientation on each facet. If $d > 1$ and the simplex is ordered by $\{v_1 < \cdots < v_d\}$, then we orient $\{v_2, \dots, v_d\}$ using the ordering $v_2 < \cdots < v_d$, and we orient the facet which omits v_i by changing the ordering by even permutations so that v_i is first. If $d = 1$ and the simplex is $\{v_1 < v_2\}$, then we orient $\{v_1\}$ by -1 and orient $\{v_2\}$ by 1.

Because Δ is a k-homology manifold, each $(d-2)$-dimensional simplex is contained in exactly two facets (as the link must be S^0). An *orientation* of Δ is a choice of orientation for each facet of Δ such that the two orientations on any $(d-2)$-dimensional simplex of Δ induced by the two facets containing it are opposite. In what follows, we fix a choice of orientation.

If k has characteristic 2, then we say that any k-homology manifold is oriented by definition.

For each facet $F = \{j_1 < \cdots < j_d\}$, the orientation on Δ defines a sign $\epsilon_F \in \{1, -1\}$, which is 1 if the permutation which takes $(j_1, \dots, j_d)$ to the ordering given by the orientation is even, and is -1 if this permutation is odd. If the characteristic of k is 2, then $\epsilon_F = 1$ by definition.

There is an explicit isomorphism $\deg\colon H^d(\Delta) \to K$, called the *degree map*. This isomorphism was constructed by Brion [Bri97], see also [KX23, Lemma 2.2]. Recall that, for a subset $F = \{j_1 < \cdots < j_d\}$ of V of size d, $[F]$ is the determinant of the matrix whose (i, m)th entry is a_{i,j_m}.

Proposition 2.1. *There is an isomorphism* $\deg\colon H^d(\Delta) \to K$ *of* K*-vector spaces such that, for any facet* F *of* Δ*, we have*

$$\deg(x_F) = \frac{\epsilon_F}{[F]}. \tag{1}$$

In particular, if k does not have characteristic 2, then the degree map associated to the opposite orientation is the negative of the original degree map.

More generally, consider d elements $\mu = (\mu_1, \dots, \mu_d)$ in $K[\Delta]$ of degree 1, with $\mu_i = \sum_{j \in V} \mu_{i,j} x_j$ for some $\mu_{i,j} \in K$. Let $k[a_{i,j}]$ denote the polynomial ring $k[a_{i,j}]_{1 \leq i \leq d,\, 1 \leq j \leq n}$ with fraction field K, and consider the k-algebra homomorphism $\mathrm{ev}_\mu\colon k[a_{i,j}] \to K$ defined by

$$\mathrm{ev}_\mu(a_{i,j}) = \mu_{i,j}.$$

We will use the following criterion for the elements of μ to be a linear system of parameters (l.s.o.p.).

Proposition 2.2. *(Stanley's criterion)* [Sta92, Proposition 4.3] *Consider* d *elements* $\mu = (\mu_1, \dots, \mu_d)$ *in* $K[\Delta]$ *of degree* 1*. Then* $\mu_1, \dots, \mu_d$ *is an l.s.o.p. if and only if* $\mathrm{ev}_\mu([F]) \neq 0$ *for each facet* F *of* Δ.

Suppose that $\mu = (\mu_1, \dots, \mu_d)$ is an l.s.o.p. Let $H_\mu(\Delta) := K[\Delta]/(\mu_1, \dots, \mu_d)$. We still have $\dim H^d_\mu(\Delta) = 1$ (see, for example, [Sch81]), and so the degree map described in Proposition 2.1 "specializes" to an isomorphism $\deg_\mu\colon H^d_\mu(\Delta) \to K$ of K-vector spaces such that, for a fixed choice of facet F of Δ,

$$\deg_\mu(x_F) = \frac{\epsilon_F}{\mathrm{ev}_\mu([F])}. \tag{2}$$

We will verify below that (2) is independent of the choice of facet F. We also have a well-defined Gorenstein quotient $\overline{H}_\mu(\Delta)$, i.e., the quotient of $H_\mu(\Delta)$ by the ideal $(y \in H_\mu(\Delta) : (y \cdot z)_d = 0 \text{ for all } z \in H_\mu(\Delta))$, where y_d denotes the degree d component of y in $H_\mu(\Delta)$. For example, as in the statement of Conjecture 1.5, let F be a subset of V of size d which is not a facet of Δ, and set $\theta_1^F = \sum_{j \notin F} a_{1,j}$. Then $\theta_F = (\theta_1^F, \theta_2, \dots, \theta_d)$ is an l.s.o.p., and we write $H_F(\Delta) := H_{\theta_F}(\Delta)$, $\overline{H}_F(\Delta) := \overline{H}_{\theta_F}(\Delta)$, and $\deg_F := \deg_{\theta_F}$.

We now describe two known techniques that can be used to compute the degree map. We first recall the following application of Cramer's rule, see, e.g., [PP23, Proposition 2.1]. Below, $\mathrm{sgn}(\pi) \in \{\pm 1\}$ denotes the sign of a permutation π.

Lemma 2.3. *Let* $\mu = (\mu_1, \dots, \mu_d)$ *be an l.s.o.p. Let* $F = \{j_1 < \cdots < j_d\}$ *be a subset of* V *of size* d.

Fix $1 \leq m \leq d$*. Then*

$$\mathrm{ev}_\mu([F]) x_{j_m} = -\sum_{v \in V \smallsetminus F} \mathrm{sgn}(\pi_v)\, \mathrm{ev}_\mu([F \cup \{v\} \smallsetminus \{j_m\}]) x_v \in H^1_\mu(\Delta), \tag{3}$$

where $\pi_v \in S_d$ *is the permutation such that the elements of* $\pi_v \cdot (j_1, \dots, j_{m-1}, v, j_{m+1}, \dots, j_d)$ *are in increasing order.*

Suppose that F and F' are facets of Δ. It is well-known that there is a sequence of facets $F = F_1, F_2, \dots, F_s = F'$, where F_j and F_{j+1} meet along a common face of dimension $d-2$ for $1 \leq j < s$. Suppose that $F = \{j_1 < \cdots < j_d\}$ and F' meet along the common face $F \smallsetminus \{j_m\}$. Then multiplying (3) by x_F/x_{j_m} and tracing through the signs yields that $\epsilon_F\, \mathrm{ev}_\mu([F']) = \epsilon_{F'}\, \mathrm{ev}_\mu([F]) \in H^d_\mu(\Delta)$. We conclude that (2) holds for any facet F of Δ.

Given a nonzero monomial $x_{j_1}^{b_{j_1}} \cdots x_{j_s}^{b_{j_s}} \in K[\Delta]$ with each $b_j > 0$, define its *support* to be the face $\{j_1, \dots, j_s\}$ of Δ. Suppose that the above monomial is not squarefree, i.e., $b_{j_m} > 1$ for some $1 \leq m \leq s$. Let F be a facet containing the support $\{j_1, \dots, j_s\}$. Then Lemma 2.3 implies that

$$x_{j_1}^{b_{j_1}} \cdots x_{j_s}^{b_{j_s}} = -\frac{1}{\mathrm{ev}_\mu([F])} \sum_{v \in V \smallsetminus F} \mathrm{sgn}(\pi_v)\, \mathrm{ev}_\mu([F \cup \{v\} \smallsetminus \{j_m\}]) \frac{x_v \cdot x_{j_1}^{b_{j_1}} \cdots x_{j_s}^{b_{j_s}}}{x_{j_m}} \in H_\mu(\Delta), \tag{4}$$

for some permutations π_v as defined in Lemma 2.3. Importantly, all nonzero monomials on the right-hand side of (4) have support strictly containing the support of $x_{j_1}^{b_{j_1}} \cdots x_{j_s}^{b_{j_s}}$. Hence we may compute the degree of any monomial by using (4) to repeatedly increase the size of the support.

We will need the following lemma. Let $R \subset K$ be the localization of $k[a_{i,j}]$ at the irreducible polynomials $\{[F] : F \text{ facet of } \Delta\}$. By Proposition 2.2, ev_μ extends to a k-algebra homomorphism $\mathrm{ev}_\mu \colon R \to K$.

Lemma 2.4. *Let $\mu = (\mu_1, \dots, \mu_d)$ be an l.s.o.p. Let $g \in k[x_1, \dots, x_n]_d$ be a polynomial of degree d. Then $\deg(g) \in R$ and $\deg_\mu(g) = \mathrm{ev}_\mu(\deg(g))$.*

Proof. It is enough to consider the case when g is a monomial. If g is squarefree, then the result follows from (2). If g is not squarefree, then the result follows by using (4) to repeatedly increase the size of the support. □

We will apply Lemma 2.4 in combination with the following simple observation. We will often use the remark below with $P = [F]$ for some non-face F of size d.

Remark 2.5. Consider an element $f \in R$. Let $P \in k[a_{i,j}]$ be an irreducible polynomial, and suppose that there is an l.s.o.p. μ with $\mathrm{ev}_\mu(P) = 0$, but $\mathrm{ev}_\mu(f) \neq 0$. We claim that $\mathrm{ord}_P(f) = 0$. Indeed, because $\mathrm{ev}_\mu(P) = 0$, P is not a scalar multiple of any $\{[F] : F \text{ facet of } \Delta\}$, so $\mathrm{ord}_P(f) \geq 0$. But P can't divide f to positive order as $\mathrm{ev}_\mu(f) \neq 0$.

We next recall a formula for the degree map due to Karu and Xiao. It is closely related to the work of Brion [Bri97] as well as a formula of Lee [Lee96, Corollary 4.5]. To state the formula, we define $\widehat{V} \coloneqq \{0\} \cup V$ and $\widehat{K} \coloneqq K(a_{i,0} : 1 \leq i \leq d)$. For a subset $\widehat{F} = \{j_1 < \cdots < j_d\}$ of $\widehat{V}$ of size d, let $[\widehat{F}]$ be the determinant of the $d \times d$ matrix whose (i,m)th entry is a_{i,j_m}.

Proposition 2.6. [KX23, Lemma 3.1, Theorem 3.2] *Let $g \in K[x_1, \dots, x_n]_d$ be a polynomial of degree d. For any facet $F = \{j_1 < \cdots < j_d\}$ of Δ, let $g_F(t_1, \dots, t_d)$ be obtained from g by setting x_i to zero for $i \notin F$ and setting $x_{j_m} = t_m$ for $1 \leq m \leq d$. Let $X_{F,m} \coloneqq (-1)^m [F \cup \{0\} \smallsetminus \{j_m\}] \in \widehat{K}$ for $1 \leq m \leq d$. Then*

$$\deg(g) = \sum_{F \text{ facet of } \Delta} \frac{\epsilon_F g_F(X_{F,1}, \dots, X_{F,d})}{[F] \prod_{m=1}^d X_{F,m}}. \tag{5}$$

In particular, the expression in (5) lies in K. When $g \in k[x_1, \dots, x_n]$ this formula specializes to a formula for all l.s.o.p.'s. Explicitly, suppose that $\mu = (\mu_1, \dots, \mu_d)$ is an l.s.o.p. Recall that we have an evaluation map $\mathrm{ev}_\mu \colon k[a_{i,j}] \to K$ defined by $\mathrm{ev}_\mu(a_{i,j}) = \mu_{i,j}$ for $1 \leq i \leq d$ and $1 \leq j \leq n$. This naturally extends to a k-algebra homomorphism $\widehat{\mathrm{ev}}_\mu \colon k[a_{i,j}]_{1 \leq i \leq d, 0 \leq j \leq n} \to \widehat{K}$ such that $\widehat{\mathrm{ev}}(a_{i,0}) = a_{i,0}$ for $1 \leq i \leq d$. Below, if $F = \{j_1 < \cdots < j_d\}$ is a facet of Δ and $1 \leq m \leq d$, observe that $\widehat{\mathrm{ev}}_\mu([F \cup \{0\} \smallsetminus \{j_m\}])$ is nonzero since it specializes (up to a sign) to $\mathrm{ev}_\mu([F])$ by setting $a_{i,0}$ to μ_{i,j_m} for $1 \leq i \leq d$. By Proposition 2.2, $\mathrm{ev}_\mu([F]) \neq 0$.

Corollary 2.7. *Suppose that $\mu = (\mu_1, \dots, \mu_d)$ is an l.s.o.p. Let $g \in k[x_1, \dots, x_n]_d$ be a polynomial of degree d. For any facet $F = \{j_1 < \cdots < j_d\}$ of Δ, let $g_F(t_1, \dots, t_d)$ be obtained from g by setting x_i to zero for $i \notin F$ and setting $x_{j_m} = t_m$ for $1 \leq m \leq d$. Let $X_{F,\mu,m} \coloneqq (-1)^m \widehat{\mathrm{ev}}_\mu([F \cup \{0\} \smallsetminus \{j_m\}]) \in \widehat{K}$ for $1 \leq m \leq d$. Then*

$$\deg_\mu(g) = \sum_{F \text{ facet of } \Delta} \frac{\epsilon_F g_F(X_{F,\mu,1}, \ldots, X_{F,\mu,d})}{\mathrm{ev}_\mu([F]) \prod_{m=1}^d X_{F,\mu,m}}.$$

Proof. Recall that $R \subset K$ is the localization of $k[a_{i,j}]$ at the irreducible polynomials $\{[F] : F \text{ is a facet of } \Delta\}$, and ev_μ maps R to K which is contained in $\widehat{K}$. With the notation of Proposition 2.6, let $\widehat{R} \subset \widehat{K}$ be the localization of $k[a_{i,j}]_{1 \le i \le d, 0 \le j \le n}$ at the irreducible polynomials $\{X_{F,m} : F \text{ facet of } \Delta,\, 0 \le m \le d\}$, where $X_{F,0} := [F]$. Then $\widehat{\mathrm{ev}}_\mu$ extends to a k-algebra homomorphism $\widehat{\mathrm{ev}}_\mu \colon \widehat{R} \to \widehat{K}$ such that ev_μ is the restriction of $\widehat{\mathrm{ev}}_\mu$ to $R \subset \widehat{R}$. By Lemma 2.4, both sides of (5) lie in $\widehat{R}$ and $\widehat{\mathrm{ev}}_\mu(\deg(g)) = \deg_\mu(g)$. The result now follows by applying $\widehat{\mathrm{ev}}_\mu$ to both sides of (5). □

Finally, we show that the degree can be computed "locally" on Δ, in an appropriate sense. The *closed star* $\mathrm{Star}_\Delta(G)$ of a face G of Δ is the simplicial complex consisting of all faces G' of Δ that contain G, together with their subfaces. Let $G = \{j_1, \ldots, j_s\}$ be a face of Δ, and let S be the set of vertices in $\mathrm{Star}_\Delta(G)$. Let K_S be the subfield of K generated over k by $a_{i,j}$, where $1 \le i \le d$ and $j \in S$. Let Δ' be another connected oriented simplicial k-homology manifold of dimension d, with vertex set V' and a face $G' = \{j'_1, \ldots, j'_s\}$. Let $K' = k(a'_{i,j})_{1 \le i \le d, j \in V'}$ be the field of coefficients for $\overline{H}(\Delta')$. Suppose that there is an isomorphism of simplicial complexes $\tau \colon \mathrm{Star}_\Delta(G) \to \mathrm{Star}_{\Delta'}(G')$ that maps j_m to j'_m for $1 \le m \le s$. Then τ allows us to identify K_S with a subfield of K'. Let $\deg_\Delta \colon H^d(\Delta) \to K$ and $\deg_{\Delta'} \colon H^d(\Delta') \to K'$ denote the degree maps for Δ and Δ' respectively.

Lemma 2.8. *With the notation above, let $x_{j_1}^{b_1} \cdots x_{j_s}^{b_s}$ be a monomial of degree d with support G. Then $\deg_\Delta(x_{j_1}^{b_1} \cdots x_{j_s}^{b_s}) \in K_S$. Using the identification of K_S with a subfield of K', we have*

$$\deg_\Delta(x_{j_1}^{b_1} \cdots x_{j_s}^{b_s}) = \epsilon \deg_{\Delta'}(x_{j'_1}^{b_1} \cdots x_{j'_s}^{b_s}),$$

where $\epsilon = 1$ if the orientations on $\mathrm{Star}_\Delta(G)$ induced by the orientations on Δ and Δ' agree, and $\epsilon = -1$ if they are opposite.

Proof. The only nonzero terms in the right-hand side of the formula for $\deg_\Delta(x_{j_1}^{b_1} \cdots x_{j_s}^{b_s})$ in (5) are those corresponding to facets in $\mathrm{Star}_\Delta(G)$, and those terms lie in $K_S(a_{i,0})_{1 \le i \le d}$ and are equal (up to a global sign) to the corresponding terms in the formula for $\deg_{\Delta'}(x_{j'_1}^{b_1} \cdots x_{j'_s}^{b_s})$. □

3. Some important special cases

In this section, we analyze several important special cases. In order to prove the main theorems, we will need a detailed understanding of the suspension of the boundary of the $(d-1)$-dimensional simplex, i.e., the complex Σ with vertex set $V = \{1, \ldots, d+2\}$ and minimal non-faces $\{1, \ldots, d\}$ and $\{d+1, d+2\}$. For this, it will be helpful to study the boundary of the d-dimensional simplex, i.e., the complex S^{d-1} with vertex set $\{1, \ldots, d+1\}$ and minimal non-face $\{1, \ldots, d+1\}$. We continue to assume that $d > 1$ unless otherwise stated.

Recall that the polynomials $\{[G] : G \subset V,\, |G| = d\}$ in $k[a_{i,j}]$ are irreducible (see, for example, Lemma 4.1). We will need the following lemma.

Lemma 3.1. *If A can be written as a k-linear combination of $\{[G] : G \subset V,\, |G| = d\}$ where at least two coefficients are nonzero, then $\mathrm{ord}_{[G]}(A) = 0$ for all $G \subset V$ of size d.*

Proof. The k-algebra generated by the irreducible polynomials $\{[G] : G \subset V,\, |G| = d\}$ in $k[a_{i,j}]$ is isomorphic to the Plücker ring, i.e., the homogeneous coordinate ring of the Grassmannian of d planes in k^n. In particular, since the Plücker relations all have degree strictly greater than 1, the polynomials $\{[G] : G \subset$

V, $|G| = d\}$ are linearly independent over k. Each $[G]$ is homogeneous of degree d. Hence A is also homogeneous of degree d. If $\operatorname{ord}_{[G']}(A) > 0$, then, by comparing degrees, $A = \lambda[G']$ for some $\lambda \in k$, contradicting the assumption that at least two coefficients are nonzero. □

We can now analyze the boundary of the d-dimensional simplex, for $d > 1$.

Example 3.2. Let S^{d-1} be the boundary of the d-dimensional simplex with vertex set $\{1, \dots, d+1\}$. By Lemma 2.3, for $1 \leq m, p \leq d+1$, we have

$$[V \smallsetminus \{p\}]x_m = (-1)^{|p-m|}[V \smallsetminus \{m\}]x_p \in H^1(\Delta). \tag{6}$$

Fix $0 \leq q \leq d/2$. A basis for $H^q(\Delta) = \overline{H}^q(\Delta)$ is x_1^q. Using (6), we compute

$$\left(\prod_{m=2}^{d} (-1)^{m-1}[V \smallsetminus \{m\}]\right) \deg(x_1^d) = [V \smallsetminus \{1\}]^{d-1} \deg(x_1 \cdots x_d),$$

and hence, for some $\epsilon \in \{\pm 1\}$, we have

$$\deg(x_1^d) = \frac{\epsilon[V \smallsetminus \{1\}]^d}{\prod_{m=1}^{d+1}[V \smallsetminus \{m\}]}.$$

Using (6), we compute

$$[V \smallsetminus \{1\}]^{d-2q} \deg(\ell^{d-2q} \cdot x_1^{2q}) = A^{d-2q} \deg(x_1^d),$$

where $A := \sum_{m=1}^{d+1}(-1)^{m-1}[V \smallsetminus \{m\}]$. By Lemma 3.1, $\operatorname{ord}_{[G]}(A) = 0$ for all facets G of Δ. Putting this together gives

$$\deg(\ell^{d-2q} \cdot x_1^{2q}) = \frac{\epsilon A^{d-2q}[V \smallsetminus \{1\}]^{2q}}{\prod_{m=1}^{d+1}[V \smallsetminus \{m\}]}.$$

Let D_q be the image of $\deg(\ell^{d-2q} \cdot x_1^{2q})$ in $K^\times/(K^\times)^2$. Then

$$D_q = \begin{cases} \epsilon \prod_{G \text{ facet}}[G] & \text{if } d \text{ is even} \\ \epsilon A \prod_{G \text{ facet}}[G] & \text{if } d \text{ is odd.} \end{cases}$$

We conclude that Conjecture 1.3 holds in this case. Observe that Conjecture 1.5 holds vacuously since S^{d-1} has no non-faces of size d.

We will also need to analyze the case of S^0, i.e., the disjoint union of two vertices. Although S^0 is not connected, $H(S^0)$ is a Gorenstein ring with a well-defined degree map, and we may verify directly that the conclusion of Theorem 1.4 (and hence Conjecture 1.3) holds.

Example 3.3. Let $d = 1$, $V = \{1, 2\}$, and consider the complex S^0 with vertex set $\{1, 2\}$ and minimal non-face $\{1, 2\}$. We orient S^0 by assigning -1 to the facet $\{1\}$ and assigning 1 to the facet $\{2\}$. We have

$$\deg(\ell) = \deg(x_1) + \deg(x_2) = -\frac{1}{a_{1,1}} + \frac{1}{a_{1,2}} = \frac{a_{1,1} - a_{1,2}}{a_{1,1}a_{1,2}}.$$

We now analyze Σ, the suspension of the boundary of the $(d-1)$-dimensional simplex. The proofs of Theorem 1.1, Theorem 1.4, and Theorem 1.6 will depend on this special case, via a use of Lemma 2.8. Let $F = \{1, \dots, d\}$ and recall that $V = \{1, \dots, d+2\}$ and $\ell = x_1 + \cdots + x_{d+2}$.

Lemma 3.4. *There are polynomials $A_{d+1}, A_{d+2} \in k[a_{i,j}]$ such that $\mathrm{ord}_{[G]}(A_{d+1}) = \mathrm{ord}_{[G]}(A_{d+2}) = 0$ for any subset G of V of size d, and*

$$[F]\ell = A_{d+1}x_{d+1} + A_{d+2}x_{d+2} \in H^1(\Sigma).$$

Moreover, there is $\epsilon \in \{\pm 1\}$ such that, for $0 < j \leq d$,

$$\deg(\ell^{d-j} \cdot x_{d+1}^j) = \frac{\epsilon A_{d+1}^{d-j}[F]^{j-1}}{\prod_{m=1}^d [F \cup \{d+1\} \smallsetminus \{m\}]} \quad \text{and} \quad \deg(\ell^{d-j} \cdot x_{d+2}^d) = \frac{-\epsilon A_{d+2}^{d-j}[F]^{j-1}}{\prod_{m=1}^d [F \cup \{d+2\} \smallsetminus \{m\}]}. \tag{7}$$

Proof. By Lemma 2.3, for $1 \leq m \leq d$, we have

$$[F]x_m = (-1)^{d+1-m}([F \cup \{d+1\} \smallsetminus \{m\}]x_{d+1} + [F \cup \{d+2\} \smallsetminus \{m\}]x_{d+2}) \in H^1(\Sigma). \tag{8}$$

For $v \in \{d+1, d+2\}$, by multiplying x_v with the product of (8) over all $1 \leq m < d$ and by using the relation $x_{d+1} \cdot x_{d+2} = 0$ in $K[\Sigma]$, we deduce that there is $\epsilon' \in \{\pm 1\}$ such that

$$\left(\prod_{m=1}^{d-1} [F \cup \{v\} \smallsetminus \{m\}] \right) x_v^d = \epsilon'[F]^{d-1} x_1 \cdots x_{d-1} \cdot x_v.$$

By Proposition 2.1, taking degrees of both sides of this equation yields

$$\epsilon_{F \cup \{v\} \smallsetminus \{d-1\}} \deg(x_v^d) = \frac{\epsilon'[F]^{d-1}}{\prod_{m=1}^d [F \cup \{v\} \smallsetminus \{m\}]}$$

If we set $\epsilon = \epsilon' \epsilon_{F \cup \{v\} \smallsetminus \{d-1\}}$, then the $j = d$ case of (7) follows since $\epsilon_{F \cup \{d+1\} \smallsetminus \{d-1\}} = -\epsilon_{F \cup \{d+2\} \smallsetminus \{d-1\}}$. Applying (8) for $1 \leq m \leq d$ yields

$$[F]\ell = A_{d+1}x_{d+1} + A_{d+2}x_{d+2} \in H^1(\Sigma), \tag{9}$$

where for $v \in \{d+1, d+2\}$, we have

$$A_v = [F] + (-1)^{d+1} \sum_{m=1}^{d} (-1)^m [F \cup \{v\} \smallsetminus \{m\}] \in k[a_{i,j}].$$

By Lemma 3.1, $\mathrm{ord}_{[G]}(A_{d+1}) = \mathrm{ord}_{[G]}(A_{d+2}) = 0$ for any subset G of V of size d. Using (9) and the relation $x_{d+1} \cdot x_{d+2} = 0$, we compute

$$[F]^{d-j} \deg(\ell^{d-j} \cdot x_v^j) = A_v^{d-j} \deg(x_v^d).$$

The result now follows from the $j = d$ case of (7). □

Although it will not be needed in what follows, we observe that Lemma 3.4 implies Conjecture 1.3 for Σ in the case $0 < q \leq d/2$. Explicitly, $\{x_{d+1}^q, x_{d+2}^q\}$ is a basis for $H^q(\Sigma) = \overline{H}^q(\Sigma)$, and (7) implies that the corresponding determinant $D_q \in K^\times/(K^\times)^2$ is equal to

$$D_q = \deg(\ell^{d-2q} \cdot x_{d+1}^{2q}) \deg(\ell^{d-2q} \cdot x_{d+2}^{2q}) = \begin{cases} -\prod_{G \text{ facet}}[G] & \text{if } d \text{ is even} \\ -A_{d+1}A_{d+2} \prod_{G \text{ facet}}[G] & \text{if } d \text{ is odd.} \end{cases}$$

See also Example 4.11.

The next lemma will be crucial to the proof of Theorem 1.4. Recall that $d > 1$ and Σ is the suspension of the boundary of the $(d-1)$-dimensional simplex.

Lemma 3.5. *For every non-face G of size d, we have $\mathrm{ord}_{[G]}(\deg(\ell^d)) = 0$.*

Proof. By Lemma 2.4 and Remark 2.5, it is enough to show that there is an l.s.o.p. μ which has $\mathrm{ev}_\mu([G]) = 0$, but $\deg_\mu(\ell^d) = \mathrm{ev}_\mu(\deg(\ell^d)) \neq 0$. Set $\mu_i = a_{i,1}x_1 + \cdots + a_{i,d}x_d$ for $1 \leq i < d$, and set $\mu_d = a_{d,d+1}x_{d+1} + a_{d,d+2}x_{d+2}$. Because $\Sigma = S^{d-2} * S^0$, we see that $H_\mu(\Sigma) = H(S^{d-2}) \otimes H(S^0)$. Furthermore, we can write $\ell = \ell_1 + \ell_2$, where $\ell_1 = x_1 + \cdots + x_d$ and $\ell_2 = x_{d+1} + x_{d+2}$. We have $\ell_1^d = 0$ and $\ell_2^2 = 0$, and we see that

$$\deg_\mu(\ell^d) = \deg_\mu(\ell_1^{d-1} \cdot \ell_2) = \deg_{S^{d-2}}(\ell_1^{d-1}) \deg_{S^0}(\ell_2).$$

Since $\deg_{S^{d-2}}(\ell_1^{d-1}) \neq 0$ and $\deg_{S^0}(\ell_2) \neq 0$ by Example 3.2 and Example 3.3 respectively, we deduce that $\deg_\mu(\ell^d) \neq 0$. It remains to show that $\mathrm{ev}_\mu([G]) = 0$. Since G is a non-face, either $G = \{1, \ldots, d\}$ or G contains $\{d+1, d+2\}$. In the former case, $\mathrm{ev}_\mu([G])$ is the determinant of a matrix whose dth row is identically zero. In the latter case, $\mathrm{ev}_\mu([G])$ is the determinant of a matrix whose last two columns are identically zero except in the dth row and hence are linearly dependent. □

4. Proofs of theorems

In this section, we prove Theorem 1.4, then Theorem 1.6, and then finally Theorem 1.1. Recall that throughout we are assuming that $d > 1$. We first prove a lemma which will be used in the proof of Theorem 1.4. The case when $p = m$ is very well known; see, for example [Bǫc64, Theorem 61.1].

Lemma 4.1. *For some $1 < p \leq m$, let N be the $m \times m$ matrix with $N_{i,j} = a_{i,j}$ if $i = 1$ and $j \leq p$ or if $i > 1$, and $N_{i,j} = 0$ for $i = 1$ and $j > p$. Then $\det N$ is an irreducible polynomial in $k[a_{i,j}]$.*

Proof. Suppose that $\det N = f \cdot g$, where $f, g \in k[a_{i,j}]$. Because $\det N$ is linear in $a_{1,1}$, we see that $a_{1,1}$ must occur in exactly one of f and g, say f. Because $p > 1$, the variables $a_{i,1}$ appear in $\det N$ for $i \geq 1$. Those variables must also only occur in f, because $a_{1,1}a_{i,1}$ does not appear in $\det N$. This implies that $a_{i,j}$ must also occur only in f for each j, because $a_{i,1}a_{i,j}$ does not appear in $\det N$. We conclude that g is a unit. □

In particular, Lemma 4.1 implies that the polynomial $\det N$ defines a valuation on K. We now begin proving Theorem 1.4. We first deal with the case when F is a facet.

Proposition 4.2. *Let F be a facet of Δ. Then*

$$\mathrm{ord}_{[F]}(\deg(\ell^d)) = -1.$$

Proof. Using Proposition 2.6 and properties of valuations, we have

$$\mathrm{ord}_{[F]}(\deg(\ell^d)) \geq \min_{G \text{ facet of } \Delta} \left(d \,\mathrm{ord}_{[F]}(X_{G,1} + \cdots + X_{G,d}) - \mathrm{ord}_{[F]}([G]) - \sum_{m=1}^{d} \mathrm{ord}_{[F]}(X_{G,m}) \right), \tag{10}$$

with equality if the minimum is achieved only once. As $X_{G,m}$, $[G]$, and $[F]$ are irreducible polynomials of the same degree which are not scalar multiples of each other (except that $[G] = [F]$ if $G = F$), we see that for $G \neq F$, the quantity in the minimum in (10) is nonnegative. Note that $\mathrm{ord}_{[F]}(X_{F,1} + \cdots + X_{F,d}) = 0$ by the proof of Lemma 3.1. Therefore the quantity in the minimum in (10) is equal to -1 when $G = F$, and so the minimum is -1 and is achieved exactly once. □

Proof of Theorem 1.4. By Proposition 4.2, it suffices to show that if F is a subset of V of size d which is not a facet, then $\mathrm{ord}_{[F]}(\deg(\ell^d)) = 0$. By Lemma 2.4 and Remark 2.5, it is enough to show that $\deg_F(\ell^d) = \mathrm{ev}_{\theta_F}(\deg(\ell^d)) \neq 0$.

First assume there is a facet F' of Δ with $|F' \cap F| \leq d-2$. Let $\overline{[F']} = \mathrm{ev}_{\theta_F}([F'])$, which is irreducible by Lemma 4.1. We use Proposition 2.6 to compute that $\mathrm{ord}_{\overline{[F']}}(\deg_F(\ell^d))$ is bounded below by

$$\min_{G \text{ facet of } \Delta} \left(d \,\mathrm{ord}_{\overline{[F']}}(X_{G,\theta_F,1} + \cdots + X_{G,\theta_F,d}) - \mathrm{ord}_{\overline{[F']}}(\mathrm{ev}_{\theta_F}([G])) - \sum_{m=1}^{d} \mathrm{ord}_{\overline{[F']}}(X_{G,\theta_F,m}) \right), \tag{11}$$

with equality if the minimum is achieved only once. If $G \neq F$, then it is easy to see that $\mathrm{ord}_{\overline{[F']}}(\mathrm{ev}_{\theta_F}([G])) = 0$, because $\overline{[F']}$ and $\mathrm{ev}_{\theta_F}([G])$ are irreducible polynomials of the same degree which are not scalar multiples. Similarly, $\mathrm{ord}_{\overline{[F']}}(X_{G,\theta_F,m}) = 0$ (this holds even if $G = F'$). So if $G \neq F'$, then the quantity in the minimum in (11) is nonnegative.

If $G = F'$, then $\mathrm{ord}_{\overline{[F']}}(\mathrm{ev}_{\theta_F}([G])) = \mathrm{ord}_{\overline{[F']}}(\overline{[F']}) = 1$. Write $F' = \{j_1 < \cdots < j_d\}$ and fix $1 \leq m \leq d$. Then for $1 \leq m' \leq d$, the coefficient of the monomial $a_{1,0}a_{2,j_1} \cdots a_{m,j_{m-1}}a_{m+1,j_{m+1}} \cdots a_{d,j_d}$ in $X_{F',\theta_F,m'}$ is 1 if $m = m'$ and is 0 otherwise, and the coefficient of this monomial in $[F']$ is zero. We deduce that $X_{F',\theta_F,1} + \cdots + X_{F',\theta_F,d}$ is nonzero with the same degree as $[F']$ and $\mathrm{ord}_{\overline{[F']}}(X_{F',\theta_F,1} + \cdots + X_{F',\theta_F,d}) = 0$. Therefore, the quantity in the minimum in (11) is -1 for $G = F'$, so we deduce that $\mathrm{ord}_{\overline{[F']}}(\deg_F(\ell^d)) = -1$. In particular, $\deg_F(\ell^d) \neq 0$.

Suppose that there is no such facet F'. Then we show that Δ must be the suspension Σ of the boundary of a $(d-1)$-dimensional simplex, i.e., the case discussed in Section 3. Let v be a vertex of Δ not in F, so every facet containing v has $d-1$ vertices from F. Let L be the link of v. Because Δ is a k-homology manifold, L is a $(d-2)$-dimensional k-homology sphere whose facets are all contained in the boundary of F, which is isomorphic to S^{d-2}. In particular, because L is pure of dimension $d-2$, L must be a subcomplex of S^{d-2}.

Suppose that L is a proper subcomplex of S^{d-2}, i.e., it does not contain some facet G of S^{d-2}. Then the map $L \hookrightarrow S^{d-2}$ factors through the contractible complex $S^{d-2} \smallsetminus G$, and so the induced map on H_{d-2} is 0. The long exact sequence in homology associated to the pair (S^{d-2}, L) begins

$$0 \to H_{d-2}(L) \to H_{d-2}(S^{d-2}) \to H_{d-2}(S^{d-2}, L) \to H_{d-3}(L) \to \cdots,$$

which implies that $H_{d-2}(L) = 0$, contradicting that L is a k-homology sphere.

We see that Δ is isomorphic to the join of S^{d-2} with a disjoint union of some vertices $\{v_1, \ldots, v_r\}$. Because the link of any facet of S^{d-2} is $\{v_1, \ldots, v_r\}$, we must have $r = 2$ in order for Δ to be a homology manifold. Therefore $\Delta = \Sigma$. The case of Σ was treated in Lemma 3.5. □

Remark 4.3. The above argument, together with the proof of Lemma 3.5, shows that if F is a non-face of size d, then $\deg_F(\ell^d) = \mathrm{ev}_{\theta_F}(\deg(\ell^d))$ is nonzero. In particular, Conjecture 1.5 holds when $q = 0$.

We now prove Theorem 1.6. We will need the following result of Novik and Swartz, which uses as input results of Gräbe and Schenzel [Grä84, Sch81]. Let $\beta_q = \dim \tilde{H}^q(\Delta; k)$, the dimension of the reduced cohomology of Δ over k. By the universal coefficient theorem, this depends only on the characteristic of k. Let $(h_0(\Delta), \ldots, h_d(\Delta))$ be the h-vector of Δ. Let $\overline{H}_\mu(\Delta)$ be the Gorenstein quotient of $K[\Delta]/(\mu_1, \ldots, \mu_d)$ for an l.s.o.p. $\mu = (\mu_1, \ldots, \mu_d)$ for $K[\Delta]$.

Proposition 4.4. [NS09, Theorem 1.3 and 1.4] *Let $\mu = (\mu_1, \ldots, \mu_d)$ be an l.s.o.p. for $K[\Delta]$. Then*

$$\dim \overline{H}^q_\mu(\Delta) = \begin{cases} h_q(\Delta) - \binom{d}{q} \sum_{p=0}^{q-1} (-1)^{q-p} \beta_{p-1} & \text{if } 0 \leq q < d \\ 1 & \text{if } q = d. \end{cases}$$

In particular, $\dim \overline{H}^q_\mu(\Delta)$ is independent of the choice of l.s.o.p. Recall from the introduction that $\theta_1^F = \sum_{j \notin F} a_{1,j} x_j$ and that $\overline{H}_F(\Delta)$ is the Gorenstein quotient of $K[\Delta]/(\theta_1^F, \ldots, \theta_d)$. Let $\varphi \colon K[\Delta] \to \overline{H}(\Delta)$ and $\varphi_F \colon K[\Delta] \to \overline{H}_F(\Delta)$ be the quotient maps. Set $\ell_F = \varphi_F(\sum_j x_j) \in \overline{H}^1_F(\Delta)$.

Lemma 4.5. *Fix some $0 \leq q \leq d/2$. The algebra $\overline{H}_F(\Delta)$ has the strong Lefschetz property in degree q if and only if multiplication by ℓ_F^{d-2q} is an isomorphism from $\overline{H}^q_F(\Delta) \to \overline{H}^{d-q}_F(\Delta)$, i.e., ℓ_F is a strong Lefschetz element.*

A similar equivalence holds for $\overline{H}(\Delta)$, i.e., $\overline{H}(\Delta)$ has the strong Lefschetz property in degree q if and only if ℓ is a strong Lefschetz element in degree q.

Proof of Lemma 4.5. If ℓ_F is a strong Lefschetz element in degree q, then clearly $\overline{H}_F(\Delta)$ has the strong Lefschetz property in degree q. For the converse, we may replace k by its algebraic closure. Then a Zariski open subset of all $y \in \overline{H}^1_F(\Delta)$ are strong Lefschetz elements in degree q. It follows that a Zariski open subset of all coefficients $(\lambda_1, \dots, \lambda_n) \in k^n$ correspond to elements $\sum \lambda_j x_j \in \overline{H}^1_F(\Delta)$ which are strong Lefschetz elements in degree q. Therefore, we can find a strong Lefschetz element $\ell_{F,\lambda} = \sum \lambda_j x_j$ with each $\lambda_j \in k^\times$. Let

$$H_{F,\lambda}(\Delta) = K[\Delta]/(\sum_{j \notin F} \lambda_j a_{1,j} x_j, \sum_j \lambda_j a_{2,j} x_j, \dots, \sum_j \lambda_j a_{d,j} x_j),$$

and let $\overline{H}_{F,\lambda}(\Delta)$ be the Gorenstein quotient. Because the $\lambda_j a_{i,j}$ are algebraically independent, $\ell_{F,\lambda}$ is a strong Lefschetz element for $\overline{H}_{F,\lambda}(\Delta)$. Let $\Phi\colon \overline{H}_F(\Delta) \to \overline{H}_{F,\lambda}$ be the graded isomorphism given by sending $a_{i,j}$ to $\lambda_j a_{i,j}$. Then we have a commutative square

$$\begin{array}{ccc} \overline{H}^q_F(\Delta) & \xrightarrow{\ell_F^{d-2q}} & \overline{H}^{d-q}_F(\Delta) \\ \downarrow \Phi & & \downarrow \Phi \\ \overline{H}^q_{F,\lambda}(\Delta) & \xrightarrow{\ell_{F,\lambda}^{d-2q}} & \overline{H}^{d-q}_{F,\lambda}(\Delta). \end{array}$$

As the bottom horizontal arrow is an isomorphism, so is the top horizontal arrow. □

Let $\mu = (\mu_1, \dots, \mu_d)$ be an l.s.o.p. for $K[\Delta]$. Let $\varphi_\mu\colon K[\Delta] \to \overline{H}_\mu(\Delta)$ be the quotient map and set $\ell_\mu = \varphi_\mu(\sum_j x_j) \in \overline{H}^1_\mu(\Delta)$. Recall that $R \subset K$ denotes the localization of $k[a_{i,j}]$ at the irreducible polynomials $\{[G] : G \text{ facet of } \Delta\}$, and $\mathrm{ev}_\mu\colon R \to K$ is the map defined by $\mathrm{ev}_\mu(a_{i,j}) = \mu_{i,j}$.

Lemma 4.6. *Let $\mu = (\mu_1, \dots, \mu_d)$ be an l.s.o.p. and let $0 \le q \le d/2$. Suppose that multiplication by ℓ_μ^{d-2q} is an isomorphism from $\overline{H}^q_\mu(\Delta) \to \overline{H}^{d-q}_\mu(\Delta)$, i.e., ℓ_μ is a strong Lefschetz element. Let $P \in k[a_{i,j}]$ be an irreducible polynomial such that $\mathrm{ev}_\mu(P) = 0$. Then there are monomials $y_1, \dots, y_p$ such that $\{\varphi(y_i)\}$ is a basis of $\overline{H}^q(\Delta)$ and $\mathrm{ord}_P(\det M) = 0$, where M is the $p \times p$ matrix whose (i,j) entry is $\deg(\ell^{d-2q} \cdot \varphi(y_i) \cdot \varphi(y_j))$. In particular, if $D_q \in K^\times/(K^\times)^2$ is the determinant of the Hodge–Riemann form on $\overline{H}^q(\Delta)$, then $\mathrm{ord}_P(D_q) = 0 \in \mathbb{Z}/2\mathbb{Z}$.*

Proof. Choose monomials $y_1, \dots, y_p$ in the degree q part of $K[\Delta]$ such that $\{\varphi_\mu(y_i)\}$ is a basis for $\overline{H}^q_\mu(\Delta)$; this is possible because $\overline{H}^q_\mu(\Delta)$ is spanned by the images of monomials.

Let M be the $p \times p$ matrix whose (i,j) entry is $\deg(\ell^{d-2q} \cdot \varphi(y_i) \cdot \varphi(y_j))$. By Lemma 2.4, each entry of M lies in R, so $\det M$ lies in R. By Remark 2.5, if we can show that $\mathrm{ev}_\mu(\det M) \neq 0$, then $\mathrm{ord}_P(\det M) = 0$.

Let M_μ be the $p \times p$ matrix whose (i,j) entry is $\deg_\mu(\ell_\mu^{d-2q} \cdot \varphi_\mu(y_i) \cdot \varphi_\mu(y_j))$. By Lemma 2.4, the (i,j) entry of M_μ is $\mathrm{ev}_\mu(\deg(\ell^{d-2q} \cdot \varphi(y_i) \cdot \varphi(y_j)))$, so $\det M_\mu = \mathrm{ev}_\mu(\det M)$. As ℓ_μ is a strong Lefschetz element for $\overline{H}^q_\mu(\Delta)$, $\det M_\mu \neq 0$, and we conclude that $\mathrm{ord}_P(\det M) = 0$.

Finally, that $\det M$ is nonzero implies that $\{\varphi(y_i)\}$ is linearly independent in $\overline{H}^q(\Delta)$. As $\dim \overline{H}^q(\Delta) = \dim \overline{H}^q_\mu(\Delta)$ by Proposition 4.4, $\{\varphi(y_i)\}$ is a basis for $\overline{H}^q(\Delta)$, so $\det M$ computes the determinant of the Hodge–Riemann form on $\overline{H}^q(\Delta)$. This completes the proof. □

Proposition 4.7. *Let F be a subset of V of size d which is not a facet, and let $0 \leq q \leq d/2$. Suppose that $\overline{H}_F(\Delta)$ has the strong Lefschetz property in degree q. Let $D_q \in K^\times/(K^\times)^2$ be the determinant of the Hodge–Riemann form on $\overline{H}^q(\Delta)$. Then* $\operatorname{ord}_{[F]}(D_q) = 0$.

Proof. By Lemma 4.5, ℓ_F is a strong Lefschetz element for $\overline{H}^q_F(\Delta)$. The result now follows from Lemma 4.6 setting $\mu = \theta_F$ and $P = [F]$. □

Lemma 4.8. *Let F be a facet of Δ. Then for each q, there is a basis for $\overline{H}^q(\Delta)$ consisting of the images of monomials in $K[\Delta]$ whose support is disjoint from F.*

Proof. Using the l.s.o.p., one can write any monomial in $\overline{H}^1(\Delta)$ in terms of the monomials corresponding to vertices not in F. As $\overline{H}(\Delta)$ is generated in degree 1, this implies that each $\overline{H}^q(\Delta)$ is spanned by monomials whose support is disjoint from F. Some subset of these monomials form a basis. □

For a facet F of Δ, let Δ' be the simplicial complex obtained by doing a stellar subdivision in the interior of F, i.e., the vertex set of Δ' is $V \cup \{n+1\} = \{1, \dots, n+1\}$, and the facets of Δ' are the facets of Δ except for F, together with $(F \cup \{n+1\}) \smallsetminus \{j\}$ for each $j \in F$. Then Δ' is an oriented connected k-homology manifold, with its orientation determined by orienting the facets of Δ' which are also facets of Δ in the same way that they are oriented in Δ.

Lemma 4.9. *For $0 < q < d$, we have* $\dim \overline{H}^q(\Delta) + 1 = \dim \overline{H}^q(\Delta')$

Proof. The geometric realization of Δ' is homeomorphic to the geometric realization of Δ, so the reduced Betti numbers do not change. Therefore, by Proposition 4.4, $\dim \overline{H}^q(\Delta') - \dim \overline{H}^q(\Delta) = h_q(\Delta') - h_q(\Delta)$. That this is 1 when $0 < q < d$ follows from the formula for how the h-vector changes under refinement in [Sta92, Theorem 3.2], or can be checked using the formula for the h-vector in terms of the f-vector. □

Note that the proof of Proposition 4.7 implies that if F is a non-face of size d and $\overline{H}_F(\Delta)$ has the strong Lefschetz property in degree q, then so does $\overline{H}(\Delta)$. If Δ has no non-faces of size d, then Δ must be isomorphic to S^{d-1}, and so Conjecture 1.3 holds for Δ by Example 3.2. When proving Theorem 1.6, we may therefore assume that $\overline{H}(\Delta)$ has the strong Lefschetz property in degree q.

Proof of Theorem 1.6. As Theorem 1.4 implies Conjecture 1.3 when $q = 0$, we may assume that $0 < q \leq d/2$.

Proposition 4.7 shows that if F is not a facet, then $\operatorname{ord}_{[F]}(D_q) = 0$. Suppose that F is a facet of Δ. By Lemma 4.8, we may choose a collection of monomials $y_1, \dots, y_p \in K[\Delta]$ of degree q whose support is disjoint from F and such that their image in $\overline{H}^q(\Delta)$ under $\varphi \colon K[\Delta] \to \overline{H}(\Delta)$ is a basis. By the version of Lemma 4.5 for $\overline{H}(\Delta)$, ℓ is a strong Lefschetz element in degree q. Let M be the $p \times p$ matrix whose (i, j)th entry is $\deg(\ell^{d-2q} \cdot \varphi(y_i) \cdot \varphi(y_j))$, so M is nonsingular and the image of $\det M$ in $K^\times/(K^\times)^2$ is D_q.

Let Δ' be the simplicial complex obtained by doing a stellar subdivision in the interior of F, with orientation as described above. We can identify $K[\Delta]/(x_F)$ with a subring of $K[\Delta']$, and hence consider the images $y'_1, \dots, y'_p$ of $y_1, \dots, y_p$ in $K[\Delta']$. Set $y'_{p+1} = x^q_{n+1}$. Let $\varphi' \colon K[\Delta'] \to \overline{H}(\Delta')$ be the quotient map, and let $\ell' = \varphi'(\sum_{j=1}^{n+1} x_j) \in \overline{H}^1(\Delta')$. Let M' be the $(p+1) \times (p+1)$ matrix whose (i, j)th entry is $\deg((\ell')^{d-2q} \cdot \varphi'(y'_i) \cdot \varphi'(y'_j))$. For $j \leq p$, we have $y'_j \cdot y'_{p+1} = 0$ in $K[\Delta']$. By Lemma 2.8, M' is a block diagonal matrix whose northwest $p \times p$ block is M and whose $(p+1, p+1)$ entry $M'_{p+1,p+1}$ is equal to the degree of $\ell^{d-2q} \cdot x^{2q}_{d+1}$ in the complex Σ considered in Section 3 (up to sign). Lemma 3.4 implies that $M'_{p+1,p+1}$ is nonzero and $\operatorname{ord}_{[F]}(M'_{p+1,p+1}) = 2q - 1$.

We see that M' is nonsingular, so $\{\varphi'(y'_1), \dots, \varphi'(y'_{p+1})\}$ is a linearly independent subset of $\overline{H}^q(\Delta')$. As $\dim \overline{H}^q(\Delta') = \dim \overline{H}^q(\Delta) + 1$ by Lemma 4.9, $\{\varphi'(y'_1), \dots, \varphi'(y'_{p+1})\}$ is a basis for $\overline{H}^q(\Delta')$. In particular, $\det M'$ computes the determinant of the Hodge–Riemann form.

As F is not a facet of Δ', Proposition 4.7 gives that $\operatorname{ord}_{[F]}(\det M')$ is even. Since $\operatorname{ord}_{[F]}(M'_{p+1,p+1}) = 2q-1$ is odd, we see that $\operatorname{ord}_{[F]}(\det M)$ is odd, as desired. □

Remark 4.10. The argument used to prove Theorem 1.6, together with Lemma 3.4, shows that, if Δ' is the stellar subdivision of Δ in the interior of a facet of Δ, then Conjecture 1.3 holds for Δ in degree q if and only if it holds for Δ' in degree q.

Example 4.11. Let Δ be a simplicial sphere obtained from S^{d-1}, the boundary of the d-dimensional simplex, by successively applying stellar subdivisions to the interiors of facets. Then Example 3.2 and Remark 4.10 imply that Conjecture 1.3 holds. In this case, for any $q > 0$, multiplication by ℓ^{q-1} maps a basis for $H^1(\Delta)$ to a basis for $H^q(\Delta)$, and it follows that D_q is independent of q when $q > 0$. In particular, when d is even, the analogue of Theorem 1.1 holds for D_q when $q > 0$. We saw special cases of this by explicit calculation in Example 3.2 and Lemma 3.4.

Proof of Theorem 1.1. Theorem 1.6 implies Corollary 1.2. Hence if F is a facet of Δ, then $\operatorname{ord}_{[F]}(D_{d/2}) = 1 \in \mathbb{Z}/2\mathbb{Z}$. Let $P \in k[a_{i,j}]$ be an irreducible polynomial that is not equal (up to multiplication by a scalar) to one of the polynomials $\{[F] : F \text{ facet of } \Delta\}$. Over $\overline{k}[a_{i,j}]$, we may factor $P = P_1^{m_1} \cdots P_r^{m_r}$, where the P_i are distinct irreducible polynomials over $\overline{k}$ and $m_i \in \mathbb{Z}_{>0}$. Note that none of the P_i are scalar multiples of $[F]$. We claim that there are monomials $y_1, \ldots, y_p$ such that $\{\varphi(y_i)\}$ is a basis of $\overline{H}^{d/2}(\Delta)$ and $\operatorname{ord}_{P_1}(\det M) = 0$, where M is the $p \times p$ matrix whose (i,j) entry is $\deg(\varphi(y_i) \cdot \varphi(y_j))$. This implies that $\operatorname{ord}_P(\det M) = 0$ and hence $\operatorname{ord}_P(D_{d/2}) = 0$. We deduce that $D_{d/2} = \lambda \prod_{F \text{ facet of } \Delta} [F] \in K^\times/(K^\times)^2$ for some $\lambda \in k^\times/(k^\times)^2$, completing the proof.

It remains to verify the claim. Let $V(P_1)$ be the vanishing locus of P_1 inside $\mathbb{A}^{dn}_{\overline{k}}$, and let $(\mu_{i,j}) \in V(P_1)$ be a $\overline{k}$-point. Set $\mu_i = \sum_j \mu_{i,j} x_j$. First suppose that $\mu = (\mu_1, \ldots, \mu_d)$ is an l.s.o.p. for $\overline{k}(a_{i,j})[\Delta]$. Observe that $\operatorname{ev}_\mu(P_1) = 0$ since $(\mu_{i,j}) \in V(P_1)$. Then the claim follows from Lemma 4.6. Note that the assumption in Lemma 4.6 that ℓ_μ is a strong Lefschetz element holds vacuously since we are in middle dimension. Hence we may assume that μ is not an l.s.o.p. By Proposition 2.2 there must be some facet F of Δ such that $(\mu_{i,j})$ is contained in the vanishing locus of $[F]$. Applying this to every $\overline{k}$-point of $V(P_1)$, we see that

$$V(P_1) \subset \bigcup_{F \text{ facet of } \Delta} V([F]).$$

As there are only finitely many facets, this implies that $V(P_1)$ is contained in $V([F])$ for some facet F. The irreducibility of $[F]$ then implies that P_1 and $[F]$ are equal up to multiplication by a scalar, a contradiction. □

5. Further discussion

Assume that ℓ is a strong Lefschetz element in all degrees, i.e., the Hodge–Riemann form on $\overline{H}^q(\Delta)$ is nondegenerate for $0 \le q \le d/2$. The *primitive part* of $\overline{H}^q(\Delta)$ is $\overline{H}^q_{\text{prim}}(\Delta) := \{y \in \overline{H}^q(\Delta) : \ell^{d-2q+1} \cdot y = 0\}$. Let $D_{\text{prim},q} \in K^\times/(K^\times)^2$ be the determinant of the induced Hodge–Riemann form on $\overline{H}^q_{\text{prim}}(\Delta)$. For $0 < q \le d/2$, multiplication by ℓ induces an injection $\overline{H}^{q-1}(\Delta) \to \overline{H}^q(\Delta)$ which splits to give an isomorphism $\overline{H}^q(\Delta) \cong \overline{H}^{q-1}(\Delta) \oplus \overline{H}^q_{\text{prim}}(\Delta)$. As this decomposition is orthogonal with respect to the Hodge–Riemann form, we have $D_q = D_{q-1} D_{\text{prim},q}$. In particular, $D_q = D_0 \prod_{q'=1}^{q} D_{\text{prim},q'}$. Since we established Conjecture 1.3 when $q = 0$ in Theorem 1.4, we conclude that Conjecture 1.3 holding for all $0 \le q \le d/2$ is equivalent to the following conjecture.

Conjecture 5.1. *Let Δ be a connected oriented simplicial k-homology manifold of dimension $d-1$ with vertex set V. Then ℓ is a strong Lefschetz element in all degrees, and, for each subset F of V of size d and $0 < q \le d/2$, we have* $\operatorname{ord}_{[F]}(D_{\text{prim},q}) = 0$.

It is natural to try to extend Conjecture 1.3 to the setting of connected oriented pseudomanifolds, where the construction of the degree map still works (see [KX23, Section 2.5]). However, a key property of homology manifolds which was used in the proof of our results, e.g. Theorem 1.1, was that the dimension of $\overline{H}^q_\mu(\Delta)$ does not depend on μ, the chosen l.s.o.p. (see Proposition 4.4). We show in the example below that this independence of the dimension can fail for pseudomanifolds.

Example 5.2. Let Δ be the standard 6 vertex triangulation of $\mathbb{RP}^2$, and let $\Delta' = \Delta * S^0$ be the suspension. Over a field of characteristic 2, Δ' is a connected oriented pseudomanifold, but it is not a homology manifold. Using Macaulay2 [GS], we checked that, if one chooses an l.s.o.p. $\mu_1, \mu_2, \mu_3, \mu_4$ with all coefficients random elements of the field with 1024 elements, the Hilbert function of $H_\mu(\Delta')$ is usually given by $(1, 4, 9, 6, 1)$, and the Hilbert function of $\overline{H}_\mu(\Delta')$ is usually given by $(1, 4, 8, 4, 1)$. If one chooses μ'_1, μ'_2, μ'_3 to be generic linear combinations of the vertices of $\mathbb{RP}^2$ and chooses μ'_4 to be a generic linear combination of the vertices of S^0, then $H_{\mu'}(\Delta') = H_{(\mu'_1,\mu'_2,\mu'_3)}(\Delta) \otimes H_{(\mu'_4)}(S^0)$, and similarly for $\overline{H}_{\mu'}(\Delta')$. We can then use Proposition 4.4 to compute that the Hilbert function of $H_{\mu'}(\Delta')$ is given by $(1, 4, 9, 7, 1)$, and the Hilbert function of $\overline{H}_{\mu'}(\Delta')$ is given by $(1, 4, 6, 4, 1)$.

Recently, Papadakis and Petrotou [PP20] introduced a powerful technique using the special behavior of differential operators in characteristic 2 to prove a strengthening of the strong Lefschetz property when k has characteristic 2. Their technique was extended by Karu and Xiao [KX23] to prove the anisotropy of the Hodge–Riemann form on $\overline{H}^q(\Delta)$: if $u \in \overline{H}^q(\Delta)$ is nonzero, then $\deg(\ell^{d-2q} \cdot u^2)$ is nonzero. The following example shows that this anisotropy property does not hold for the rings $\overline{H}_F(\Delta)$, so it seems like this technique cannot be used to prove the strong Lefschetz property for $\overline{H}_F(\Delta)$.

Example 5.3. Let $d = 2$, and consider Σ as in Section 3, i.e., Σ has vertex set $\{1, 2, 3, 4\}$ and minimal non-faces $\{1, 2\}$ and $\{3, 4\}$. Consider $H_F(\Sigma) = \overline{H}_F(\Sigma)$, where $F = \{1, 2\}$. Then $\theta_{F,1} = a_{1,3}x_3 + a_{1,4}x_4$, so the relation $x_3 \cdot x_4 = 0$ in $K[\Sigma]$ implies that $x_3^2 = 0$ in $H_F(\Sigma)$. As $x_3 \neq 0$ in $H_F(\Sigma)$, anisotropy fails for $H_F(\Sigma)$.

It would be interesting to extend Theorem 1.1 by computing D_q is other cases. This is related to the following question: for which choices of l.s.o.p. μ and $0 \leq q \leq d/2$ is the image ℓ_μ of $\sum_j x_j$ in $\overline{H}^1_\mu(\Delta)$ a strong Lefschetz element for $\overline{H}^q_\mu(\Delta)$, i.e., when is the hypothesis in Lemma 4.6 satisfied?

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