DETERMINANTS OF HODGE–RIEMANN FORMS

MATT LARSON, ISABELLA NOVIK AND ALAN STAPLEDON

Abstract. We calculate the determinant of the bilinear form in middle degree of the generic artinian reduction of the Stanley–Reisner ring of an odd-dimensional simplicial sphere. This proves the odd multiplicity conjecture of Papadakis and Petrotou and implies that this determinant is a complete invariant of the simplicial sphere. We extend this result to odd-dimensional connected oriented simplicial homology manifolds. In characteristic 2, we prove a generalization to the Hodge–Riemann forms of any connected simplicial homology manifold. To prove the latter theorem we establish the strong Lefschetz property for certain quotients of the Stanley–Reisner rings of connected simplicial pseudomanifolds.

1. Introduction

Let Δ be a simplicial complex with vertex set $V = \{1, \dots, n\}$ of dimension $d - 1 > 0$. Let k be a field, and set $K = k(a_{i,j})_{1 \le i \le d,\, 1 \le j \le n}$. We assume that Δ is a connected homology manifold over k, i.e., Δ is connected, and the link of every nonempty face G of Δ has the same homology as a sphere of dimension $d - |G| - 1$ over k. Let $K[\Delta]$ be the Stanley–Reisner ring of Δ, and set $\theta_i = a_{i,1}x_1 + \cdots + a_{i,n}x_n \in K[\Delta]$ for $i \in \{1, \dots, d\}$, so $\theta_1, \dots, \theta_d$ is a linear system of parameters for $K[\Delta]$. Let $H(\Delta) = K[\Delta]/(\theta_1, \dots, \theta_d)$ be the generic artinian reduction of $K[\Delta]$.

Assume that Δ is oriented. Then there is a distinguished isomorphism $\deg\colon H^d(\Delta) \to K$ [Bri97], see Section 2. Let $\overline{H}(\Delta)$ be the Gorenstein quotient of $H(\Delta)$, i.e., the quotient by the ideal $(y \in H(\Delta) : (yz)_d = 0 \text{ for all } z \in H(\Delta))$, where y_d denotes the degree d component of y in $H(\Delta)$. If Δ is a homology sphere over k, i.e., a homology manifold with the same homology over k as a sphere of dimension $d - 1$, then $\overline{H}(\Delta) = H(\Delta)$. By construction, $\overline{H}(\Delta)$ is an artinian Gorenstein ring: for each q, the bilinear form $\overline{H}^q(\Delta) \times \overline{H}^{d-q}(\Delta) \to K$ given by $(y, z) \mapsto \deg(yz)$ is nondegenerate.

Suppose that d is even. Let $D_{d/2} \in K^\times/(K^\times)^2$ be the determinant of the nondegenerate bilinear form on $\overline{H}^{d/2}(\Delta)$. That is, choose a basis $y_1, \dots, y_p$ for $\overline{H}^{d/2}(\Delta)$, and let M be the symmetric matrix whose (i, j)th entry is $\deg(y_i y_j)$. Then $D_{d/2}$ is the image of $\det M$ in $K^\times/(K^\times)^2$; choosing a different basis for $\overline{H}^{d/2}(\Delta)$ only changes $\det M$ by a square, so $D_{d/2}$ is well-defined. For a subset $F = \{j_1 < \cdots < j_d\}$ of V of size d, set $[F]$ to be the determinant of the $d \times d$ matrix whose (i, m)th entry is a_{i,j_m}.

Theorem 1.1. *Let d be even, and let Δ be a connected oriented simplicial k-homology manifold of dimension $d - 1$. Then*

$$D_{d/2} = \lambda \prod_{F \text{ facet of } \Delta} [F] \in K^\times/(K^\times)^2$$

for some $\lambda \in k^\times/(k^\times)^2$.

Papadakis and Petrotou proved Theorem 1.1 for 1-dimensional simplicial spheres [PP23, Proposition 5.1].

Let F be a subset of V of size d. As $[F]$ is an irreducible polynomial (see Lemma 4.2), it defines a valuation $\operatorname{ord}_{[F]}\colon K^\times \to \mathbb{Z}$ given by the order of vanishing along the hypersurface defined by $[F]$. This descends to a homomorphism $\operatorname{ord}_{[F]}\colon K^\times/(K^\times)^2 \to \mathbb{Z}/2\mathbb{Z}$. We immediately deduce the following corollary to Theorem 1.1. It implies that the determinant of the bilinear form on $\overline{H}^{d/2}(\Delta)$ is a complete invariant of the connected oriented simplicial k-homology manifold Δ.

Corollary 1.2. *Let d be even, and let Δ be a connected oriented simplicial k-homology manifold of dimension $d-1$ with vertex set V. Let F be a subset of V of size d. Then*

$$\operatorname{ord}_{[F]}(D_{d/2}) = \begin{cases} 1 & \text{if } F \text{ is a facet of } \Delta \\ 0 & \text{otherwise.} \end{cases}$$

When Δ is a simplicial sphere, Corollary 1.2 was conjectured by Papadakis and Petrotou [PP23, Conjecture 5.4], who called it the *odd multiplicity conjecture*. This conjecture has motivated our work.

We prove a generalization of the odd multiplicity conjecture. Assume that $\operatorname{char} k = 2$, or $\operatorname{char} k = 0$ and the integral homology of the link of any face (including the empty face) of Δ has no 2-torsion. Let $\ell = \sum_{j=1}^{n} x_j \in \overline{H}^1(\Delta)$. For $0 \leq q \leq d/2$, define the Hodge–Riemann form $\overline{H}^q(\Delta) \times \overline{H}^q(\Delta) \to K$ via $(y, z) \mapsto \deg(\ell^{d-2q} yz)$. When d is even and $q = d/2$, the Hodge–Riemann form is the bilinear form on $\overline{H}^{d/2}(\Delta)$ considered above. Let D_q be the determinant of the Hodge–Riemann form on $\overline{H}^q(\Delta)$.

Theorem 1.3. *Let Δ be a connected oriented simplicial k-homology manifold of dimension $d-1$ with vertex set V, and let $0 \leq q \leq d/2$. Assume that $\operatorname{char} k = 2$, or $\operatorname{char} k = 0$ and the integral homology of the link of any face (including the empty face) of Δ has no 2-torsion. Let F be a subset of V of size d. Then*

$$\operatorname{ord}_{[F]}(D_q) = \begin{cases} 1 & \text{if } F \text{ is a facet of } \Delta \\ 0 & \text{otherwise.} \end{cases}$$

When d is even and $q = d/2$, Theorem 1.3 follows from Corollary 1.2. The nondegeneracy of the Hodge–Riemann form, which is part of Theorem 1.3, is equivalent to the map $\overline{H}^q(\Delta) \to \overline{H}^{d-q}(\Delta)$ given by multiplication by ℓ^{d-2q} being an isomorphism. By Lemma 5.1, this is equivalent to $\overline{H}(\Delta)$ having the strong Lefschetz property in degree q, i.e., that there is some $y \in \overline{H}^1(\Delta)$ such that the map $\overline{H}^q(\Delta) \to \overline{H}^{d-q}(\Delta)$ given by multiplication by y^{d-2q} is an isomorphism.

In particular, Theorem 1.3 is a generalization of the algebraic g-conjecture for Δ (that $\overline{H}(\Delta)$ has the strong Lefschetz property), and it implies that the Hodge–Riemann form in any degree is a complete invariant of Δ. A proof of the algebraic g-conjecture for connected oriented simplicial pseudomanifolds was announced in [APP21], see also [Adi18, KX23, PP20]. We have been heavily inspired by the recent progress on the algebraic g-conjecture, and, in particular, the key insight that one should study the generic artinian reduction of $K[\Delta]$ and the corresponding degree map. See also [APP24].

Theorem 1.3 follows from a strengthening of the algebraic g-conjecture for less generic artinian reductions of $K[\Delta]$. Let F be a subset of V of size d which is not a facet of Δ, and set $\theta_1^F = \sum_{j \notin F} a_{1,j} x_j$. Then $\theta_1^F, \theta_2, \ldots, \theta_d$ is still a linear system of parameters for $K[\Delta]$ (see Proposition 2.2). Let $H_F(\Delta) = K[\Delta]/(\theta_1^F, \theta_2, \ldots, \theta_d)$. Then there is a distinguished isomorphism $\deg_F \colon H_F^d(\Delta) \to K$ (see Section 2). Set $\overline{H}_F(\Delta)$ to be the Gorenstein quotient of $H_F(\Delta)$. For example, if Δ is a homology sphere over k, then $H_F(\Delta) = \overline{H}_F(\Delta)$. We deduce Theorem 1.3 from the following theorem.

Theorem 1.4. *Let Δ be a connected oriented simplicial k-homology manifold of dimension $d-1$, and let $0 \leq q \leq d/2$. Assume that $\operatorname{char} k = 2$, or $\operatorname{char} k = 0$ and the integral homology of the link of any face (including the empty face) of Δ has no 2-torsion. Then for every non-face F of size d, $\overline{H}_F(\Delta)$ has the strong Lefschetz property in degree q.*

We use the characteristic 2 method of Papadakis and Petrotou [PP20], as refined by Karu and Xiao [KX23]. In the case of $\overline{H}(\Delta)$, this method was used to show that the Hodge–Riemann form is *anisotropic* [KX23, Theorem 4.4]. This can fail in the case of $\overline{H}_F(\Delta)$ (see Example 5.3), but we show that anisotropy "almost" holds (Proposition 5.6) and use this to deduce Theorem 1.4.

When k has characteristic 0 and Δ is a polytopal sphere, i.e., is the boundary of a simplicial polytope, Stanley's proof of this case of the algebraic g-conjecture [Sta80] can be adapted to give a simple proof of Theorem 1.4, see Remark 7.2.

Recall that a pure simplicial complex Δ of dimension $d-1$ is a *pseudomanifold* if every $(d-2)$-dimensional face lies in exactly two facets, and each connected component of the geometric realization of Δ remains connected after we remove its $(d-3)$-skeleton. Any k-homology manifold is a pseudomanifold, and the constructions of $H(\Delta)$, $\overline{H}(\Delta)$, deg, $H_F(\Delta)$, $\overline{H}_F(\Delta)$, and $\deg_F$ above are valid for pseudomanifolds. For connected oriented pseudomanifolds, we establish Theorem 1.3 and Theorem 1.4 when $q = 0$ for all fields k (see Theorem 4.1 and Remark 4.4 respectively), and, if we further assume that $\operatorname{char} k = 2$, then we establish Theorem 1.3 when $q = 1$ (see Remark 8.4) and Theorem 1.4 in all degrees (see Theorem 6.3). We conjecture that all the results above hold for connected oriented pseudomanifolds over any field k (Conjecture 8.1). In Section 8, we discuss the sole obstruction to using our methods to prove this conjecture when $\operatorname{char} k = 2$.

Our paper is organized as follows. In Section 2, we recall the construction and properties of the degree map. In Section 3, we compute some special cases which will be used in the proofs of the main theorems. In Section 4, we prove the $q = 0$ case of Theorems 1.3 and 1.4. In Sections 5 and 6, we prove Theorem 1.4 when $\operatorname{char} k = 2$. In Section 7, we prove the main theorems. In Section 8, we give some examples and discuss possible extensions.

Throughout, we fix a connected oriented simplicial pseudomanifold Δ over k of dimension $d - 1$ with vertex set V. We will sometimes further assume that Δ is a k-homology manifold, and we will occasionally omit the connected, oriented, and simplicial hypotheses. If G is a face of Δ with vertices $\{j_1, \ldots, j_r\}$, we write $x_G \coloneqq x_{j_1} \cdots x_{j_r}$ for the corresponding monomial in $K[\Delta]$. We will sometimes abuse notation and use x_G to denote its image in $H(\Delta)$ or $\overline{H}(\Delta)$. See [Sta96] for any undefined terminology.

We will assume throughout that $d > 1$. If $d = 1$, the (not connected) case of a simplicial sphere of dimension 0, i.e., Δ consists of two points, is discussed in Example 3.3.

Acknowledgements. We thank Ed Swartz for suggesting Example 8.2. The work of the second author is partially supported by NSF grant DMS-2246399. The first and third author thank the Institute for Advanced Study, where part of this work was conducted, for a pleasant environment.

2. Degree maps

We now discuss degree maps on artinian reductions of Stanley–Reisner rings of connected oriented simplicial pseudomanifolds over k. The normalization of the degree map will be crucial in what follows, as the results of the introduction can fail if we use an arbitrary isomorphism from $H^d(\Delta)$ to K. Explicitly, two such isomorphisms vary by multiplication by a nonzero element $\omega \in K$, and if $p = \dim \overline{H}^q(\Delta)$ is odd, then the determinant of a nondegenerate bilinear form on $\overline{H}^q(\Delta)$ will vary by multiplication by $\omega^p = \omega \in K^\times/(K^\times)^2$.

We first discuss orientations over k in the case when the characteristic of k is not 2. If $d > 1$, then an *orientation* on a $(d-1)$-dimensional simplex is a choice of ordering of the vertices, up to changing the ordering by an even permutation. If $d = 1$, then an orientation on a $(d-1)$-dimensional simplex is a choice of $\epsilon \in \{1, -1\}$. An orientation on a $(d-1)$-dimensional simplex induces an orientation on each facet. If $d > 1$ and the simplex is ordered by $\{v_1 < \cdots < v_d\}$, then we orient $\{v_2, \ldots, v_d\}$ using the ordering $v_2 < \cdots < v_d$, and we orient the facet which omits v_i by changing the ordering by even permutations so that v_i is first. If $d = 1$ and the simplex is $\{v_1 < v_2\}$, then we orient $\{v_1\}$ by -1 and orient $\{v_2\}$ by 1.

An *orientation* of Δ over k is a choice of orientation for each facet of Δ such that the two orientations on any $(d-2)$-dimensional simplex of Δ induced by the two facets containing it are opposite. In what follows, we fix a choice of orientation.

If k has characteristic 2, then we say that any pseudomanifold is oriented over k by definition.

For each facet $F = \{j_1 < \cdots < j_d\}$, the orientation on Δ defines a sign $\epsilon_F \in \{1, -1\}$, which is 1 if the permutation which takes $(j_1, \dots, j_d)$ to the ordering given by the orientation is even, and is -1 if this permutation is odd. If the characteristic of k is 2, then $\epsilon_F = 1$ by definition.

There is an explicit isomorphism $\deg\colon H^d(\Delta) \to K$, called the *degree map*. This isomorphism was constructed by Brion [Bri97], see also [KX23, Lemma 2.2]. Recall that, for a subset $F = \{j_1 < \cdots < j_d\}$ of V of size d, $[F]$ is the determinant of the matrix whose (i, m)th entry is a_{i,j_m}.

Proposition 2.1. *There is an isomorphism* $\deg\colon H^d(\Delta) \to K$ *of* K*-vector spaces such that, for any facet* F *of* Δ*, we have*

$$\deg(x_F) = \frac{\epsilon_F}{[F]}. \tag{1}$$

In particular, if k does not have characteristic 2, then the degree map associated to the opposite orientation is the negative of the original degree map.

More generally, consider d elements $\mu = (\mu_1, \dots, \mu_d)$ in $K[\Delta]$ of degree 1, with $\mu_i = \sum_{j \in V} \mu_{i,j} x_j$ for some $\mu_{i,j} \in K$. Let $k[a_{i,j}]$ denote the polynomial ring $k[a_{i,j}]_{1 \le i \le d,\, 1 \le j \le n}$ with fraction field K, and consider the k-algebra homomorphism $\mathrm{ev}_\mu\colon k[a_{i,j}] \to K$ defined by

$$\mathrm{ev}_\mu(a_{i,j}) = \mu_{i,j}.$$

We will use the following criterion for the elements of μ to be a linear system of parameters (l.s.o.p.).

Proposition 2.2. [Sta96, Lemma III.2.4] *Consider* d *elements* $\mu = (\mu_1, \dots, \mu_d)$ *in* $K[\Delta]$ *of degree* 1*. Then* $\mu_1, \dots, \mu_d$ *is an l.s.o.p. if and only if* $\mathrm{ev}_\mu([F]) \neq 0$ *for each facet* F *of* Δ*.*

Suppose that $\mu = (\mu_1, \dots, \mu_d)$ is an l.s.o.p. Let $H_\mu(\Delta) \coloneqq K[\Delta]/(\mu_1, \dots, \mu_d)$. We still have $\dim H^d_\mu(\Delta) = 1$ (see, for example, [TW00, Corollary 3.2]), and so the degree map described in Proposition 2.1 "specializes" to an isomorphism $\deg_\mu\colon H^d_\mu(\Delta) \to K$ of K-vector spaces such that, for a fixed choice of facet F of Δ,

$$\deg_\mu(x_F) = \frac{\epsilon_F}{\mathrm{ev}_\mu([F])}. \tag{2}$$

We will verify below that (2) is independent of the choice of facet F. We also have a well-defined Gorenstein quotient $\overline{H}_\mu(\Delta)$, i.e., the quotient of $H_\mu(\Delta)$ by the ideal $(y \in H_\mu(\Delta) : (yz)_d = 0 \text{ for all } z \in H_\mu(\Delta))$, where y_d denotes the degree d component of y in $H_\mu(\Delta)$. For example, as in the statement of Theorem 1.4, let F be a subset of V of size d which is not a facet of Δ, and set $\theta_1^F = \sum_{j \notin F} a_{1,j} x_j$. Then $\theta_F = (\theta_1^F, \theta_2, \dots, \theta_d)$ is an l.s.o.p., and we write $H_F(\Delta) \coloneqq H_{\theta_F}(\Delta)$, $\overline{H}_F(\Delta) \coloneqq \overline{H}_{\theta_F}(\Delta)$, and $\deg_F \coloneqq \deg_{\theta_F}$.

We now describe two known techniques that can be used to compute the degree map. We first recall the following application of Cramer's rule, see, e.g., [PP23, Proposition 2.1]. Below, $\mathrm{sgn}(\pi) \in \{\pm 1\}$ denotes the sign of a permutation π.

Lemma 2.3. *Let* $\mu = (\mu_1, \dots, \mu_d)$ *be an l.s.o.p. Let* $F = \{j_1 < \cdots < j_d\}$ *be a subset of* V *of size* d*. Fix* $1 \le m \le d$*. Then*

$$\mathrm{ev}_\mu([F]) x_{j_m} = -\sum_{v \in V \smallsetminus F} \mathrm{sgn}(\pi_v)\, \mathrm{ev}_\mu([F \cup \{v\} \smallsetminus \{j_m\}]) x_v \in H^1_\mu(\Delta), \tag{3}$$

where $\pi_v \in S_d$ *is the permutation such that the elements of* $\pi_v(j_1, \dots, j_{m-1}, v, j_{m+1}, \dots, j_d)$ *are in increasing order.*

Suppose that F and F' are facets of Δ. Since Δ is a connected pseudomanifold, there is a sequence of facets $F = F_1, F_2, \dots, F_s = F'$, where F_j and F_{j+1} meet along a common face of dimension $d - 2$ for $1 \le j < s$. Suppose that $F = \{j_1 < \cdots < j_d\}$ and F' meet along the common face $F \smallsetminus \{j_m\}$. Then multiplying (3) by x_F/x_{j_m} and tracing through the signs yields that $\epsilon_F \operatorname{ev}_\mu([F']) = \epsilon_{F'} \operatorname{ev}_\mu([F]) \in H^d_\mu(\Delta)$. We conclude that (2) holds for any facet F of Δ.

Given a nonzero monomial $x_{j_1}^{b_{j_1}} \cdots x_{j_s}^{b_{j_s}} \in K[\Delta]$ with each $b_j > 0$, define its *support* to be the face $\{j_1, \ldots, j_s\}$ of Δ. Suppose that the above monomial is not squarefree, i.e., $b_{j_m} > 1$ for some $1 \le m \le s$. Let F be a facet containing the support $\{j_1, \ldots, j_s\}$. Then Lemma 2.3 implies that

$$x_{j_1}^{b_{j_1}} \cdots x_{j_s}^{b_{j_s}} = -\frac{1}{\operatorname{ev}_\mu([F])} \sum_{v \in V \smallsetminus F} \operatorname{sgn}(\pi_v) \operatorname{ev}_\mu([F \cup \{v\} \smallsetminus \{j_m\}]) \frac{x_v x_{j_1}^{b_{j_1}} \cdots x_{j_s}^{b_{j_s}}}{x_{j_m}} \in H_\mu(\Delta), \tag{4}$$

for some permutations π_v as defined in Lemma 2.3. Importantly, all nonzero monomials on the right-hand side of (4) have support strictly containing the support of $x_{j_1}^{b_{j_1}} \cdots x_{j_s}^{b_{j_s}}$. Hence we may compute the degree of any monomial by using (4) to repeatedly increase the size of the support.

We will need the following lemma. Let $R \subset K$ be the localization of $k[a_{i,j}]$ at the irreducible polynomials $\{[F] : F \text{ facet of } \Delta\}$. By Proposition 2.2, ev_μ extends to a k-algebra homomorphism $\operatorname{ev}_\mu \colon R \to K$.

Lemma 2.4. *Let $\mu = (\mu_1, \ldots, \mu_d)$ be an l.s.o.p. Let $g \in k[x_1, \ldots, x_n]_d$ be a polynomial of degree d. Then $\deg(g) \in R$ and $\deg_\mu(g) = \operatorname{ev}_\mu(\deg(g))$.*

Proof. It is enough to consider the case when g is a monomial. If g is squarefree, then the result follows from (2). If g is not squarefree, then the result follows by using (4) to repeatedly increase the size of the support. □

We will apply Lemma 2.4 in combination with the following simple observation. We will often use the remark below with $P = [F]$ for some non-face F of size d.

Remark 2.5. Consider an element $f \in R$. Let $P \in k[a_{i,j}]$ be an irreducible polynomial, and suppose that there is an l.s.o.p. μ with $\operatorname{ev}_\mu(P) = 0$, but $\operatorname{ev}_\mu(f) \neq 0$. We claim that $\operatorname{ord}_P(f) = 0$. Indeed, because $\operatorname{ev}_\mu(P) = 0$, P is not a scalar multiple of any $\{[F] : F \text{ facet of } \Delta\}$, so $\operatorname{ord}_P(f) \ge 0$. But P cannot divide f to positive order as $\operatorname{ev}_\mu(f) \neq 0$.

We next recall a formula for the degree map due to Karu and Xiao. It is closely related to the work of Brion [Bri97] and that of Lee [Lee96]. To state the formula, we define $\widehat{V} := \{0\} \cup V$ and $\widehat{K} := K(a_{i,0} : 1 \le i \le d)$. For a subset $\widehat{F} = \{j_1 < \cdots < j_d\}$ of $\widehat{V}$ of size d, let $[\widehat{F}]$ be the determinant of the $d \times d$ matrix whose (i, m)th entry is a_{i,j_m}.

Proposition 2.6. [KX23, Lemma 3.1, Theorem 3.2] *Let $g \in K[x_1, \ldots, x_n]_d$ be a polynomial of degree d. For any facet $F = \{j_1 < \cdots < j_d\}$ of Δ, let $g_F(t_1, \ldots, t_d)$ be obtained from g by setting x_i to zero for $i \notin F$ and setting $x_{j_m} = t_m$ for $1 \le m \le d$. Let $X_{F,m} := (-1)^m [F \cup \{0\} \smallsetminus \{j_m\}] \in \widehat{K}$ for $1 \le m \le d$. Then*

$$\deg(g) = \sum_{F \text{ facet of } \Delta} \frac{\epsilon_F g_F(X_{F,1}, \ldots, X_{F,d})}{[F] \prod_{m=1}^d X_{F,m}}. \tag{5}$$

In particular, the expression in (5) lies in K. When $g \in k[x_1, \ldots, x_n]$ this formula specializes to a formula for all l.s.o.p.'s. Explicitly, suppose that $\mu = (\mu_1, \ldots, \mu_d)$ is an l.s.o.p. Recall that we have an evaluation map $\operatorname{ev}_\mu \colon k[a_{i,j}] \to K$ defined by $\operatorname{ev}_\mu(a_{i,j}) = \mu_{i,j}$ for $1 \le i \le d$ and $1 \le j \le n$. This naturally extends to a k-algebra homomorphism $\widehat{\operatorname{ev}}_\mu \colon k[a_{i,j}]_{1 \le i \le d, 0 \le j \le n} \to \widehat{K}$ such that $\widehat{\operatorname{ev}}(a_{i,0}) = a_{i,0}$ for $1 \le i \le d$. For the statement below, observe that if $F = \{j_1 < \cdots < j_d\}$ is a facet of Δ and $1 \le m \le d$, then $\widehat{\operatorname{ev}}_\mu([F \cup \{0\} \smallsetminus \{j_m\}])$ is nonzero since it specializes (up to a sign) to $\operatorname{ev}_\mu([F])$ by setting $a_{i,0}$ to μ_{i,j_m} for $1 \le i \le d$, and by Proposition 2.2, $\operatorname{ev}_\mu([F]) \neq 0$.

Corollary 2.7. *Suppose that $\mu = (\mu_1, \dots, \mu_d)$ is an l.s.o.p. Let $g \in k[x_1, \dots, x_n]_d$ be a polynomial of degree d. For any facet $F = \{j_1 < \cdots < j_d\}$ of Δ, let $g_F(t_1, \dots, t_d)$ be obtained from g by setting x_i to zero for $i \notin F$ and setting $x_{j_m} = t_m$ for $1 \leq m \leq d$. Let $X_{F,\mu,m} \coloneqq (-1)^m \widehat{\mathrm{ev}}_\mu([F \cup \{0\} \smallsetminus \{j_m\}]) \in \widehat{K}$ for $1 \leq m \leq d$. Then*

$$\deg_\mu(g) = \sum_{F \text{ facet of } \Delta} \frac{\epsilon_F g_F(X_{F,\mu,1}, \dots, X_{F,\mu,d})}{\mathrm{ev}_\mu([F]) \prod_{m=1}^d X_{F,\mu,m}}.$$

Proof. Recall that $R \subset K$ is the localization of $k[a_{i,j}]$ at the irreducible polynomials $\{[F] : F \text{ is a facet of } \Delta\}$, and ev_μ maps R to K which is contained in $\widehat{K}$. With the notation of Proposition 2.6, let $\widehat{R} \subset \widehat{K}$ be the localization of $k[a_{i,j}]_{1 \leq i \leq d, 0 \leq j \leq n}$ at the irreducible polynomials $\{X_{F,m} : F \text{ facet of } \Delta, 0 \leq m \leq d\}$, where $X_{F,0} \coloneqq [F]$. Then $\widehat{\mathrm{ev}}_\mu$ extends to a k-algebra homomorphism $\widehat{\mathrm{ev}}_\mu \colon \widehat{R} \to \widehat{K}$ such that ev_μ is the restriction of $\widehat{\mathrm{ev}}_\mu$ to $R \subset \widehat{R}$. By Lemma 2.4, both sides of (5) lie in $\widehat{R}$ and $\widehat{\mathrm{ev}}_\mu(\deg(g)) = \deg_\mu(g)$. The result now follows by applying $\widehat{\mathrm{ev}}_\mu$ to both sides of (5). □

We next show that the degree can be computed "locally" on Δ, in an appropriate sense. The *closed star* $\mathrm{Star}_\Delta(G)$ of a face G of Δ is the simplicial complex consisting of all faces G' of Δ that contain G, together with their subfaces. Let $G = \{j_1, \dots, j_s\}$ be a face of Δ, and let S be the set of vertices in $\mathrm{Star}_\Delta(G)$. Let K_S be the subfield of K generated over k by $a_{i,j}$, where $1 \leq i \leq d$ and $j \in S$. Let Δ' be another connected oriented simplicial pseudomanifold over k of dimension $d-1$, with vertex set V' and a face $G' = \{j'_1, \dots, j'_s\}$. Let $K' = k(a'_{i,j})_{1 \leq i \leq d, j \in V'}$ be the field of coefficients for $\overline{H}(\Delta')$. Suppose that there is an isomorphism of simplicial complexes $\tau \colon \mathrm{Star}_\Delta(G) \to \mathrm{Star}_{\Delta'}(G')$ that maps j_m to j'_m for $1 \leq m \leq s$. Then τ allows us to identify K_S with a subfield of K'. Let $\deg_\Delta \colon H^d(\Delta) \to K$ and $\deg_{\Delta'} \colon H^d(\Delta') \to K'$ denote the degree maps for Δ and Δ' respectively.

Lemma 2.8. *With the notation above, let $x_{j_1}^{b_1} \cdots x_{j_s}^{b_s}$ be a monomial of degree d with support G. Then $\deg_\Delta(x_{j_1}^{b_1} \cdots x_{j_s}^{b_s}) \in K_S$. Using the identification of K_S with a subfield of K', we have*

$$\deg_\Delta(x_{j_1}^{b_1} \cdots x_{j_s}^{b_s}) = \epsilon \deg_{\Delta'}(x_{j'_1}^{b_1} \cdots x_{j'_s}^{b_s}),$$

where $\epsilon = 1$ if the orientations on $\mathrm{Star}_\Delta(G)$ induced by the orientations on Δ and Δ' agree, and $\epsilon = -1$ if they are opposite.

Proof. The only nonzero terms in the right-hand side of the formula for $\deg_\Delta(x_{j_1}^{b_1} \cdots x_{j_s}^{b_s})$ in (5) are those corresponding to facets in $\mathrm{Star}_\Delta(G)$, and those terms lie in $K_S(a_{i,0})_{1 \leq i \leq d}$ and are equal (up to a global sign) to the corresponding terms in the formula for $\deg_{\Delta'}(x_{j'_1}^{b_1} \cdots x_{j'_s}^{b_s})$. □

For the remainder of the section, we assume that k has characteristic 2. For $1 \leq i \leq d$ and $1 \leq j \leq n$, let $\partial_{a_{i,j}} \colon K \to K$ be the partial derivative with respect to $a_{i,j}$. Since k has characteristic 2,

$$\partial_{a_{i,j}} \lambda^2 = 0 \tag{6}$$

for all $\lambda \in K$. For a sequence $I = (i_1, \dots, i_r)$ in V, let $\partial_I = \partial_{a_{1,i_1}} \cdots \partial_{a_{d,i_r}}$; note that this depends on the order of I. Let $x_I = x_{i_1} \cdots x_{i_r}$. If we write $x_I = x_1^{b_1} \cdots x_n^{b_n}$ for some nonnegative integers b_i, then we define $\sqrt{x_I}$ to be $x_1^{b_1/2} \cdots x_n^{b_n/2}$ if each b_i is even, and otherwise we define $\sqrt{x_I}$ to be 0. The following result of Karu and Xiao was conjectured in [PP20, Conjecture 14.1].

Proposition 2.9. [KX23, Theorem 4.1] *Let Δ be a pseudomanifold of dimension $d-1$, and assume $\mathrm{char}\, k = 2$. Let I and J be sequences of elements of V of size d. Then*

$$\partial_I \deg(x_J) = \deg(\sqrt{x_I x_J})^2.$$

Recall that for a non-face F of Δ of size d, $\theta_1^F = \sum_{j \notin F} a_{1,j} x_j$, $\theta_F = (\theta_1^F, \theta_2, \ldots, \theta_d)$, $\overline{H}_F(\Delta) = \overline{H}_{\theta_F}(\Delta)$, and $\deg_F = \deg_{\theta_F}$. We have the following corollary.

Corollary 2.10. *Let Δ be a pseudomanifold of dimension $d-1$, and assume that* $\operatorname{char} k = 2$. *Let F be a non-face of Δ of size d, and let I and J be sequences of elements of V of size d. Assume that the first element of I does not lie in F. Then*

$$\partial_I \deg_F(x_J) = \deg_F(\sqrt{x_I x_J})^2.$$

Proof. Recall that $R \subset K$ is the localization of $k[a_{i,j}]$ at $\{[G] : G \text{ facet of } \Delta\}$, and $\operatorname{ev}_{\theta_F} \colon R \to K$. By Lemma 2.4, $\deg(x_J) \in R$ and $\deg_F(x_J) = \operatorname{ev}_{\theta_F}(\deg(x_J))$. Observe that ∂_I restricts to a map from R to itself. Since the first element of I does not lie in F,

$$\partial_I \operatorname{ev}_{\theta_F}(\lambda) = \operatorname{ev}_{\theta_F}(\partial_I \lambda)$$

for any $\lambda \in R$. Using Proposition 2.9 and Lemma 2.4, we compute

$$\partial_I \deg_F(x_J) = \partial_I \operatorname{ev}_{\theta_F}(\deg(x_J)) = \operatorname{ev}_{\theta_F}(\partial_I \deg(x_J)) = \operatorname{ev}_{\theta_F}(\deg(\sqrt{x_I x_J})^2) = \deg_F(\sqrt{x_I x_J})^2. \qquad \square$$

The following analogue of [KX23, Corollary 4.2] will be useful in what follows. Although the proof is identical to that of [KX23, Corollary 4.2], we recall it for the benefit of the reader.

Corollary 2.11. *Let Δ be a pseudomanifold of dimension $d-1$, and assume that* $\operatorname{char} k = 2$. *Let F be a non-face of Δ of size d, let $h \in K[x_1, \ldots, x_n]_q$ for some $0 \le q \le d/2$, and let I and J be sequences of elements of V of size d and $d - 2q$ respectively. Assume that the first element of I does not lie in F. Then*

$$\partial_I \deg_F(h^2 x_J) = \deg_F(h\sqrt{x_I x_J})^2.$$

Proof. Write $h = \sum_L \lambda_L x_L$ for some sequences L of elements of V of size q and some $\lambda_L \in K$. By (6), ∂_I commutes with multiplication by elements of K^2. Using Corollary 2.10, we compute

$$\begin{aligned} \partial_I \deg_F(h^2 x_J) &= \sum_L \partial_I(\lambda_L^2 \deg_F(x_L^2 x_J)) \\ &= \sum_L \lambda_L^2 \partial_I \deg_F(x_L^2 x_J) \\ &= \sum_L \lambda_L^2 \deg_F(x_L \sqrt{x_I x_J})^2 \\ &= \deg_F(h\sqrt{x_I x_J})^2. \qquad \square \end{aligned}$$

For another recent adaptation of [KX23, Theorem 4.1] to a nongeneric situation, we refer the reader to [Oba24, Lemma 4.2 and Corollary 4.3].

3. Some important special cases

In this section, we analyze several important special cases. In order to prove the main theorems, we will need a detailed understanding of the suspension of the boundary of the $(d-1)$-dimensional simplex, i.e., the complex Σ with vertex set $V = \{1, \ldots, d+2\}$ and minimal non-faces $\{1, \ldots, d\}$ and $\{d+1, d+2\}$. For this, it will be helpful to study the boundary of the d-dimensional simplex, i.e., the complex S^{d-1} with vertex set $\{1, \ldots, d+1\}$ and minimal non-face $\{1, \ldots, d+1\}$. We continue to assume that $d > 1$ unless otherwise stated. Let k be any field.

Recall that the polynomials $\{[G] : G \subset V,\ |G| = d\}$ in $k[a_{i,j}]$ are irreducible (see, for example, Lemma 4.2). We will need the following lemma.

Lemma 3.1. *If A can be written as a k-linear combination of $\{[G] : G \subset V, |G| = d\}$ where at least two coefficients are nonzero, then* $\operatorname{ord}_{[G]}(A) = 0$ *for all $G \subset V$ of size d.*

Proof. The k-algebra generated by the irreducible polynomials $\{[G] : G \subset V, |G| = d\}$ in $k[a_{i,j}]$ is isomorphic to the Plücker ring, i.e., the homogeneous coordinate ring of the Grassmannian of d-planes in k^n. In particular, since the Plücker relations all have degree strictly greater than 1, the polynomials $\{[G] : G \subset V, |G| = d\}$ are linearly independent over k. Each $[G]$ is homogeneous of degree d. Hence A is also homogeneous of degree d. If $\operatorname{ord}_{[G']}(A) > 0$, then, by comparing degrees, $A = \lambda[G']$ for some $\lambda \in k$, contradicting the assumption that at least two coefficients are nonzero. ☐

We can now analyze the boundary of the d-dimensional simplex, for $d > 1$.

Example 3.2. Let S^{d-1} be the boundary of the d-dimensional simplex with vertex set $\{1, \dots, d+1\}$. By Lemma 2.3, for $1 \le m, p \le d+1$, we have

$$[V \setminus \{p\}]x_m = (-1)^{|p-m|}[V \setminus \{m\}]x_p \in H^1(S^{d-1}). \tag{7}$$

Fix $0 \le q \le d/2$. A basis for $H^q(S^{d-1}) = \overline{H}^q(S^{d-1})$ is x_1^q. Using (7), we compute

$$\left(\prod_{m=2}^{d}(-1)^{m-1}[V \setminus \{m\}]\right)\deg(x_1^d) = [V \setminus \{1\}]^{d-1}\deg(x_1 \cdots x_d),$$

and hence, for some $\epsilon \in \{\pm 1\}$, we have

$$\deg(x_1^d) = \frac{\epsilon[V \setminus \{1\}]^d}{\prod_{m=1}^{d+1}[V \setminus \{m\}]}.$$

Using (7), we compute

$$[V \setminus \{1\}]^{d-2q}\deg(\ell^{d-2q}x_1^{2q}) = A^{d-2q}\deg(x_1^d),$$

where $A := \sum_{m=1}^{d+1}(-1)^{m-1}[V \setminus \{m\}]$. By Lemma 3.1, $\operatorname{ord}_{[G]}(A) = 0$ for all facets G of S^{d-1}. Putting this together gives

$$\deg(\ell^{d-2q}x_1^{2q}) = \frac{\epsilon A^{d-2q}[V \setminus \{1\}]^{2q}}{\prod_{m=1}^{d+1}[V \setminus \{m\}]}.$$

Let D_q be the image of $\deg(\ell^{d-2q}x_1^{2q})$ in $K^\times/(K^\times)^2$. Then

$$D_q = \begin{cases} \epsilon\prod_{G \text{ facet}}[G] & \text{if } d \text{ is even} \\ \epsilon A\prod_{G \text{ facet}}[G] & \text{if } d \text{ is odd.} \end{cases}$$

We conclude that Theorem 1.3 holds in this case (without any assumptions on k). Observe that Theorem 1.4 holds vacuously since S^{d-1} has no non-faces of size d.

We will also need to analyze the case of S^0, i.e., the disjoint union of two vertices. Although S^0 is not connected, $H(S^0)$ is a Gorenstein ring with a well-defined degree map, and we may verify directly that the conclusion of Theorem 1.3 holds in this case (without any assumptions on k).

Example 3.3. Let $d = 1$, $V = \{1, 2\}$, and consider the complex S^0 with vertex set $\{1, 2\}$ and minimal non-face $\{1, 2\}$. We orient S^0 by assigning -1 to the facet $\{1\}$ and assigning 1 to the facet $\{2\}$. We have

$$\deg(\ell) = \deg(x_1) + \deg(x_2) = -\frac{1}{a_{1,1}} + \frac{1}{a_{1,2}} = \frac{a_{1,1} - a_{1,2}}{a_{1,1}a_{1,2}}.$$

We now analyze Σ, the suspension of the boundary of the $(d-1)$-dimensional simplex, i.e., the complex with $V = \{1, \dots, d+2\}$ and minimal non-faces $\{1, \dots, d\}$ and $\{d+1, d+2\}$; in particualr, $\ell = x_1 + \cdots + x_{d+2}$. The proofs of Theorem 1.1 and Theorem 1.3 will depend on this special case, via a use of Lemma 2.8.

Lemma 3.4. *Let $F = \{1, \dots, d\}$. There are polynomials $A_{d+1}, A_{d+2} \in k[a_{i,j}]$ such that $\mathrm{ord}_{[G]}(A_{d+1}) = \mathrm{ord}_{[G]}(A_{d+2}) = 0$ for any subset G of V of size d, and*

$$[F]\ell = A_{d+1}x_{d+1} + A_{d+2}x_{d+2} \in H^1(\Sigma).$$

Moreover, there is $\epsilon \in \{\pm 1\}$ such that, for $0 < j \le d$,

$$\deg(\ell^{d-j}x_{d+1}^j) = \frac{\epsilon A_{d+1}^{d-j}[F]^{j-1}}{\prod_{m=1}^d [F \cup \{d+1\} \smallsetminus \{m\}]} \quad \text{and} \quad \deg(\ell^{d-j}x_{d+2}^j) = \frac{-\epsilon A_{d+2}^{d-j}[F]^{j-1}}{\prod_{m=1}^d [F \cup \{d+2\} \smallsetminus \{m\}]}. \tag{8}$$

Proof. By Lemma 2.3, for $1 \le m \le d$, we have

$$[F]x_m = (-1)^{d+1-m}([F \cup \{d+1\} \smallsetminus \{m\}]x_{d+1} + [F \cup \{d+2\} \smallsetminus \{m\}]x_{d+2}) \in H^1(\Sigma). \tag{9}$$

For $v \in \{d+1, d+2\}$, by multiplying x_v with the product of (9) over all $1 \le m < d$ and by using the relation $x_{d+1}x_{d+2} = 0$ in $K[\Sigma]$, we deduce that there is $\epsilon' \in \{\pm 1\}$ such that

$$\left(\prod_{m=1}^{d-1} [F \cup \{v\} \smallsetminus \{m\}] \right) x_v^d = \epsilon' [F]^{d-1} x_1 \cdots x_{d-1} x_v.$$

By Proposition 2.1, taking degrees of both sides of this equation yields

$$\epsilon_{F \cup \{v\} \smallsetminus \{d\}} \deg(x_v^d) = \frac{\epsilon' [F]^{d-1}}{\prod_{m=1}^d [F \cup \{v\} \smallsetminus \{m\}]}.$$

If we set $\epsilon = \epsilon' \epsilon_{F \cup \{v\} \smallsetminus \{d\}}$, then the $j = d$ case of (8) follows since $\epsilon_{F \cup \{d+1\} \smallsetminus \{d\}} = -\epsilon_{F \cup \{d+2\} \smallsetminus \{d\}}$. Applying (9) for $1 \le m \le d$ yields

$$[F]\ell = A_{d+1}x_{d+1} + A_{d+2}x_{d+2} \in H^1(\Sigma), \tag{10}$$

where for $v \in \{d+1, d+2\}$, we have

$$A_v = [F] + (-1)^{d+1} \sum_{m=1}^d (-1)^m [F \cup \{v\} \smallsetminus \{m\}] \in k[a_{i,j}].$$

By Lemma 3.1, $\mathrm{ord}_{[G]}(A_{d+1}) = \mathrm{ord}_{[G]}(A_{d+2}) = 0$ for any subset G of V of size d. Using (10) and the relation $x_{d+1}x_{d+2} = 0$, we compute

$$[F]^{d-j} \deg(\ell^{d-j}x_v^j) = A_v^{d-j} \deg(x_v^d).$$

The result now follows from the $j = d$ case of (8). □

Although it will not be needed in what follows, we observe that Lemma 3.4 implies Theorem 1.3 (without any assumptions on k) for Σ in the case $0 < q \le d/2$. Explicitly, $\{x_{d+1}^q, x_{d+2}^q\}$ is a basis for $H^q(\Sigma) = \overline{H}^q(\Sigma)$, and (8) implies that the corresponding determinant $D_q \in K^\times/(K^\times)^2$ is equal to

$$D_q = \deg(\ell^{d-2q}x_{d+1}^{2q}) \deg(\ell^{d-2q}x_{d+2}^{2q}) = \begin{cases} -\prod_{G \text{ facet}} [G] & \text{if } d \text{ is even} \\ -A_{d+1}A_{d+2} \prod_{G \text{ facet}} [G] & \text{if } d \text{ is odd.} \end{cases}$$

The next lemma will be crucial to the proof of Theorem 1.3 when $q = 0$. Recall that $d > 1$ and Σ is the suspension of the boundary of the $(d-1)$-dimensional simplex.

Lemma 3.5. *For every non-face G of Σ of size d, we have $\mathrm{ord}_{[G]}(\deg(\ell^d)) = 0$.*

Proof. By Lemma 2.4 and Remark 2.5, it is enough to show that there is an l.s.o.p. μ which has $\mathrm{ev}_\mu([G]) = 0$, but $\deg_\mu(\ell^d) = \mathrm{ev}_\mu(\deg(\ell^d)) \neq 0$. Set $\mu_i = a_{i,1}x_1 + \cdots + a_{i,d}x_d$ for $1 \leq i < d$, and set $\mu_d = a_{d,d+1}x_{d+1} + a_{d,d+2}x_{d+2}$. Because $\Sigma = S^{d-2} * S^0$, we see that $H_\mu(\Sigma) = H(S^{d-2}) \otimes H(S^0)$. Furthermore, we can write $\ell = \ell_1 + \ell_2$, where $\ell_1 = x_1 + \cdots + x_d$ and $\ell_2 = x_{d+1} + x_{d+2}$. We have $\ell_1^d = 0$ and $\ell_2^2 = 0$, and we see that

$$\deg_\mu(\ell^d) = \deg_\mu(\ell_1^{d-1}\ell_2) = \deg_{S^{d-2}}(\ell_1^{d-1}) \deg_{S^0}(\ell_2).$$

Since $\deg_{S^{d-2}}(\ell_1^{d-1}) \neq 0$ and $\deg_{S^0}(\ell_2) \neq 0$ by Example 3.2 and Example 3.3 respectively, we deduce that $\deg_\mu(\ell^d) \neq 0$. It remains to show that $\mathrm{ev}_\mu([G]) = 0$. Since G is a non-face, either $G = \{1, \ldots, d\}$ or G contains $\{d+1, d+2\}$. In the former case, $\mathrm{ev}_\mu([G])$ is the determinant of a matrix whose dth row is identically zero. In the latter case, $\mathrm{ev}_\mu([G])$ is the determinant of a matrix whose last two columns are identically zero except in the dth row and hence are linearly dependent. □

4. The degree zero case

In this section we prove the following result that settles the $q = 0$ case of Theorem 1.3. This case works over a field of any characteristic.

Theorem 4.1. *Let Δ be a connected oriented pseudomanifold of dimension $d-1$ with vertex set V. Let F be a subset of V of size d. Then*

$$\mathrm{ord}_{[F]}(\deg(\ell^d)) = \begin{cases} -1 & \text{if } F \text{ is a facet of } \Delta \\ 0 & \text{otherwise.} \end{cases}$$

Recall that throughout we are assuming that $d > 1$. We first prove a lemma which will be used in the proof of Theorem 4.1. The case when $p = m$ is very well-known; see, for example, [Boc64, Theorem 61.1].

Lemma 4.2. *For some $1 < p \leq m$, let N be the $m \times m$ matrix with $N_{i,j} = a_{i,j}$ if $i = 1$ and $j \leq p$ or if $i > 1$, and $N_{i,j} = 0$ for $i = 1$ and $j > p$. Then $\det N$ is an irreducible polynomial in $k[a_{i,j}]$.*

Proof. Suppose that $\det N = fg$, where $f, g \in k[a_{i,j}]$. Because $\det N$ is linear in $a_{1,1}$, we see that $a_{1,1}$ must occur in exactly one of f and g, say f. Because $p > 1$, the variables $a_{i,1}$ appear in $\det N$ for $i \geq 1$. Those variables must also only occur in f, because $a_{1,1}a_{i,1}$ does not appear in $\det N$. This implies that $a_{i,j}$ must also occur only in f for each j, because $a_{i,1}a_{i,j}$ does not appear in $\det N$. We conclude that g is a unit. □

In particular, Lemma 4.2 implies that the polynomial $\det N$ defines a valuation on K. We now begin proving Theorem 4.1. We first deal with the case when F is a facet.

Proposition 4.3. *Let F be a facet of Δ. Then*

$$\mathrm{ord}_{[F]}(\deg(\ell^d)) = -1.$$

Proof. Using Proposition 2.6 and properties of valuations, we have

$$\mathrm{ord}_{[F]}(\deg(\ell^d)) \geq \min_{G \text{ facet of } \Delta} \left(d\, \mathrm{ord}_{[F]}(X_{G,1} + \cdots + X_{G,d}) - \mathrm{ord}_{[F]}([G]) - \sum_{m=1}^{d} \mathrm{ord}_{[F]}(X_{G,m}) \right), \tag{11}$$

with equality if the minimum is achieved only once. As $X_{G,m}$, $[G]$, and $[F]$ are irreducible polynomials of the same degree which are not scalar multiples of each other (except that $[G] = [F]$ if $G = F$), we see that for $G \neq F$, the quantity in the minimum in (11) is nonnegative. Note that $\mathrm{ord}_{[F]}(X_{F,1} + \cdots + X_{F,d}) = 0$ by Lemma 3.1. Therefore the quantity in the minimum in (11) is equal to -1 when $G = F$, and so the minimum is -1 and is achieved exactly once. □

Proof of Theorem 4.1. By Proposition 4.3, it suffices to show that if F is a subset of V of size d which is not a facet, then $\mathrm{ord}_{[F]}(\deg(\ell^d)) = 0$. By Lemma 2.4 and Remark 2.5, it is enough to show that $\deg_F(\ell^d) = \mathrm{ev}_{\theta_F}(\deg(\ell^d)) \neq 0$.

First assume there is a facet F' of Δ with $|F' \cap F| \leq d-2$. Let $\overline{[F']} = \mathrm{ev}_{\theta_F}([F'])$, which is irreducible by Lemma 4.2. We use Corollary 2.7 to compute that $\mathrm{ord}_{\overline{[F']}}(\deg_F(\ell^d))$ is bounded below by

$$\min_{G \text{ facet of } \Delta} \left(d \,\mathrm{ord}_{\overline{[F']}}(X_{G,\theta_F,1} + \cdots + X_{G,\theta_F,d}) - \mathrm{ord}_{\overline{[F']}}(\mathrm{ev}_{\theta_F}([G])) - \sum_{m=1}^{d} \mathrm{ord}_{\overline{[F']}}(X_{G,\theta_F,m}) \right), \tag{12}$$

with equality if the minimum is achieved only once. If $G \neq F'$, then $\mathrm{ord}_{\overline{[F']}}(\mathrm{ev}_{\theta_F}([G])) = 0$, because $\overline{[F']}$ and $\mathrm{ev}_{\theta_F}([G])$ are irreducible polynomials of the same degree which are not scalar multiples. Similarly, $\mathrm{ord}_{\overline{[F']}}(X_{G,\theta_F,m}) = 0$ (this holds even if $G = F'$). So if $G \neq F'$, then the quantity in the minimum in (12) is nonnegative.

If $G = F'$, then $\mathrm{ord}_{\overline{[F']}}(\mathrm{ev}_{\theta_F}([G])) = \mathrm{ord}_{\overline{[F']}}(\overline{[F']}) = 1$. Write $F' = \{j_1 < \cdots < j_d\}$ and fix $1 \leq m \leq d$. Then for $1 \leq m' \leq d$, the coefficient of the monomial $a_{1,0}a_{2,j_1} \cdots a_{m,j_{m-1}}a_{m+1,j_{m+1}} \cdots a_{d,j_d}$ in $X_{F',\theta_F,m'}$ is 1 if $m = m'$ and is 0 otherwise, and the coefficient of this monomial in $[F']$ is zero. We deduce that $X_{F',\theta_F,1} + \cdots + X_{F',\theta_F,d}$ is nonzero with the same degree as $[F']$ and $\mathrm{ord}_{\overline{[F']}}(X_{F',\theta_F,1} + \cdots + X_{F',\theta_F,d}) = 0$. Therefore, the quantity in the minimum in (12) is -1 for $G = F'$, so we deduce that $\mathrm{ord}_{\overline{[F']}}(\deg_F(\ell^d)) = -1$. In particular, $\deg_F(\ell^d) \neq 0$.

Suppose that there is no such facet F'. Then we show that Δ must be the suspension Σ of the boundary of a $(d-1)$-dimensional simplex, i.e., the case discussed in Section 3. Let v be a vertex of Δ not in F, so every facet containing v has $d-1$ vertices from F. Let L be the link of v. Then L is pure of dimension $d-2$ and has the property that any face of dimension $d-3$ is contained in exactly two faces of dimension $d-2$. Because all facets of L are contained in the boundary of F and L is pure, L is isomorphic to a subcomplex of S^{d-2}. The only subcomplex of S^{d-2} which is pure of dimension $d-2$ and has the property that any face of dimension $d-3$ is contained in exactly two faces of dimension $d-2$ is all of S^{d-2}.

We see that Δ is isomorphic to the join of S^{d-2} with a disjoint union of some vertices $\{v_1, \ldots, v_r\}$. Then the link of any facet of S^{d-2} is $\{v_1, \ldots, v_r\}$. Since every $(d-2)$-dimensional face of Δ lies in exactly two facets, we must have $r = 2$. Therefore $\Delta = \Sigma$. The case of Σ was treated in Lemma 3.5. □

Remark 4.4. The above argument, together with the proof of Lemma 3.5, shows that if F is a non-face of size d, then $\deg_F(\ell^d) = \mathrm{ev}_{\theta_F}(\deg(\ell^d))$ is nonzero. In particular, $\overline{H}_F(\Delta)$ has the strong Lefschetz property in degree 0 over any field.

5. Almost anisotropy

In this section and the next, we prove Theorem 1.4 when the characteristic of k is 2. We use the method introduced by Papadakis and Petrotou [PP20] based on the special behavior of differential operators in characteristic 2, as refined by Karu and Xiao [KX23]. In the case of $\overline{H}(\Delta)$, their approach establishes that the Hodge–Riemann forms are anisotropic. While, in general, anisotropy fails for $\overline{H}_F(\Delta)$ (see Example 5.3), the same approach allows us to prove anisotropy "away from a 1-dimensional subspace" (Proposition 5.6). Furthermore, as we will show in the next section, this weaker property is enough to deduce the strong Lefschetz property.

Fix a non-face F of Δ of size d. Recall from the introduction that $\theta_1^F = \sum_{j \notin F} a_{1,j}x_j$ and that $\overline{H}_F(\Delta)$ is the Gorenstein quotient of $K[\Delta]/(\theta_1^F, \ldots, \theta_d)$. Let $\ell_F = \sum_{j=1}^n x_j \in \overline{H}_F^1(\Delta)$. For $0 \leq q \leq d/2$, define the Hodge–Riemann form $\overline{H}_F^q(\Delta) \times \overline{H}_F^q(\Delta) \to K$ via $(y, z) \mapsto \deg_F(\ell_F^{d-2q}yz)$. The nondegeneracy of the

Hodge–Riemann form is equivalent to the map $\overline{H}_F^q(\Delta) \to \overline{H}_F^{d-q}(\Delta)$ given by multiplication by ℓ_F^{d-2q} being an isomorphism, and, by Lemma 5.1 below, this is equivalent to the strong Lefschetz property in degree q.

Lemma 5.1. *Fix some $0 \le q \le d/2$. The algebra $\overline{H}_F(\Delta)$ has the strong Lefschetz property in degree q if and only if multiplication by ℓ_F^{d-2q} is an isomorphism from $\overline{H}_F^q(\Delta) \to \overline{H}_F^{d-q}(\Delta)$, i.e., ℓ_F is a strong Lefschetz element.*

Proof. If ℓ_F is a strong Lefschetz element in degree q, then clearly $\overline{H}_F(\Delta)$ has the strong Lefschetz property in degree q. For the converse, we may replace k by its algebraic closure. Then a Zariski open subset of all $y \in \overline{H}_F^1(\Delta)$ are strong Lefschetz elements in degree q. It follows that a Zariski open subset of all coefficients $(\lambda_1, \dots, \lambda_n) \in k^n$ correspond to elements $\sum \lambda_j x_j \in \overline{H}_F^1(\Delta)$ which are strong Lefschetz elements in degree q. Therefore, we can find a strong Lefschetz element $\ell_{F,\lambda} = \sum \lambda_j x_j$ with each $\lambda_j \in k^\times$. Let

$$H_{F,\lambda}(\Delta) = K[\Delta]/(\sum_{j \notin F} \lambda_j a_{1,j} x_j, \sum_j \lambda_j a_{2,j} x_j, \dots, \sum_j \lambda_j a_{d,j} x_j),$$

and let $\overline{H}_{F,\lambda}(\Delta)$ be the Gorenstein quotient. Because the $\lambda_j a_{i,j}$ are algebraically independent, $\ell_{F,\lambda}$ is a strong Lefschetz element for $\overline{H}_{F,\lambda}(\Delta)$. Let $\Phi\colon \overline{H}_F(\Delta) \to \overline{H}_{F,\lambda}$ be the graded isomorphism given by sending $a_{i,j}$ to $\lambda_j a_{i,j}$. Then we have a commutative square

$$\begin{array}{ccc} \overline{H}_F^q(\Delta) & \xrightarrow{\ell_F^{d-2q}} & \overline{H}_F^{d-q}(\Delta) \\ \downarrow \Phi & & \downarrow \Phi \\ \overline{H}_{F,\lambda}^q(\Delta) & \xrightarrow{\ell_{F,\lambda}^{d-2q}} & \overline{H}_{F,\lambda}^{d-q}(\Delta). \end{array}$$

As the bottom horizontal arrow is an isomorphism, so is the top horizontal arrow. □

A similar equivalence holds for $\overline{H}(\Delta)$, i.e., $\overline{H}(\Delta)$ has the strong Lefschetz property in degree q if and only if ℓ is a strong Lefschetz element in degree q.

Say that $\overline{H}_F(\Delta)$ has the *weak Lefschetz property* in degree q if there is an element $y \in \overline{H}_F^1(\Delta)$ such that multiplication by y induces a map of full rank from $\overline{H}_F^q(\Delta)$ to $\overline{H}_F^{q+1}(\Delta)$. It is well-known (see, e.g., [MZ08, Lemma 2.3 and Remark 2.4]) that an artinian Gorenstein ring which is generated in degree 1 (such as $\overline{H}_F(\Delta)$) has the weak Lefschetz property in all degrees if and only if it has the weak Lefschetz property in middle degree (i.e., degree $\lfloor d/2 \rfloor$); furthermore, multiplication by u induces an injection from $\overline{H}_F^q(\Delta)$ to $\overline{H}_F^{q+1}(\Delta)$ for $q < d/2$ and a surjection for $q > d/2 - 1$. The proof of Lemma 5.1 gives the following result.

Lemma 5.2. *For any q, the algebra $\overline{H}_F(\Delta)$ has the weak Lefschetz property in degree q if and only if multiplication by ℓ_F is a map of full rank from $\overline{H}_F^q(\Delta)$ to $\overline{H}_F^{q+1}(\Delta)$, i.e., ℓ_F is a weak Lefschetz element in degree q.*

For the rest of this section, let k be a field of characteristic 2, and fix a non-face F of Δ of size d. Recall that by Lemma 5.1, Theorem 1.4 holds in degree q if and only if the Hodge–Riemann form on $\overline{H}_F^q(\Delta)$ is nondegenerate. We will show that when the latter condition holds, the induced quadratic form is "almost" anisotropic, in the sense that there is at most one nonzero vector up to scaling for which the quadratic form is zero.

Recall that a quadratic form Q on a vector space V is *anisotropic* if $Q(v) \neq 0$ for all nonzero v in V. Fix $0 \le q \le d/2$. Karu and Xiao proved the anisotropy of the (quadratic form associated to the) Hodge–Riemann form on $\overline{H}^q(\Delta)$ [KX23, Theorem 4.4], i.e., $\deg(\ell^{d-2q} z^2) \neq 0$ for nonzero $z \in \overline{H}^q(\Delta)$. Our goal is to analogously establish "almost" anisotropy for the Hodge–Riemann form on $\overline{H}^q_F(\Delta)$. The following example shows that anisotropy need not hold.

Example 5.3. Let $d = 2$, and consider Σ as in Section 3, i.e., Σ has vertex set $\{1, 2, 3, 4\}$ and minimal non-faces $\{1, 2\}$ and $\{3, 4\}$. Consider $H_F(\Sigma) = \overline{H}_F(\Sigma)$, where $F = \{1, 2\}$. Then $\theta_1^F = a_{1,3}x_3 + a_{1,4}x_4$, so the relation $x_3x_4 = 0$ in $K[\Sigma]$ implies that $x_3^2 = 0$ in $H_F(\Sigma)$. As $x_3 \neq 0$ in $H_F(\Sigma)$, anisotropy fails for $H_F(\Sigma)$.

Let $W_q \subset \overline{H}^q_F(\Delta)$ be the span of all monomials whose support is not contained in F. We will mainly be interested in the case when $q = \lfloor d/2 \rfloor$.

Lemma 5.4. *The codimension of W_q in $\overline{H}^q_F(\Delta)$ is at most 1. Consider a linear form $g = \sum_{i=1}^n \lambda_i x_i$ for some $\lambda_i \in k$ such that $\lambda_v \neq 0$ for some $v \in F$. If g^q lies in W_q, then $W_q = \overline{H}^q_F(\Delta)$.*

Proof. Let w be a vertex not in F. We claim that $\{g\} \cup \{x_v : v \notin F \cup \{w\}\}$ is a basis of $H^1_F(\Delta)$. Assuming this claim, monomials of degree q in this basis span $\overline{H}^q_F(\Delta)$. Every such monomial except g^q lies in W_q, proving that W_q has codimension at most 1, and that $W_q = \overline{H}^q_F(\Delta)$ if g^q lies in W_q.

It remains to establish the claim. We need to show that $\theta_1^F, \theta_2, \dots, \theta_d$ together with $\{g\} \cup \{x_v : v \notin F \cup \{w\}\}$ are linearly independent, and hence form a basis of $K[x_1, \dots, x_n]_1$. Without loss of generality, assume that $F = \{1, \dots, d\}$ and $w = d + 1$. Let M denote the $(d + 1) \times (d + 1)$ matrix whose rows record the coefficients of $\theta_1^F, \theta_2, \dots, \theta_d, g$ with respect to the vertices $\{1, \dots, d + 1\}$. It is enough to show that $\det M \neq 0$. Let M' be the submatrix obtained by removing the first row and last column. Since the only nonzero entry in the first row of M is the last entry, it suffices to show that $\det M' \neq 0$. This follows since all entries of M' are generic except the last row, which is nonzero by assumption. □

Let $W_q^\perp = \{z \in \overline{H}^q_F(\Delta) : \deg_F(\ell_F^{d-2q} zw) = 0 \text{ for all } w \in W_q\}$. If we assume the Hodge–Riemann form on $\overline{H}^q_F(\Delta)$ is nondegenerate in degree q, then $W_q^\perp$ is the orthogonal complement of W_q and the dimension of $W_q^\perp$ is at most 1 by Lemma 5.4.

Example 5.5. In Proposition 8.3, we will show that $\overline{H}^1_F(\Delta) = H^1_F(\Delta)$. In this case, as θ_1^F gives a linear relation between $\{x_w : w \notin F\}$ in $H^1_F(\Delta)$, W_1 has codimension 1. In Example 5.3, $W_1 = W_1^\perp = \text{Span}(x_3) \subset H^1_F(\Delta)$.

We have the following application of Corollary 2.11.

Proposition 5.6. *Let $z \in \overline{H}^q_F(\Delta)$. If $\deg_F(\ell_F^{d-2q} z^2) = 0$, then $z \in W_q^\perp$.*

Proof. Let $J = (j_1, \dots, j_q)$ be a sequence of elements of V such that the support of x_J is not contained in F. We may assume that $j_1 \notin F$. Suppose that we can show that $\deg_F(\ell_F^{d-2q} x_J z) = 0$. Then applying this to all such J implies that $z \in W_q^\perp$.

Let L be a sequence of elements of V of size $\lfloor d/2 \rfloor - q$, and let v be a vertex of V. Let $(J_1 | \cdots | J_r)$ denote the concatenation of sequences of vertices $J_1, \dots, J_r$. Define

$$I = \begin{cases} (J|J|L|L) & \text{if } d \text{ is even} \\ (J|J|L|L|v) & \text{if } d \text{ is odd.} \end{cases}$$

If d is even, then Corollary 2.11 implies that

$$0 = \partial_I \deg_F(\ell_F^{d-2q} z^2) = \deg_F(\ell_F^{d/2-q} x_J x_L z)^2.$$

Since this holds for all L, we deduce that $\deg_F(\ell_F^{d-2q} x_J z) = 0$.

If d is odd, then Corollary 2.11 implies that

$$0 = \partial_I \deg_F(\ell_F^{d-2q} z^2) = \sum_{i=1}^{n} \deg_F(\ell_F^{(d-1)/2-q} x_J x_L \sqrt{x_i x_v} z)^2 = \deg_F(\ell_F^{(d-1)/2-q} x_J x_L x_v z)^2.$$

Since this holds for all L and v, we deduce that $\deg_F(\ell_F^{d-2q} x_J z) = 0$. $\square$

Corollary 5.7. *Assume that either $W_q^{\perp} = 0$, or the Hodge–Riemann form on $\overline{H}_F^q(\Delta)$ is nondegenerate and $W_q^{\perp} \not\subset W_q$. Then the Hodge–Riemann form on $\overline{H}_F^q(\Delta)$ is anisotropic.*

Proof. If $W_q^{\perp} = 0$, then Proposition 5.6 implies that this form is anisotropic. Assume that the Hodge–Riemann form on $\overline{H}_F^q(\Delta)$ is nondegenerate and $W_q^{\perp} \not\subset W_q$. Then $\dim W_q^{\perp} = \operatorname{codim} W_q = 1$ by Lemma 5.4. Let $W_q^{\perp} = \operatorname{span}(z)$, and assume that $\deg_F(\ell_F^{d-2q} z^2) = 0$. Then $z \in (W_q^{\perp})^{\perp} = W_q$, contradicting our assumption that $W_q^{\perp}$ is not contained in W_q. Hence $\deg_F(\ell_F^{d-2q} z^2) \neq 0$ and the result follows from Proposition 5.6. $\square$

Proposition 5.8. *Assume that the Hodge–Riemann form on $\overline{H}_F^q(\Delta)$ is nondegenerate. Let U be a subspace of $\overline{H}_F^q(\Delta)$ where the restriction of the Hodge–Riemann form on $\overline{H}_F^q(\Delta)$ to U is degenerate. Then $U \subset W_q$.*

Proof. If $W_q^{\perp} = 0$, then Corollary 5.7 implies that the Hodge–Riemann form in degree q is anisotropic, and so the restriction to any subspace is anisotropic and therefore nondegenerate. By Lemma 5.4, we may therefore assume that $W_q^{\perp}$ is one-dimensional, generated by y_1. By Corollary 5.7, we may assume that $y_1 \in W_q$. By Proposition 5.6, if $y_1 \notin U$, then the restriction of the Hodge–Riemann form to U is anisotropic and hence nondegenerate, a contradiction. Therefore U must contain y_1.

Assume that U is not contained in W_q. Then the codimension of $U \cap W_q$ in U is 1, and so we may extend y_1 to a basis $y_1, y_2, \ldots, y_r$ of U with $y_1, \ldots, y_{r-1} \in W_q$. Since $y_1 \in W_1^{\perp}$, $\deg_F(\ell_F^{d-2q} y_1 y_i) = 0$ for $1 \leq i < r$. As $(W_q^{\perp})^{\perp} = W_q$ and $y_r \notin W_q$, it follows that $\deg_F(\ell_F^{d-2q} y_1 y_r) \neq 0$. Let M be the $r \times r$ matrix whose (i,j)th entry is $\deg_F(\ell_F^{d-2q} y_i y_j)$. Let M' be the $(r-2) \times (r-2)$ submatrix given by rows $\{2, \ldots, r-1\}$ and columns $\{2, \ldots, r-1\}$. We see that

$$\det(M) = \deg_F(\ell_F^{d-2q} y_1 y_r)^2 \det(M').$$

As the span of $y_2, \ldots, y_{r-1}$ does not contain $W_q^{\perp}$, Proposition 5.6 implies that the Hodge–Riemann form restricted to the span of $y_2, \ldots, y_{r-2}$ is anisotropic, and hence nondegenerate. In particular, $\det(M') \neq 0$. We conclude that $\det(M) \neq 0$, contradicting that the Hodge–Riemann form is degenerate when restricted to U. $\square$

6. Strong Lefschetz in characteristic 2

In this section, we complete the proof of the strong Lefschetz property for $\overline{H}_F(\Delta)$ in characteristic 2. We use some ideas from [PP20, Section 9.1], which are in turn inspired by ideas of Swartz, e.g., [Swa09, Proposition 4.24].

Throughout this section, we assume that k has characteristic 2. Let $d > 1$ be arbitrary. Let $S(\Delta)$ denote the suspension of Δ, which has vertex set $V \cup \{n+1, n+2\} = \{1, \ldots, n+1, n+2\}$. Since Δ is a connected pseudomanifold, so is $S(\Delta)$. Let $\hat{F} = F \cup \{n+1\}$; note that $\hat{F}$ is a non-face of $S(\Delta)$. Let $\hat{K} = k(a_{i,j})_{1 \leq i \leq d+1, 1 \leq j \leq n+2}$.

We will consider a slightly different l.s.o.p. for $S(\Delta)$. Set $\hat{\theta}_1^F = a_{d+1,n+1}^{-1} \sum_{j \notin \hat{F}} a_{1,j} x_j$. For $1 < i \leq d$, set

$$\hat{\theta}_i = a_{i,n+1} x_{n+1} + a_{i,n+2} x_{n+2} + \sum_{j=1}^{n} (a_{i,j} + a_{d+1,j} a_{i,n+1} a_{d+1,n+1}^{-1}) x_j,$$

and set $\theta_{d+1} = \sum_{j=1}^{n+2} a_{d+1,j}x_j$. Let $\hat{\theta}_F = (\hat{\theta}_1^F, \hat{\theta}_2, \ldots, \hat{\theta}_d, \theta_{d+1})$. Let $\overline{H}_{\hat{\theta}_F}(S(\Delta))$ be the Gorenstein quotient of $\hat{K}[S(\Delta)]/\hat{\theta}_F$. Note that the coefficients of $\hat{\theta}_F$ are generic except that, when $j \in \hat{F}$, the coefficient of x_j in $\hat{\theta}_1^F$ is 0. In particular, $\overline{H}_{\hat{\theta}_F}(S(\Delta))$ is isomorphic as a graded ring to $\overline{H}_{\hat{F}}(S(\Delta))$.

In the statement of the proposition below, we extend $\deg_F$ to an isomorphism of $\hat{K}$-vector spaces from $\overline{H}^d_F(\Delta) \otimes_K \hat{K}$ to $\hat{K}$.

Proposition 6.1. *There is an isomorphism of $\hat{K}$-algebras*

$$\overline{\varphi}\colon \overline{H}_{\hat{\theta}_F}(S(\Delta))/\operatorname{ann}(x_{n+1}) \xrightarrow{\sim} \overline{H}_F(\Delta) \otimes_K \hat{K}$$

given by $\overline{\varphi}(x_i) = x_i$ for $i \le n$, $\overline{\varphi}(x_{n+1}) = \sum_{j=1}^n a_{d+1,j}a_{d+1,n+1}^{-1}x_j$, and $\overline{\varphi}(x_{n+2}) = 0$. Let φ be the composition

$$\overline{H}_{\hat{\theta}_F}(S(\Delta)) \to \overline{H}_{\hat{\theta}_F}(S(\Delta))/\operatorname{ann}(x_{n+1}) \xrightarrow{\overline{\varphi}} \overline{H}_F(\Delta) \otimes_K \hat{K}.$$

Then $\deg_{\hat{\theta}_F}(zx_{n+1}) = \deg_F(\varphi(z))$ for all $z \in \overline{H}^d_{\hat{\theta}_F}(S(\Delta))$.

Proof. First observe that there is a map $H_{\hat{\theta}_F}(S(\Delta)) \to H_F(\Delta) \otimes_K \hat{K}$ defined by $x_i \mapsto x_i$ for $i \le n$, $x_{n+1} \mapsto a_{d+1,n+1}^{-1}\sum_{j=1}^n a_{d+1,j}x_j$, and $x_{n+2} \mapsto 0$. We check that the induced map $\varphi'\colon H_{\hat{\theta}_F}(S(\Delta)) \to \overline{H}_F(\Delta) \otimes_K \hat{K}$ descends to $\overline{H}_{\hat{\theta}_F}(S(\Delta))/\operatorname{ann}(x_{n+1})$. For this, we need to show that if we have $0 \le q \le d$ and $y \in H^q_{\hat{\theta}_F}(S(\Delta))$ with $\deg_{\hat{\theta}_F}(x_{n+1}yz) = 0$ for all $z \in H^{d-q}_{\hat{\theta}_F}(S(\Delta))$, then $\varphi'(y) = 0$. To do so, it suffices to prove that

$$\deg_{\hat{\theta}_F}(x_{n+1}w) = \deg_F(\varphi'(w)) \quad \text{for all } w \in H^d_{\hat{\theta}_F}(S(\Delta)).$$

We claim that $H^d_{\hat{\theta}_F}(S(\Delta))$ is spanned by squarefree monomials in $x_1, \ldots, x_n$. It is well-known that it is spanned by squarefree monomials in $x_1, \ldots, x_n, x_{n+1}, x_{n+2}$ supported on $(d-1)$-dimensional faces of $S(\Delta)$; this follows by using (4) to repeatedly increase the size of the support. Let $G = \{i_1, \ldots, i_{d-1}, i_d\}$ be one such face with $i_d \in \{n+1, n+2\}$ and let $G' = (G \smallsetminus \{i_d\}) \cup \{n+1, n+2\} \subset \{1, \ldots, n+2\}$. Then G' has size $d+1$, and Lemma 2.3 allows us to express $x_{i_d} \in H^1_{\hat{\theta}_F}(S(\Delta))$ as a linear combination of variables in $\{x_p : p \notin G'\}$. Substituting such an expression for x_{i_d} in the monomial x_G shows that x_G lies in the span of squarefree monomials in $x_1, \ldots, x_n$, as claimed.

For each squarefree monomial $x_{j_1} \cdots x_{j_d}$ in $x_1, \ldots, x_n$, $\deg_{\hat{\theta}_F}(x_{j_1} \cdots x_{j_d}x_{n+1})$ equals

$$\det\begin{pmatrix} a_{1,j_1}a_{d+1,n+1}^{-1} & \cdots & a_{j_d,1}a_{d+1,n+1}^{-1} & 0 \\ a_{2,j_1} + a_{d+1,j_1}a_{2,n+1}a_{d+1,n+1}^{-1} & \cdots & a_{2,j_d} + a_{d+1,j_d}a_{2,n+1}a_{d+1,n+1}^{-1} & a_{2,n+1} \\ \vdots & \cdots & \vdots & \vdots \\ a_{d+1,j_1} & \cdots & a_{d+1,j_d} & a_{d+1,n+1} \end{pmatrix}^{-1},$$

where if $j_p \in F$ we interpret a_{1,j_p} as 0. Using row operations to make the last column 0 except for the $(d+1, d+1)$ entry, we see that this is equal to $\deg_F(x_{j_1} \cdots x_{j_d})$. We conclude that $\overline{\varphi}$ is well-defined.

Multiplication by x_{n+1} induces a degree 1 graded isomorphism $\overline{H}_{\hat{\theta}_F}(S(\Delta))/\operatorname{ann}(x_{n+1}) \cong (x_{n+1}) \subset \overline{H}_{\hat{\theta}_F}(S(\Delta))$. It follows that $\overline{H}_{\hat{\theta}_F}(S(\Delta))/\operatorname{ann}(x_{n+1})$ is an artinian Gorenstein algebra of socle degree d. Since $\overline{\varphi}$ is a surjective graded map between artinian Gorenstein algebras whose socles are in the same degree, it is an isomorphism. □

Proposition 6.1 then implies that the ideal (x_{n+1}) in $\overline{H}_{\hat{\theta}_F}(S(\Delta))$ can be identified with $\overline{H}_F(\Delta) \otimes_K \hat{K}[-1]$. Following the strategy of [PP20, Section 9.1], we apply the results in Section 5 to $S(\Delta)$ in order to prove the weak Lefschetz property for Δ.

Proposition 6.2. *Let F be a non-face of Δ of size d. Then $\overline{H}_F(\Delta)$ has the weak Lefschetz property.*

Proof. We can check this after extending scalars to $\hat{K}$, see, e.g., [PP20, Proposition 13.3]. We assume the setup of Proposition 6.1, and set $\varphi(x_{n+1}) = x$. Let U_q denote the degree q part of the ideal (x_{n+1}) in $\overline{H}_{\hat{\theta}_F}(S(\Delta))$. Let W_q denote the subspace of $\overline{H}^q_{\hat{\theta}_F}(S(\Delta))$ spanned by monomials whose support is not contained in $\hat{F}$.

First assume that d is odd. Because $\overline{H}_F(\Delta)$ is generated in degree 1, it suffices to prove that multiplication by x induces an isomorphism from $\overline{H}_F^{(d-1)/2}(\Delta) \otimes_K \hat{K}$ to $\overline{H}_F^{(d+1)/2}(\Delta) \otimes_K \hat{K}$. This, in turn, is equivalent to the form $(y, z) \mapsto \deg_F(xyz)$ on $\overline{H}_F^{(d-1)/2}(\Delta) \otimes_K \hat{K}$ being nondegenerate. By Proposition 6.1,

$$\deg_F(xyz) = \deg_{\hat{\theta}_F}(x_{n+1}^2 \tilde{y}\tilde{z}),$$

where $\tilde{y}$ and $\tilde{z}$ are any lifts of y and z to $\overline{H}_{\hat{\theta}_F}(S(\Delta))$. Recall that we can identify $\overline{H}_F^{(d-1)/2}(\Delta) \otimes_K \hat{K}$ with $U_{(d+1)/2}$ such that y and z correspond to $\tilde{y}x_{n+1}$ and $\tilde{z}x_{n+1}$, respectively. Thus, we can identify the form $(y, z) \mapsto \deg_F(xyz)$ on $\overline{H}_F^{(d-1)/2}(\Delta) \otimes_K \hat{K}$ with the form $(y', z') \mapsto \deg_{\hat{\theta}_F}(y'z')$ on $U_{(d+1)/2}$, i.e., the restriction of the Hodge–Riemann form on $\overline{H}_{\hat{\theta}_F}^{(d+1)/2}(S(\Delta))$ to $U_{(d+1)/2}$. Our goal is to show that this form is nondegenerate.

Note that the Hodge–Riemann form on $\overline{H}_{\hat{\theta}_F}^{(d+1)/2}(S(\Delta))$ is nondegenerate. Assume that its restriction to $U_{(d+1)/2}$ is degenerate. Then Proposition 5.8 implies that $U_{(d+1)/2} \subset W_{(d+1)/2}$. Note that $x_{n+1}^{(d+1)/2} \in U_{(d+1)/2}$. Then Lemma 5.4 implies that $W_{(d+1)/2} = \overline{H}_{\hat{\theta}_F}^{(d+1)/2}(S(\Delta))$, so Corollary 5.7 implies that the Hodge–Riemann form on $\overline{H}_{\hat{\theta}_F}^{(d+1)/2}(S(\Delta))$ is anisotropic, contradicting the assumption that the restriction to $U_{(d+1)/2}$ is degenerate.

Next assume that d is even. The argument is similar to the one above. Let $\ell_{\hat{F}}$ be the image of $\sum_{i=1}^{n+2} x_i$ in $\overline{H}^1_{\hat{\theta}_F}(S(\Delta))$. Because $\overline{H}_F(\Delta)$ is generated in degree 1, it suffices to prove that multiplication by x induces an injection from $\overline{H}_F^{d/2-1}(\Delta) \otimes_K \hat{K}$ to $\overline{H}_F^{d/2}(\Delta) \otimes_K \hat{K}$. This follows if we can show that multiplication by $x\varphi(\ell_{\hat{F}})$ is injective, or, equivalently, that the form $(y, z) \mapsto \deg_F(x\varphi(\ell_{\hat{F}})yz)$ on $\overline{H}_F^{d/2-1}(\Delta) \otimes_K \hat{K}$ is nondegenerate. By Proposition 6.1,

$$\deg_F(x\varphi(\ell_{\hat{F}})yz) = \deg_{\hat{\theta}_F}(x_{n+1}^2 \ell_{\hat{F}} \tilde{y}\tilde{z}),$$

where $\tilde{y}$ and $\tilde{z}$ are any lifts of y and z to $\overline{H}_{\hat{\theta}_F}(S(\Delta))$. Recall that we can identify $\overline{H}_F^{d/2-1}(\Delta) \otimes_K \hat{K}$ with $U_{d/2}$ such that y and z correspond to $\tilde{y}x_{n+1}$ and $\tilde{z}x_{n+1}$ respectively. Thus, we can identify the form $(y, z) \mapsto \deg_F(x\varphi(\ell_{\hat{F}})yz)$ on $\overline{H}_F^{(d-1)/2}(\Delta) \otimes_K \hat{K}$ with the form $(y', z') \mapsto \deg_{\hat{\theta}_F}(\ell_{\hat{F}} y'z')$ on $U_{d/2}$, i.e., the restriction of the Hodge–Riemann form on $\overline{H}_{\hat{\theta}_F}^{d/2}(S(\Delta))$ to $U_{d/2}$. Our goal is to show that this form is nondegenerate.

Since we have established weak Lefschetz for even-dimensional pseudomanifolds, and, in particular, for $S(\Delta)$, Lemma 5.1 implies that the Hodge–Riemann form on $\overline{H}_{\hat{\theta}_F}^{d/2}(S(\Delta))$ is nondegenerate. The rest of the argument now proceeds just as above. Explicitly, assume that its restriction to $U_{d/2}$ is degenerate. Then Proposition 5.8 implies that $U_{d/2} \subset W_{d/2}$. Note that $x_{n+1}^{d/2} \in U_{d/2}$. Then Lemma 5.4 implies that $W_{d/2} = \overline{H}_{\hat{\theta}_F}^{d/2}(S(\Delta))$, and Corollary 5.7 implies that the Hodge–Riemann form on $\overline{H}_{\hat{\theta}_F}^{d/2}(S(\Delta))$ is anisotropic, contradicting the assumption that the restriction to $U_{d/2}$ is degenerate. ☐

We are now ready to prove the strong Lefschetz property for $\overline{H}_F(\Delta)$ in characteristic 2.

Theorem 6.3. *Let Δ be a connected simplicial pseudomanifold of dimension $d-1$, and let $0 \leq q \leq d/2$. Assume that* $\operatorname{char} k = 2$. *Then for every non-face F of size d, $\overline{H}_F(\Delta)$ has the strong Lefschetz property in degree q.*

Proof. We want to show that the map $\ell_F^{d-2q} \colon \overline{H}_F^q(\Delta) \to \overline{H}_F^{d-q}(\Delta)$ is an isomorphism. This is equivalent to showing that the Hodge–Riemann form $(y,z) \to \deg_F(\ell_F^{d-2q} yz)$ is nondegenerate on $\overline{H}_F^q(\Delta)$.

Let $m = \lfloor d/2 \rfloor$. By Proposition 6.2 and Lemma 5.2, ℓ_F is a weak Lefschetz element, and so the Hodge–Riemann form on $\overline{H}_F^m(\Delta)$ is nondegenerate. Moreover,

$$\ell_F^{m-q} \colon \overline{H}_F^q(\Delta) \to \overline{H}_F^m(\Delta)$$

is injective and compatible with the Hodge–Riemann forms. For example, in the case when d is odd, for all $y, z \in \overline{H}_F^q(\Delta)$ we have

$$\deg_F(\ell_F^{d-2q} yz) = \deg_F(\ell_F(\ell_F^{m-q} y)(\ell_F^{m-q} z)).$$

Let U be the degree m part of the ideal (ℓ_F^{m-q}) in $\overline{H}_F(\Delta)$. In order to complete the proof, it suffices to show that the restriction of the Hodge–Riemann form on $\overline{H}_F^m(\Delta)$ to U is nondegenerate.

Suppose that the restriction to U is degenerate. Proposition 5.8 implies that $U \subset W_m$, where W_m is the subspace of $\overline{H}_F^q(\Delta)$ spanned by monomials whose support is not contained in F. Since $\ell_F^m \in U$, Lemma 5.4 implies that $W_m = \overline{H}_F^m(\Delta)$. Corollary 5.7 implies that the Hodge–Riemann form on $\overline{H}_F^m(\Delta)$ is anisotropic, contradicting the assumption that the restriction to U is degenerate. □

As Example 5.3 shows, the Hodge–Riemann forms on $\overline{H}_F(\Delta)$ are not, in general, anisotropic. We note one case where the proof of Theorem 6.3 can be used to deduce anisotropy. Examples of simplicial complexes with small non-faces include, for instance, all flag complexes. In this case, all minimal non-faces have size two.

Proposition 6.4. *Let Δ be a pseudomanifold, and let k be a field of characteristic* 2. *Let F be a non-face of Δ of size d, and suppose that F contains a non-face of size at most $m = \lfloor d/2 \rfloor$. Then, for every $0 \leq q \leq m$, the Hodge–Riemann form on $\overline{H}_F^q(\Delta)$ is anisotropic.*

Proof. To simplify the notation, assume that $F = \{1, 2, \ldots, d\}$ and that $G = \{1, 2, \ldots, r\}$ is a non-face of Δ for some $r \leq m$. Let $W_m \subset \overline{H}_F^m(\Delta)$ be the span of monomials whose support is not contained in F. As in the proof of Theorem 6.3, by Lemma 5.4 and Corollary 5.7, it suffices to show that $x_1^m \in W_m$.

Using (3), for each $2 \leq i \leq r$, we can express $x_i \in H_F^1(\Delta)$ as the sum of $\lambda_i x_1$ and some linear combination of $\{x_p : d+2 \leq p \leq n\}$. Here, up to a sign, λ_i equals $\operatorname{ev}_{\theta_F}([\{1, \ldots, d+1\} \smallsetminus \{i\}]) / \operatorname{ev}_{\theta_F}([\{2, \ldots, d+1\}])$, and, in particular, is nonzero. Now, since $\{1, 2, \ldots, r\}$ is a non-face of Δ, the product $x_1^{m-r+1} x_2 \ldots x_r$ is equal to zero. Substituting the above expressions for $x_2, \ldots, x_r$ in this product, implies that $\lambda_2 \cdots \lambda_r x_1^m$ is a linear combination of monomials in $\{x_p : d+2 \leq p \leq n\}$, so x_1^m belongs to W_m. □

7. Proofs of theorems

In this section, we prove Theorem 1.4, then Theorem 1.3, and then finally Theorem 1.1.

Let Δ be a connected oriented simplicial k-homology manifold of dimension $d-1$. We will need the following result of the second author and Swartz, which uses as input results of Gräbe and Schenzel [Grä84, Sch81]. Let $\beta_q = \dim \tilde{H}_q(\Delta; k)$, the dimension of the reduced homology of Δ over k. By the universal coefficient theorem, this depends only on the characteristic of k. Let $(h_0(\Delta), \ldots, h_d(\Delta))$ be the h-vector of Δ. Let $\overline{H}_\mu(\Delta)$ be the Gorenstein quotient of $K[\Delta]/(\mu_1, \ldots, \mu_d)$ for an l.s.o.p. $\mu = (\mu_1, \ldots, \mu_d)$ for $K[\Delta]$.

Proposition 7.1. [NS09, Theorem 1.3 and 1.4] *Let Δ be a connected oriented simplicial k-homology manifold of dimension $d-1$. Let $\mu = (\mu_1, \ldots, \mu_d)$ be an l.s.o.p. for $K[\Delta]$. Then*

$$\dim \overline{H}^q_\mu(\Delta) = \begin{cases} h_q(\Delta) + \binom{d}{q} \sum_{p=0}^{q-1} (-1)^{q-p}\beta_p & \text{if } 0 \le q < d \\ 1 & \text{if } q = d. \end{cases}$$

In particular, $\dim \overline{H}^q_\mu(\Delta)$ is independent of the choice of l.s.o.p.

Proof of Theorem 1.4. When $\operatorname{char} k = 2$, the result follows immediately from Theorem 6.3. Now suppose that $\operatorname{char} k = 0$, and the integral homology of the link of any face (including the empty face) of Δ has no 2-torsion.

The assumption that the integral homology of the link of every face has no 2-torsion implies that Δ is a homology manifold over $\mathbb{F}_2$, and that $\dim H_q(\Delta; k) = \dim H_q(\Delta; \mathbb{F}_2)$ for all q. Let $\overline{H}_{F,2}(\Delta)$ be the Gorenstein quotient of $\mathbb{F}_2(a_{i,j})[\Delta]/(\theta_1^F, \ldots, \theta_d)$. By Proposition 7.1, $\dim \overline{H}^q_F(\Delta) = \dim \overline{H}^q_{F,2}(\Delta)$ for each q.

Fix $0 \le q \le d/2$, and let $S = \{m_1, \ldots, m_p\}$ denote the set of monomials of degree q in $K[\Delta]$. Let M_0 be the $p \times p$ matrix whose (i,j)th entry is $\deg_F(\ell^{d-2q} m_i m_j)$. Then $\overline{H}_F(\Delta)$ has strong Lefschetz property in degree q if and only if the rank of M_0 is equal to $\dim \overline{H}^q_F(\Delta)$.

Let M_2 be the $p \times p$ matrix whose (i,j)th entry is $\deg_{F,2}(\ell^{d-2q} m_i m_j)$, i.e., the degree in $\overline{H}_{F,2}(\Delta)$. Because $\overline{H}_{F,2}(\Delta)$ has the strong Lefschetz property by Theorem 6.3, the rank of M_2 is equal to $\dim \overline{H}^q_F(\Delta)$.

Note that $\deg_F(\ell^{d-2q} m_i m_j)$ lies in $\mathbb{Z}(a_{i,j})$, and that $\deg_{F,2}(\ell^{d-2q} m_i m_j)$ is obtained by reducing it modulo 2. Hence $\dim \overline{H}^q_F(\Delta) \ge \operatorname{rank}(M_0) \ge \operatorname{rank}(M_2) = \dim \overline{H}^q_{F,2}(\Delta)$. Since $\dim \overline{H}^q_F(\Delta) = \dim \overline{H}^q_{F,2}(\Delta)$, the rank of M_0 is equal to $\dim \overline{H}^q_F(\Delta)$, as desired. □

Remark 7.2. When k has characteristic 0 and Δ is a polytopal sphere, Theorem 1.4 can be deduced from Stanley's proof of this case of the algebraic g-conjecture [Sta80]. Indeed, we can assume $k = \mathbb{Q}$ and choose a realization of Δ as the boundary of a convex polytope P in $\mathbb{R}^d$ whose vertices are rational. Because F is a non-face, using an affine transformation, we may assume that the vertices of F are contained in the hyperplane where the first coordinate of $\mathbb{R}^d$ vanishes and that the origin is in the interior of P. Then the strong Lefschetz property for $H_F(\Delta) = \overline{H}_F(\Delta)$ follows from the Hard Lefschetz theorem applied to the projective toric variety corresponding to the fan over P.

We now begin the proof of Theorem 1.3. Let $\mu = (\mu_1, \ldots, \mu_d)$ be an l.s.o.p. for $K[\Delta]$. Let ℓ_μ denote the image of $\sum_{j=1}^n x_j$ in $\overline{H}^1_\mu(\Delta)$. Recall that $R \subset K$ denotes the localization of $k[a_{i,j}]$ at the irreducible polynomials $\{[G] : G \text{ facet of } \Delta\}$, and $\operatorname{ev}_\mu \colon R \to K$ is the map defined by $\operatorname{ev}_\mu(a_{i,j}) = \mu_{i,j}$.

Lemma 7.3. *Let Δ be a connected oriented simplicial k-homology manifold. Let $\mu = (\mu_1, \ldots, \mu_d)$ be an l.s.o.p. and let $0 \le q \le d/2$. Suppose that multiplication by ℓ_μ^{d-2q} is an isomorphism from $\overline{H}^q_\mu(\Delta) \to \overline{H}^{d-q}_\mu(\Delta)$, i.e., ℓ_μ is a strong Lefschetz element in degree q. Let $P \in k[a_{i,j}]$ be an irreducible polynomial such that $\operatorname{ev}_\mu(P) = 0$. Then there are monomials $y_1, \ldots, y_p$ whose images form a basis of $\overline{H}^q(\Delta)$ and $\operatorname{ord}_P(\det M) = 0$, where M is the $p \times p$ matrix whose (i,j) entry is $\deg(\ell^{d-2q} y_i y_j)$. In particular, if $D_q \in K^\times/(K^\times)^2$ is the determinant of the Hodge–Riemann form on $\overline{H}^q(\Delta)$, then $\operatorname{ord}_P(D_q) = 0 \in \mathbb{Z}/2\mathbb{Z}$.*

Proof. Choose monomials $y_1, \ldots, y_p$ in the degree q part of $K[\Delta]$ such that their images form a basis for $\overline{H}^q_\mu(\Delta)$; this is possible because $\overline{H}^q_\mu(\Delta)$ is spanned by the images of monomials.

Let M be the $p \times p$ matrix whose (i,j) entry is $\deg(\ell^{d-2q} y_i y_j)$. By Lemma 2.4, each entry of M lies in R, so $\det M$ lies in R. By Remark 2.5, if we can show that $\operatorname{ev}_\mu(\det M) \neq 0$, then $\operatorname{ord}_P(\det M) = 0$.

Let M_μ be the $p \times p$ matrix whose (i,j) entry is $\deg_\mu(\ell_\mu^{d-2q} y_i y_j)$. By Lemma 2.4, the (i,j) entry of M_μ is $\mathrm{ev}_\mu(\deg(\ell^{d-2q} y_i y_j))$, so $\det M_\mu = \mathrm{ev}_\mu(\det M)$. As ℓ_μ is a strong Lefschetz element for $\overline{H}^q_\mu(\Delta)$, $\det M_\mu \neq 0$, and we conclude that $\mathrm{ord}_P(\det M) = 0$.

Finally, that $\det M$ is nonzero implies that the images of $y_1, \ldots, y_p$ are linearly independent in $\overline{H}^q(\Delta)$. As $\dim \overline{H}^q(\Delta) = \dim \overline{H}^q_\mu(\Delta)$ by Proposition 7.1, $\{y_1, \ldots, y_p\}$ is a basis for $\overline{H}^q(\Delta)$, so $\det M$ computes the determinant of the Hodge–Riemann form on $\overline{H}^q(\Delta)$. This completes the proof. □

Proposition 7.4. *Let Δ be a connected oriented simplicial k-homology manifold. Let F be a subset of V of size d which is not a facet, and let $0 \leq q \leq d/2$. Suppose that $\overline{H}_F(\Delta)$ has the strong Lefschetz property in degree q. Let $D_q \in K^\times/(K^\times)^2$ be the determinant of the Hodge–Riemann form on $\overline{H}^q(\Delta)$. Then $\mathrm{ord}_{[F]}(D_q) = 0$.*

Proof. By Lemma 5.1, ℓ_F is a strong Lefschetz element for $\overline{H}^q_F(\Delta)$. The result now follows from Lemma 7.3 setting $\mu = \theta_F$ and $P = [F]$. □

Lemma 7.5. *Let F be a subset of V of size d. Then for each q, there is a basis for $\overline{H}^q(\Delta)$ consisting of the images of monomials in $K[\Delta]$ whose support is disjoint from F.*

Proof. By Lemma 2.3, one can write any monomial in $\overline{H}^1(\Delta)$ in terms of the monomials corresponding to vertices not in F. As $\overline{H}(\Delta)$ is generated in degree 1, this implies that each $\overline{H}^q(\Delta)$ is spanned by monomials whose support is disjoint from F. Some subset of these monomials forms a basis. □

For a facet F of Δ, let Δ' be the simplicial complex obtained by doing a stellar subdivision in the interior of F, i.e., the vertex set of Δ' is $V \cup \{n+1\} = \{1, \ldots, n+1\}$, and the facets of Δ' are the facets of Δ except for F, together with $(F \cup \{n+1\}) \smallsetminus \{j\}$ for each $j \in F$. Then Δ' is a connected oriented simplicial k-homology manifold with its orientation determined by orienting the facets of Δ' which are also facets of Δ in the same way that they are oriented in Δ.

Note that the proof of Proposition 7.4 implies that if F is a non-face of size d and $\overline{H}_F(\Delta)$ has the strong Lefschetz property in degree q, then so does $\overline{H}(\Delta)$. If Δ has no non-faces of size d, then Δ must be isomorphic to S^{d-1}, and so Theorem 1.3 holds for Δ by Example 3.2. When proving Theorem 1.3, we may therefore assume that $\overline{H}(\Delta)$ has the strong Lefschetz property in degree q.

Proof of Theorem 1.3. As Theorem 4.1 implies Theorem 1.3 when $q = 0$, we may assume that $0 < q \leq d/2$.

Theorem 1.4 and Proposition 7.4 show that if F is not a facet, then $\mathrm{ord}_{[F]}(D_q) = 0$. Suppose that F is a facet of Δ. By Lemma 7.5, we may choose a collection of monomials $y_1, \ldots, y_p \in K[\Delta]$ of degree q whose support is disjoint from F and such that their image in $\overline{H}^q(\Delta)$ is a basis. By the version of Lemma 5.1 for $\overline{H}(\Delta)$, ℓ is a strong Lefschetz element in degree q. Let M be the $p \times p$ matrix whose (i,j)th entry is $\deg(\ell^{d-2q} y_i y_j)$, so M is nonsingular and the image of $\det M$ in $K^\times/(K^\times)^2$ is D_q.

Let Δ' be the simplicial complex obtained by doing a stellar subdivision in the interior of F, with orientation as described above and degree map $\deg'$. We can identify $K[\Delta]/(x_F)$ with a subring of $K[\Delta']$, and hence consider the images $y'_1, \ldots, y'_p$ of $y_1, \ldots, y_p$ in $K[\Delta']$. Set $y'_{p+1} = x^q_{n+1}$.

We claim the the images of $y'_1, \ldots, y'_{p+1}$ span $\overline{H}^q(\Delta')$. Indeed, by Lemma 7.5, $\overline{H}(\Delta')$ is spanned as a k-vector space by monomials whose support is disjoint from F. The latter consists of monomials supported away from $F \cup \{n+1\}$, together with powers of x_{n+1}. Let z be a monomial of degree q whose support is disjoint from $F \cup \{n+1\}$. Then $z = \sum_{i=1}^p \lambda_i y_i$ in $\overline{H}^q(\Delta)$ for some $\lambda_i \in K$. Consider $z' := z - \sum_{i=1}^p \lambda_i y'_i$ in $\overline{H}^q(\Delta')$. By Lemma 2.8, the product of z' with any monomial of degree $d-q$ supported away from $F \cup \{n+1\}$ is zero. Also, $z' x^{d-q}_{n+1} = 0$. We deduce that $z' = 0$, and the claim follows.

Let ℓ' be the image of $\sum_{j=1}^{n+1} x_j$ in $\overline{H}^1(\Delta')$, and let M' be the $(p+1)\times(p+1)$ matrix whose (i,j)th entry is $\deg'((\ell')^{d-2q}y_i'y_j')$. For $j \le p$, we have $y_j'y_{p+1}' = 0$ in $K[\Delta']$. By Lemma 2.8, M' is a block diagonal matrix whose northwest $p\times p$ block is M and whose $(p+1,p+1)$ entry $M'_{p+1,p+1}$ is equal to the degree of $\ell^{d-2q}x_{d+1}^{2q}$ in the complex Σ considered in Section 3 (up to sign). Lemma 3.4 implies that $M'_{p+1,p+1}$ is nonzero and $\mathrm{ord}_{[F]}(M'_{p+1,p+1}) = 2q-1$.

We see that M' is nonsingular, so $\{y_1', \dots, y_{p+1}'\}$ is a basis of $\overline{H}^q(\Delta')$. In particular, $\det M'$ computes the determinant of the Hodge–Riemann form.

As F is not a facet of Δ', Theorem 1.4 and Proposition 7.4 give that $\mathrm{ord}_{[F]}(\det M')$ is even. Since $\mathrm{ord}_{[F]}(M'_{p+1,p+1}) = 2q-1$ is odd, we see that $\mathrm{ord}_{[F]}(\det M)$ is odd, as desired. □

Proof of Theorem 1.1. Note that for even d and $q = d/2$, the above proof of Theorem 1.3 works over all characteristics because we are in the middle dimension, and so the assumption that ℓ_F is a strong Lefschetz element holds vacuously. This case of Theorem 1.3 implies Corollary 1.2. Hence if F is a facet of Δ, then $\mathrm{ord}_{[F]}(D_{d/2}) = 1 \in \mathbb{Z}/2\mathbb{Z}$. Let $P \in k[a_{i,j}]$ be an irreducible polynomial that is not equal (up to multiplication by a scalar) to one of the polynomials $\{[F] : F \text{ facet of } \Delta\}$. Over $\overline{k}[a_{i,j}]$, we may factor $P = P_1^{m_1}\cdots P_r^{m_r}$, where the P_i are distinct irreducible polynomials over $\overline{k}$ and $m_i \in \mathbb{Z}_{>0}$. Note that none of the P_i are scalar multiples of $[F]$. We claim that there are monomials $y_1, \dots, y_p$ such that their images form a basis of $\overline{H}^{d/2}(\Delta)$ and $\mathrm{ord}_{P_1}(\det M) = 0$, where M is the $p\times p$ matrix whose (i,j) entry is $\deg(y_iy_j)$. This implies that $\mathrm{ord}_P(\det M) = 0$ and hence $\mathrm{ord}_P(D_{d/2}) = 0$. We deduce that $D_{d/2} = \lambda \prod_{F \text{ facet of } \Delta}[F] \in K^\times/(K^\times)^2$ for some $\lambda \in k^\times/(k^\times)^2$, completing the proof.

It remains to verify the claim. Let $V(P_1)$ be the vanishing locus of P_1 inside $\mathbb{A}^{dn}_{\overline{k}}$, and let $(\mu_{i,j}) \in V(P_1)$ be a $\overline{k}$-point. Set $\mu_i = \sum_j \mu_{i,j}x_j$. First suppose that $\mu = (\mu_1, \dots, \mu_d)$ is an l.s.o.p. for $\overline{k}(a_{i,j})[\Delta]$. Observe that $\mathrm{ev}_\mu(P_1) = 0$ since $(\mu_{i,j}) \in V(P_1)$. Then the claim follows from Lemma 7.3. Note that the assumption in Lemma 7.3 that ℓ_μ is a strong Lefschetz element holds vacuously since we are in middle dimension. Hence we may assume that μ is not an l.s.o.p. By Proposition 2.2 there must be some facet F of Δ such that $(\mu_{i,j})$ is contained in the vanishing locus of $[F]$. Applying this to every $\overline{k}$-point of $V(P_1)$, we see that

$$V(P_1) \subset \bigcup_{F \text{ facet of } \Delta} V([F]).$$

As there are only finitely many facets, this implies that $V(P_1)$ is contained in $V([F])$ for some facet F. The irreducibility of $[F]$ then implies that P_1 and $[F]$ are equal up to multiplication by a scalar, a contradiction. □

8. Further discussion

We conjecture an extension of Theorem 1.3 to pseudomanifolds in arbitrary characteristic.

Conjecture 8.1. *Let k be a field of arbitrary characteristic. Let Δ be a connected oriented simplicial pseudomanifold over k of dimension $d-1$ with vertex set V.*

For all $0 \le q \le d/2$, let D_q be the determinant of the Hodge–Riemann form on $\overline{H}^q_F(\Delta)$. Let F be a subset of V of size d. Then

$$\mathrm{ord}_{[F]}(D_q) = \begin{cases} 1 & \text{if } F \text{ is a facet of } \Delta \\ 0 & \text{otherwise.} \end{cases}$$

When Δ is a k-homology manifold, the proof of Theorem 1.3 shows that if $\overline{H}_F(\Delta')$ has the strong Lefschetz property whenever $\Delta' = \Delta$ or Δ' is the stellar subdivision of Δ at the interior of a facet, then Conjecture 8.1 holds for Δ.

However, an additional ingredient is needed to establish Conjecture 8.1 for pseudomanifolds. A key property of homology manifolds which was used in the proofs of our results, e.g., Lemma 7.3, was that the dimension of $\overline{H}^q_\mu(\Delta)$ does not depend on μ, the chosen l.s.o.p. (see Proposition 7.1). We show in the example below that this independence of the dimension can fail for pseudomanifolds.

Example 8.2. Let Δ be the standard 6 vertex triangulation of $\mathbb{RP}^2$, and let $\Delta' = \Delta * S^0$ be the suspension. Over a field of characteristic 2, Δ' is an oriented pseudomanifold, but it is not a homology manifold. Using Macaulay2 [GS], we checked that, if one chooses an l.s.o.p. $\mu_1, \mu_2, \mu_3, \mu_4$ with all coefficients random elements of the field with 1024 elements, the Hilbert function of $H_\mu(\Delta')$ is usually given by $(1, 4, 9, 6, 1)$, and the Hilbert function of $\overline{H}_\mu(\Delta')$ is usually given by $(1, 4, 8, 4, 1)$. If one chooses μ_1', μ_2', μ_3' to be generic linear combinations of the vertices of $\mathbb{RP}^2$ and chooses μ_4' to be a generic linear combination of the vertices of S^0, then $H_{\mu'}(\Delta') = H_{(\mu_1',\mu_2',\mu_3')}(\Delta) \otimes H_{(\mu_4')}(S^0)$, and similarly for $\overline{H}_{\mu'}(\Delta')$. We can then use Proposition 7.1 to compute that the Hilbert function of $H_{\mu'}(\Delta')$ is given by $(1, 4, 9, 7, 1)$, and the Hilbert function of $\overline{H}_{\mu'}(\Delta')$ is given by $(1, 4, 6, 4, 1)$.

However, we do not know an example of a pseudomanifold Δ where $\dim \overline{H}^q(\Delta) \neq \dim \overline{H}^q_F(\Delta)$. It follows from Theorem 6.3 that the equality $\dim \overline{H}^q(\Delta) = \dim \overline{H}^q_F(\Delta)$ holding for all Δ would imply Conjecture 8.1 in the characteristic 2 case. We now show that this equality holds when $q = 1$ (in any characteristic).

Proposition 8.3. *Let k be a field of arbitrary characteristic. Let Δ be a connected oriented simplicial pseudomanifold over k of dimension $d-1$. Let μ be any l.s.o.p. of $K[\Delta]$. Then $\overline{H}^1_\mu(\Delta) = H^1_\mu(\Delta)$. In particular,* $\dim \overline{H}^1_\mu(\Delta) = n - d$ *independently of μ.*

Proof. Let $z = \sum_{i=1}^n \lambda_i x_i \in H^1_\mu(\Delta)$ be such that $z x_G = 0$ for all codimension 1 faces G of Δ. To prove the statement, we need to show that $z = 0$.

Let F' be any facet of Δ. Since μ is an l.s.o.p and F' is a facet, by Proposition 2.2, $\mathrm{ev}_\mu([F']) \neq 0$. It follows from Lemma 2.3 that $\{x_v : v \notin F'\}$ is a basis of $H^1_\mu(\Delta)$. By using this basis, we can assume $\lambda_i = 0$ for $i \in F'$.

Consider any codimension 1 face $G \subset F'$. Since Δ is a pseudomanifold, there is the unique facet F'' of Δ such that $F' \cap F'' = G$; we let u denote the unique vertex of $F'' \smallsetminus G$. Then $0 = z x_G = \lambda_u x_{F''}$. As $x_{F''}$ is nonzero because it has nonzero degree, it follows that $\lambda_u = 0$. We conclude that $\lambda_v = 0$ for all $v \in F''$. Continuing in this way and using the fact that Δ is a connected pseudomanifold, and hence that Δ is strongly connected, we deduce that $\lambda_v = 0$ for all $v \in V$. Thus, $z = 0$, as desired. ☐

Remark 8.4. Proposition 8.3 implies that Conjecture 8.1 holds when k has characteristic 2 and $q = 1$. By a specialization argument similar to the proof of Theorem 1.4, we see that Conjecture 8.1 also holds when k has characteristic 0 and $q = 1$.

Assume that ℓ is a strong Lefschetz element in all degrees, i.e., the Hodge–Riemann form on $\overline{H}^q(\Delta)$ is nondegenerate for $0 \leq q \leq d/2$. The *primitive part* of $\overline{H}^q(\Delta)$ is $\overline{H}^q_{\mathrm{prim}}(\Delta) := \{y \in \overline{H}^q(\Delta) : \ell^{d-2q+1} y = 0\}$. Let $D_{\mathrm{prim},q} \in K^\times/(K^\times)^2$ be the determinant of the induced Hodge–Riemann form on $\overline{H}^q_{\mathrm{prim}}(\Delta)$. For $0 < q \leq d/2$, multiplication by ℓ induces an injection $\overline{H}^{q-1}(\Delta) \to \overline{H}^q(\Delta)$ which splits to give an isomorphism $\overline{H}^q(\Delta) \cong \overline{H}^{q-1}(\Delta) \oplus \overline{H}^q_{\mathrm{prim}}(\Delta)$. As this decomposition is orthogonal with respect to the Hodge–Riemann form, we have $D_q = D_{q-1} D_{\mathrm{prim},q}$. In particular, $D_q = D_0 \prod_{q'=1}^q D_{\mathrm{prim},q'}$. Since we established Conjecture 8.1

when $q = 0$ in Theorem 4.1, we conclude that Conjecture 8.1 holding for all q is equivalent to the following conjecture.

Conjecture 8.5. *Let k be a field of arbitrary characteristic. Let Δ be a connected oriented simplicial pseudomanifold over k of dimension $d-1$ with vertex set V. Then ℓ is a strong Lefschetz element in all degrees, and, for each subset F of V of size d and $0 < q \leq d/2$, we have* $\mathrm{ord}_{[F]}(D_{\mathrm{prim},q}) = 0$.

References

[Adi18] Karim Adiprasito, *Combinatorial Lefschetz theorems beyond positivity*, arXiv:1812.10454v4.

[APP21] Karim Adiprasito, Stavros Argyrios Papadakis, and Vasiliki Petrotou, *Anisotropy, biased pairings, and the Lefschetz property for pseudomanifolds and cycles*, arXiv:2101.07245v2.

[APP24] ———, *The volume intrinsic to a commutative graded algebra*, arXiv:2407.11916v1.

[Boc64] Maxime Bocher, *Introduction to higher algebra*, Dover Publications, Inc., New York, 1964. MR 172882

[Bri97] Michel Brion, *The structure of the polytope algebra*, Tohoku Math. J. (2) **49** (1997), no. 1, 1–32.

[Grä84] Hans-Gert Gräbe, *The canonical module of a Stanley-Reisner ring*, J. Algebra **86** (1984), no. 1, 272–281.

[GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at `http://www2.macaulay2.com`.

[KX23] Kalle Karu and Elizabeth Xiao, *On the anisotropy theorem of Papadakis and Petrotou*, Algebr. Comb. **6** (2023), no. 5, 1313–1330.

[Lee96] Carl W. Lee, *P.l.-spheres, convex polytopes, and stress*, Discrete Comput. Geom. **15** (1996), no. 4, 389–421.

[MZ08] Juan Migliore and Fabrizio Zanello, *The strength of the weak Lefschetz property*, Illinois J. Math. **52** (2008), no. 4, 1417–1433. MR 2595775

[NS09] Isabella Novik and Ed Swartz, *Gorenstein rings through face rings of manifolds*, Compos. Math. **145** (2009), no. 4, 993–1000.

[Oba24] Ryoshun Oba, *Multigraded strong Lefschetz property for balanced simplicial complexes*, 2024, arXiv:2408.17110v1.

[PP20] Stavros Argyrios Papadakis and Vasiliki Petrotou, *The characteristic 2 anisotropicity of simplicial spheres*, arXiv:2012.09815v1.

[PP23] ———, *The generic anisotropy of simplicial 1-spheres*, Israel J. Math. **254** (2023), no. 1, 141–153.

[Sch81] Peter Schenzel, *On the number of faces of simplicial complexes and the purity of Frobenius*, Math. Z. **178** (1981), no. 1, 125–142.

[Sta80] Richard P. Stanley, *The number of faces of a simplicial convex polytope*, Adv. in Math. **35** (1980), no. 3, 236–238. MR 563925

[Sta96] ———, *Combinatorics and commutative algebra*, second ed., Progress in Mathematics, vol. 41, Birkhäuser Boston, Inc., Boston, MA, 1996.

[Swa09] Ed Swartz, *Face enumeration—from spheres to manifolds*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 3, 449–485. MR 2505437

[TW00] Tiong-Seng Tay and Walter Whiteley, *A homological interpretation of skeletal ridigity*, Adv. in Appl. Math. **25** (2000), no. 1, 102–151. MR 1773196

Princeton University and the Institute for Advanced Study
Email address: `mattlarson@princeton.edu`

University of Washington
Email address: `novik@uw.edu`

Sydney Mathematics Research Institute
Email address: `astapldn@gmail.com`