K-RINGS OF WONDERFUL VARIETIES AND MATROIDS

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ABSTRACT. We study the K-ring of the wonderful variety of a hyperplane arrangement and give a combinatorial presentation that depends only on the underlying matroid. We use this combinatorial presentation to define the K-ring of an arbitrary loopless matroid. We construct an exceptional isomorphism, with integer coefficients, to the Chow ring of the matroid that satisfies a Hirzebruch–Riemann–Roch-type formula, generalizing a recent construction of Berget, Eur, Spink, and Tseng for the permutohedral variety (the wonderful variety of a Boolean arrangement). As an application, we give combinatorial formulas for Euler characteristics of arbitrary line bundles on wonderful varieties. We give analogous constructions and results for augmented wonderful varieties, and for Deligne–Mumford–Knudsen moduli spaces of stable rational curves with marked points.

CONTENTS

1. INTRODUCTION

Let L be a finite dimensional vector space over a field $\mathbb{F}$, and let $\mathcal{A} = \{H_e \mid e \in E\}$ be a finite multiset of hyperplanes in L intersecting only at the origin. The wonderful variety $W_\mathcal{A}$ is a smooth compactification of $\mathbb{P}(L) \backslash \bigcup_{H \in \mathcal{A}} \mathbb{P}(H)$, originally studied by De Concini and Procesi [DCP95]. The augmented wonderful variety $W_\mathcal{A}^{\mathrm{aug}}$ is a smooth compactification of L, introduced in [BHM⁺22a], that contains $W_\mathcal{A}$ as a divisor. The Chow rings of these spaces have been extensively studied and have combinatorial presentations that depend only on the underlying matroid. As a result, such rings are naturally defined for arbitrary, not necessarily realizable, matroids. Presentations for the Chow ring of $W_\mathcal{A}$ appear in [DCP95, FY04, BES21], and this ring is used to prove log

concavity of the characteristic polynomial of $\mathcal{A}$ (and more generally of any matroid) in [AHK18]. A presentation of the Chow ring of $W_{\mathcal{A}}^{\mathrm{aug}}$ appears in [BHM$^+$22a], and this ring is used to prove the top-heavy conjecture and nonnegativity of the Kazhdan–Lusztig polynomials of matroids [BHM$^+$22b].

Our goal is to study the K-rings of these spaces. We first give presentations for the K-rings using generators analogous to the Feichtner–Yuzvinsky generators for the Chow rings. Despite the fact that the (non-homogeneous) relations among the Feichtner–Yuzvinsky generators in the K-ring are different from the (homogeneous) relations among the Feichtner–Yuzvinsky generators in the Chow ring, we construct an integral isomorphism from the K-ring to the Chow ring which satisfies a Hirzebruch–Riemann–Roch-type formula, generalizing recent results for Boolean arrangements [BEST22, EHL22]. The K-rings admit additional structures, such as Adams operations, an Euler characteristic map, and Serre duality, which leads us to new results and new questions. Most of our results can be extended to the moduli space $\overline{\mathcal{M}}_{0,n}$ of stable rational curves with n marked points, which is closely related to the wonderful variety for the braid arrangement $\mathcal{B}_{n-1}$.

1.1. **Definitions of the varieties.** For any $S \subset E$, let
$$L_S := \bigcap_{e \in S} H_e \qquad \text{and} \qquad L^S := L/L_S.$$

The dimension of L^S is called the **rank** of S, and the dimension of L_S is called the **corank**. A subset $F \subset E$ is called a **flat** if it is maximal within its rank, or, equivalently, if $L_F \subset H_e \Rightarrow e \in F$.

The **wonderful variety** $W_{\mathcal{A}}$ is defined as the closure of the image of the rational map
$$\mathbb{P}(L) \dashrightarrow \prod_F \mathbb{P}(L^F),$$

and the **augmented wonderful variety** $W_{\mathcal{A}}^{\mathrm{aug}}$ is defined as the closure of the image of the rational map
$$\mathbb{P}(L \oplus \mathbb{F}) \dashrightarrow \prod_F \mathbb{P}(L^F \oplus \mathbb{F}),$$

where both products are over the set of nonempty flats. For any nonempty flat F, let
$$\pi_F : W_{\mathcal{A}} \to \mathbb{P}(L^F) \qquad \text{and} \qquad \pi_F^{\mathrm{aug}} : W_{\mathcal{A}}^{\mathrm{aug}} \to \mathbb{P}(L^F \oplus \mathbb{F})$$

be the natural projections. We will write
$$\mathcal{L}_F := \pi_F^* \mathcal{O}_{\mathbb{P}(L^F)}(1) \qquad \text{and} \qquad \mathcal{L}_F^{\mathrm{aug}} := (\pi_F^{\mathrm{aug}})^* \mathcal{O}_{\mathbb{P}(L^F \oplus \mathbb{F})}(1).$$

The map $\pi_E : W_{\mathcal{A}} \to \mathbb{P}(L)$ is an iterated blow-up. It is obtained by blowing up first the points $\mathbb{P}(L_F)$ for all corank 1 flats F, then the strict transforms of the lines $\mathbb{P}(L_F)$ for all corank 2 flats F, and so on. For any nonempty proper flat F, let $D_F \subset W_{\mathcal{A}}$ be the closure of the preimage under the map π_E of the locus
$$\mathbb{P}(L_F) \setminus \bigcup_{F \subsetneq G} \mathbb{P}(L_G).$$

Similarly, the map $\pi_E^{\mathrm{aug}} : W_{\mathcal{A}}^{\mathrm{aug}} \to \mathbb{P}(L \oplus \mathbb{F})$ is an iterated blow-up, obtained by blowing up first the points $\mathbb{P}(L_F \oplus \{0\})$ for all corank 1 flats F, then the strict transforms of the lines $\mathbb{P}(L_F \oplus \{0\})$ for all corank 2 flats F, and so on. For any proper flat F, let $D_F^{\mathrm{aug}} \subset W_{\mathcal{A}}^{\mathrm{aug}}$ be the closure of the preimage under the map π_E^{aug} of the locus

$$\mathbb{P}(L_F \oplus \{0\}) \setminus \bigcup_{F \subsetneq G} \mathbb{P}(L_G \oplus \{0\}).$$

In addition, for any $e \in E$, let $D_e^{\mathrm{aug}} \subset W_{\mathcal{A}}^{\mathrm{aug}}$ be the strict transform of $\mathbb{P}(H_e \oplus \mathbb{F})$. We have a canonical isomorphism $W_{\mathcal{A}} \cong D_{\varnothing}^{\mathrm{aug}} \subset W_{\mathcal{A}}^{\mathrm{aug}}$, which induces identifications $\mathcal{L}_F = \mathcal{L}_F^{\mathrm{aug}}|_{W_{\mathcal{A}}}$ and $D_F = D_F^{\mathrm{aug}} \cap W_{\mathcal{A}}$ for all nonempty proper F.

1.2. Feichtner–Yuzvinsky presentations.

Let $A(W_{\mathcal{A}})$ and $A(W_{\mathcal{A}}^{\mathrm{aug}})$ denote the Chow rings of $W_{\mathcal{A}}$ and $W_{\mathcal{A}}^{\mathrm{aug}}$, respectively. Let $T_{\mathcal{A}} := \mathbb{Z}[x_F \mid F \text{ a flat}] \otimes \mathbb{Z}[y_e \mid e \in E]$. Consider the map

$$\Phi_{\mathrm{FY}}^{\mathrm{aug}} : T_{\mathcal{A}} \to A(W_{\mathcal{A}}^{\mathrm{aug}})$$

that sends x_F to $[D_F^{\mathrm{aug}}]$ for all proper F, x_E to $-c_1(\mathcal{L}_E^{\mathrm{aug}})$, and y_e to $[D_e^{\mathrm{aug}}]$ for all $e \in E$. Composing with the pullback along the inclusion $W_{\mathcal{A}} \subset W_{\mathcal{A}}^{\mathrm{aug}}$, we obtain a map

$$\Phi_{\mathrm{FY}} : T_{\mathcal{A}} \to A(W_{\mathcal{A}})$$

that sends x_F to $[D_F]$ for all nonempty proper F and y_e to zero.

We define the following ideals in $T_{\mathcal{A}}$:

(1)
$$\mathcal{I}_1 := \left\langle \sum_F x_F \right\rangle$$

$$\mathcal{I}_2 := \left\langle y_e - \sum_{e \notin F} x_F \;\middle|\; e \in E \right\rangle$$

$$\mathcal{I}_3 := \langle x_F x_G \mid F \text{ and } G \text{ incomparable} \rangle$$

$$\mathcal{I}_4 := \langle y_e \mid e \in E \rangle$$

$$\mathcal{I}_4^{\mathrm{aug}} := \langle y_e x_F \mid e \notin F \rangle.$$

Theorem 1.1. *Let $\mathcal{A}$ be a hyperplane arrangement.*

(1) The map Φ_{FY} is surjective with kernel $\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4$ [FY04, Corollary 2].

(2) The map $\Phi_{\mathrm{FY}}^{\mathrm{aug}}$ is surjective with kernel $\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4^{\mathrm{aug}}$ [BHM$^+$22a, Remark 2.13].

Remark 1.2. The original Feichtner–Yuzvinsky presentation of $A(W_{\mathcal{A}})$ does not include the variable $x_{\varnothing}$. Many later sources omit both $x_{\varnothing}$ and x_E, since the remaining x_F are sufficient to generate and correspond to boundary divisors on $W_{\mathcal{A}}$ (or to rays of the Bergman fan). In [AHK18], $-\Phi_{\mathrm{FY}}(x_E)$ is called α and $-\Phi_{\mathrm{FY}}(x_{\varnothing})$ is called β. Similarly, the presentation of $A(W_{\mathcal{A}}^{\mathrm{aug}})$ in [BHM$^+$22a] does not include the variable x_E.

Next, consider the map

$$\Psi_{\mathrm{FY}}^{\mathrm{aug}} : T_{\mathcal{A}} \to K(W_{\mathcal{A}}^{\mathrm{aug}})$$

that sends x_F to $[\mathcal{O}_{D_F^{\mathrm{aug}}}]$ for all proper F, x_E to $1 - [\mathcal{L}_E^{\mathrm{aug}}]$, and y_e to $[\mathcal{O}_{D_e^{\mathrm{aug}}}]$ for all $e \in E$. Composing with the pullback along the inclusion $W_{\mathcal{A}} \subset W_{\mathcal{A}}^{\mathrm{aug}}$, we obtain a map

$$\Psi_{\mathrm{FY}} : T_{\mathcal{A}} \to K(W_{\mathcal{A}})$$

that sends x_F to $[\mathcal{O}_{D_F}]$ for all nonempty proper F and y_e to zero.

We define the following ideals in $T_{\mathcal{A}}$:

(2)
$$\mathcal{I}_1' := \left\langle 1 - \prod_F (1 - x_F) \right\rangle$$

$$\mathcal{I}_2' := \left\langle -(1 - y_e) + \prod_{e \notin F} (1 - x_F) \ \middle|\ e \in E \right\rangle.$$

Note that the generators of $\mathcal{I}_1'$ and $\mathcal{I}_2'$ are inhomogeneous, and their lowest order terms coincide with the generators of $\mathcal{I}_1$ and $\mathcal{I}_2$. Our first main result is the following K-theoretic analogue of Theorem 1.1.

Theorem 1.3. *Let $\mathcal{A}$ be a hyperplane arrangement.*

(1) *The map Ψ_{FY} is surjective with kernel $\mathcal{I}_1' + \mathcal{I}_2' + \mathcal{I}_3 + \mathcal{I}_4$.*
(2) *The map $\Psi_{\mathrm{FY}}^{\mathrm{aug}}$ is surjective with kernel $\mathcal{I}_1' + \mathcal{I}_2' + \mathcal{I}_3 + \mathcal{I}_4^{\mathrm{aug}}$.*

Remark 1.4. The difference between the presentations in Theorems 1.1 and 1.3 is that the homogeneous ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are replaced by the inhomogeneous ideals $\mathcal{I}_1'$ and $\mathcal{I}_2'$. Indeed, the Chow rings of these varieties are isomorphic to the associated graded of the coniveau filtrations on the K-rings.

1.3. **Exceptional isomorphisms and simplicial presentations.** For any nonempty flat F, we define the following Chow and K-classes:

$$h_F := c_1(\mathcal{L}_F) \in A(W_{\mathcal{A}}) \qquad\qquad h_F^{\mathrm{aug}} := c_1(\mathcal{L}_F^{\mathrm{aug}}) \in A(W_{\mathcal{A}}^{\mathrm{aug}})$$

$$\eta_F := 1 - [\mathcal{L}_F^{-1}] \in K(W_{\mathcal{A}}) \qquad \eta_F^{\mathrm{aug}} := 1 - [(\mathcal{L}_F^{\mathrm{aug}})^{-1}] \in K(W_{\mathcal{A}}^{\mathrm{aug}}).$$

Remark 1.5. The Chow classes $\{h_F\}$ and $\{h_F^{\mathrm{aug}}\}$ may be interpreted in terms of divisors on the permutohedral or stellahedral variety coming from simplices and are therefore known as **simplicial generators**. In the non-augmented setting, these classes were studied in [Yuz02, BES21], and the definitions and basic properties immediately generalize to the augmented setting. See [BES21, Section 3.2] for more details. We will similarly refer to $\{\eta_F\}$ and $\{\eta_F^{\mathrm{aug}}\}$ as the simplicial generators of the K-ring. The statement that they generate their respective K-rings is true but not obvious (Corollary 1.9).

Our next result, which is somewhat surprising from the vantage point of Theorems 1.1 and 1.3, says that there exist isomorphisms from the K-ring to the Chow ring of the varieties $W_{\mathcal{A}}$ and $W_{\mathcal{A}}^{\mathrm{aug}}$ that satisfy analogues of the Hirzebruch–Riemann–Roch formula, in which $\frac{1}{1-h_E}$ and $\frac{1}{1-h_E^{\mathrm{aug}}}$ play the roles of the respective Todd classes. The special case in which $\mathcal{A}$ is the Boolean arrangement (i.e., the number of hyperplanes is equal to the dimension of L) appeared in [BEST22, Theorem

D] in the non-augmented setting and in [EHL22, Theorem 1.8] in the augmented setting. Our techniques give new proofs of these results.

Theorem 1.6. *There are isomorphisms* $\zeta_{\mathcal{A}} : K(W_{\mathcal{A}}) \to A(W_{\mathcal{A}})$ *and* $\zeta_{\mathcal{A}}^{\mathrm{aug}} : K(W_{\mathcal{A}}^{\mathrm{aug}}) \to A(W_{\mathcal{A}}^{\mathrm{aug}})$ *characterized by the property that* $\zeta_{\mathcal{A}}(\eta_F) = h_F$ *and* $\zeta_{\mathcal{A}}^{\mathrm{aug}}(\eta_F^{\mathrm{aug}}) = h_F^{\mathrm{aug}}$. *For any classes* $\xi \in K(W_{\mathcal{A}})$ *and* $\xi^{\mathrm{aug}} \in K(W_{\mathcal{A}}^{\mathrm{aug}})$, *we have*

$$\chi(W_{\mathcal{A}}, \xi) = \deg_{W_{\mathcal{A}}} \left(\frac{\zeta_{\mathcal{A}}(\xi)}{1 - h_E} \right) \qquad and \qquad \chi(W_{\mathcal{A}}^{\mathrm{aug}}, \xi^{\mathrm{aug}}) = \deg_{W_{\mathcal{A}}^{\mathrm{aug}}} \left(\frac{\zeta_{\mathcal{A}}^{\mathrm{aug}}(\xi^{\mathrm{aug}})}{1 - h_E^{\mathrm{aug}}} \right).$$

Remark 1.7. The isomorphisms of Theorem 1.6 are not related to the Chern character homomorphisms; in particular, they do not coincide with the respective Chern characters after tensoring with the rational numbers. If we replaced $\zeta_{\mathcal{A}}$ (respectively $\zeta_{\mathcal{A}}^{\mathrm{aug}}$) with the Chern character, a similar formula would hold with $\frac{1}{1-h_E}$ (respectively $\frac{1}{1-h_E^{\mathrm{aug}}}$) replaced by the Todd class.

Let $S_{\mathcal{A}} := \mathbb{Z}[u_F \mid F \text{ a nonempty flat}\}$, and consider the map

$$\Phi_{\nabla}^{\mathrm{aug}} : S_{\mathcal{A}} \to A(W_{\mathcal{A}}^{\mathrm{aug}})$$

that sends u_F to h_F^{aug} for all F. Composing with the pullback along the inclusion $W_{\mathcal{A}} \subset W_{\mathcal{A}}^{\mathrm{aug}}$, we obtain a map

$$\Phi_{\nabla} : S_{\mathcal{A}} \to A(W_{\mathcal{A}})$$

that sends u_F to h_F for all F. For any pair of flats F and G, let $F \vee G$ denote the smallest flat containing both F and G. Consider the following ideals in $S_{\mathcal{A}}$:

$$\mathcal{J}_1 := \langle (u_F - u_{F \vee G})(u_G - u_{F \vee G}) \mid F, G \text{ arbitrary} \rangle$$

(3) $$\mathcal{J}_2 := \langle u_F \mid \mathrm{rk}\, F = 1 \rangle$$

$$\mathcal{J}_2^{\mathrm{aug}} := \langle u_F^2 \mid \mathrm{rk}\, F = 1 \rangle + \langle u_F(u_G - u_{F \vee G}) \mid \mathrm{rk}\, F = 1, G \text{ arbitrary} \rangle.$$

Theorem 1.8. *(Theorem A.2) Let* $\mathcal{A}$ *be a hyperplane arrangement. Then*

    (1) The map Φ_{∇} *is surjective with kernel* $\mathcal{J}_1 + \mathcal{J}_2$.
    (2) The map $\Phi_{\nabla}^{\mathrm{aug}}$ *is surjective with kernel* $\mathcal{J}_1 + \mathcal{J}_2^{\mathrm{aug}}$.

For the K-theoretic analogue of Theorem 1.8, consider the map

$$\Psi_{\nabla}^{\mathrm{aug}} : S_{\mathcal{A}} \to K(W_{\mathcal{A}}^{\mathrm{aug}})$$

that sends u_F to η_F^{aug} for all F. Composing with the pullback along the inclusion $W_{\mathcal{A}} \subset W_{\mathcal{A}}^{\mathrm{aug}}$, we obtain a map

$$\Psi_{\nabla} : S_{\mathcal{A}} \to K(W_{\mathcal{A}})$$

that sends u_F to η_F for all F. Theorems 1.6 and 1.8 immediately imply the following, which we refer to as the **simplicial presentations** of $K(W_{\mathcal{A}})$ and $K(W_{\mathcal{A}}^{\mathrm{aug}})$.

Corollary 1.9. *Let* $\mathcal{A}$ *be a hyperplane arrangement.*

    (1) The map Ψ_{∇} *is surjective with kernel* $\mathcal{J}_1 + \mathcal{J}_2$.
    (2) The map $\Psi_{\nabla}^{\mathrm{aug}}$ *is surjective with kernel* $\mathcal{J}_1 + \mathcal{J}_2^{\mathrm{aug}}$.

1.4. **The K-ring of $\overline{\mathcal{M}}_{0,n}$.** Our results generalize to $\overline{\mathcal{M}}_{0,n}$, the Deligne–Mumford–Knudsen compactification of the moduli space of stable rational curves with n marked points, for $n \geq 3$. For $i \in \{1, \ldots, n\}$, there is a line bundle $\mathbb{L}_i$ on $\overline{\mathcal{M}}_{0,n}$ whose fiber over a point is the cotangent space of the ith marked point in the corresponding curve. The first Chern class of $\mathbb{L}_i$ is denoted ψ_i.

Kapranov [Kap93] showed that each $\mathbb{L}_i$ is a base-point-free line bundle whose complete linear system induces a birational map $\overline{\mathcal{M}}_{0,n} \to \mathbb{P}^{n-3}$. For every subset S of $[n] := \{1, \ldots, n\}$, we have a forgetful map $f_S : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,S} = \overline{\mathcal{M}}_{0,|S|}$. We therefore obtain a map

$$\overline{\mathcal{M}}_{0,n} \to \prod_{S \subset [n-1]} \mathbb{P}^{|S|-2}$$

by composing $f_{S \cup n}$ with the map induced by the complete linear system of $\mathbb{L}_n$ on $\overline{\mathcal{M}}_{0,S \cup n}$. This map is a closed embedding.

Consider the braid arrangement $\mathcal{B}_{n-1}$ in $\mathbb{F}^{n-1}/\mathbb{F} \cdot (1, \ldots, 1)$, whose hyperplanes are normal to $e_i - e_j$ for $i < j \in [n-1]$. The lattice of flats of $\mathcal{B}_{n-1}$ may be identified with the collection of partitions of the set $[n-1]$. For any subset $S \subset [n-1]$ of cardinality at least 2, let F_S denote the flat corresponding to the partition of $[n-1]$ into S along with a bunch of singletons. Forgetting the flats of $\mathcal{B}_{n-1}$ not of the form F_S for some S, we have a map

$$W_{\mathcal{B}_{n-1}} \to \prod_S \mathbb{P}(L^{F_S})$$

whose image is $\overline{\mathcal{M}}_{0,n}$ under the embedding described previously [DCP95, Section 4.3]. The relation between $\overline{\mathcal{M}}_{0,n}$ and $W_{\mathcal{B}_{n-1}}$ allows us to deduce results about $\overline{\mathcal{M}}_{0,n}$ from our study of wonderful varieties.

Let $\mathcal{L}_S = f_{S \cup n}^* \mathbb{L}_n$. This bundle is trivial when $|S| = 2$, and the bundles corresponding to sets of cardinality at least 3 form a basis for the Picard group of $\overline{\mathcal{M}}_{0,n}$. Let

$$c_S := c_1(\mathcal{L}_S) = f_{S \cup n}^* \psi_n \in A^1(\overline{\mathcal{M}}_{0,n}).$$

Theorem 1.10. *There is an isomorphism $\zeta_n : K(\overline{\mathcal{M}}_{0,n}) \to A(\overline{\mathcal{M}}_{0,n})$ that sends $1 - [\mathcal{L}_S^{-1}]$ to c_S. For any $\xi \in K(\overline{\mathcal{M}}_{0,n})$, the Euler characteristic of ξ is equal to the degree of $\frac{\zeta_n(\xi)}{1 - c_{[n-1]}}$.*

1.5. **Matroids.** The presentations of $A(W_{\mathcal{A}})$ and $A(W_{\mathcal{A}}^{\mathrm{aug}})$ depend only on the matroid associated with $\mathcal{A}$, which led to the definitions of Chow rings and augmented Chow rings of arbitrary loopless matroids [FY04, BHM+22a]. Similarly, our descriptions of $K(W_{\mathcal{A}})$ and $K(W_{\mathcal{A}}^{\mathrm{aug}})$, along with their Euler characteristic functionals, only depend on the matroid associated with $\mathcal{A}$. With the exception of the latter part of Section 8, we will always assume that all matroids are loopless.

Let M be a matroid on the ground set E. Let Σ_M be the **Bergman fan** of M, which has rays $\{\rho_F \mid F \text{ a nonempty proper flat}\}$. Let Σ_M^{aug} be the **augmented Bergman fan** [BHM+22a] of M, which has rays $\{\rho_F^{\mathrm{aug}} \mid F \text{ a proper flat}\} \cup \{\rho_e^{\mathrm{aug}} \mid e \in E\}$. These two fans are related by the fact that Σ_M is isomorphic to the star of the ray $\rho_\varnothing$ in Σ_M^{aug} [BHM+22a, Proposition 2.7(2)]. The rings $A(M)$ and $A^{\mathrm{aug}}(M)$ are defined to be the Chow rings of the toric varieties X_{Σ_M} and $X_{\Sigma_M^{\mathrm{aug}}}$, respectively.

If M is the matroid associated with a hyperplane arrangement $\mathcal{A}$, then, after choosing a linear functional defining each hyperplane in $\mathcal{A}$, we obtain canonical inclusions $W_{\mathcal{A}} \subset X_{\Sigma_M}$ and $W_{\mathcal{A}}^{\mathrm{aug}} \subset X_{\Sigma_M^{\mathrm{aug}}}$ with the properties that $D_F = W_{\mathcal{A}} \cap D_{\rho_F}$ for any nonempty proper flat F, $D_F^{\mathrm{aug}} = W_{\mathcal{A}}^{\mathrm{aug}} \cap D_{\rho_F^{\mathrm{aug}}}$ for any proper flat F, and $D_e^{\mathrm{aug}} = W_{\mathcal{A}}^{\mathrm{aug}} \cap D_{\rho_e^{\mathrm{aug}}}$ for any $e \in E$. The restriction maps $A(M) \to A(W_{\mathcal{A}})$ and $A^{\mathrm{aug}}(M) \to A(W_{\mathcal{A}}^{\mathrm{aug}})$ are both isomorphisms [FY04, BHM$^+$22a].

Consider the polynomial ring $T_M := \mathbb{Z}[x_F \mid F \text{ a flat}] \otimes \mathbb{Z}[y_e \mid e \in E]$ along with the homomorphism $\Phi_{\mathrm{FY}}^{\mathrm{aug}} : T_M \to A^{\mathrm{aug}}(M)$ taking x_F to $[D_{\rho_F^{\mathrm{aug}}}]$ for all proper flats F, x_E to $-\sum_{F \neq E}[D_{\rho_F^{\mathrm{aug}}}]$, and y_e to $[D_{\rho_e^{\mathrm{aug}}}]$. Composing with the pullback along the inclusion $X_{\Sigma_M} \cong D_{\rho_\varnothing^{\mathrm{aug}}} \subset X_{\Sigma_M^{\mathrm{aug}}}$, we obtain a homomorphism $\Phi_{\mathrm{FY}} : T_M \to A(M)$ that sends x_F to D_{ρ_F} for all nonempty proper F and y_e to zero. Consider also the polynomial ring $S_M := \mathbb{Z}[u_F \mid F \text{ a nonempty flat}]$. We define $\Phi_{\nabla} : S_M \to A(M)$ and $\Phi_{\nabla}^{\mathrm{aug}} : S_M \to A^{\mathrm{aug}}(M)$ via the formulas

$$(4) \qquad \Phi_{\nabla}(u_F) := -\sum_{F \subset G} \Phi_{\mathrm{FY}}(x_G) \qquad \text{and} \qquad \Phi_{\nabla}^{\mathrm{aug}}(u_F) := -\sum_{F \subset G} \Phi_{\mathrm{FY}}^{\mathrm{aug}}(x_G).$$

Remark 1.11. When M is the matroid associated with the arrangement $\mathcal{A}$, the following diagrams (along with their augmented analogues) commute, thus justifying our repeated use of the notation Φ_{FY} and Φ_{∇}:

$$
\begin{array}{ccc}
T_M & \xrightarrow{\;=\;} & T_{\mathcal{A}} \\
\downarrow{\scriptstyle\Phi_{\mathrm{FY}}} & & \downarrow{\scriptstyle\Phi_{\mathrm{FY}}} \\
A(M) & \xrightarrow{\;\cong\;} & A(W_{\mathcal{A}})
\end{array}
\qquad\qquad
\begin{array}{ccc}
S_M & \xrightarrow{\;=\;} & S_{\mathcal{A}} \\
\downarrow{\scriptstyle\Phi_{\nabla}} & & \downarrow{\scriptstyle\Phi_{\nabla}} \\
A(M) & \xrightarrow{\;\cong\;} & A(W_{\mathcal{A}}).
\end{array}
$$

Commutativity of the first diagram is immediate from the definitions, while commutativity of the second diagram is proved by reducing to the case of the Boolean matroid [BES21, Section 3.2].

Define the ideals $\mathcal{I}_1, \mathcal{I}_1', \mathcal{I}_2, \mathcal{I}_2', \mathcal{I}_3, \mathcal{I}_4, \mathcal{I}_4^{\mathrm{aug}} \subset T_M$ and $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_2^{\mathrm{aug}} \subset S_M$ as in Equations (1), (2), and (3).

Theorem 1.12. *Let M be a loopless matroid.*

(1) The map Φ_{FY} is surjective with kernel $\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4$ [FY04].

(2) The map $\Phi_{\mathrm{FY}}^{\mathrm{aug}}$ is surjective with kernel $\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4^{\mathrm{aug}}$ [BHM$^+$22a].

(3) The map Φ_{∇} is surjective with kernel $\mathcal{J}_1 + \mathcal{J}_2$ [BES21].

(4) The map $\Phi_{\nabla}^{\mathrm{aug}}$ is surjective with kernel $\mathcal{J}_1 + \mathcal{J}_2^{\mathrm{aug}}$ (Theorem A.2).

Similarly, we define $K(M)$ and $K^{\mathrm{aug}}(M)$ to be the K-rings of X_{Σ_M} and $X_{\Sigma_M^{\mathrm{aug}}}$, respectively.

Proposition 1.13. *If M is the matroid associated with $\mathcal{A}$, then the inclusions $W_{\mathcal{A}} \subset X_{\Sigma_M}$ and $W_{\mathcal{A}}^{\mathrm{aug}} \subset X_{\Sigma_M^{\mathrm{aug}}}$ induce isomorphisms $K(M) \cong K(W_{\mathcal{A}})$ and $K^{\mathrm{aug}}(M) \cong K(W_{\mathcal{A}}^{\mathrm{aug}})$.*

Let $\mathcal{L}_E^{\mathrm{aug}}$ be the line bundle on $X_{\Sigma_M^{\mathrm{aug}}}$ whose first Chern class is equal to $\sum_{F \neq E}[D_{\rho_F^{\mathrm{aug}}}]$. We define the map $\Psi_{\mathrm{FY}}^{\mathrm{aug}} : T_M \to K^{\mathrm{aug}}(M)$ by sending x_F to $[\mathcal{O}_{D_{\rho_F^{\mathrm{aug}}}}]$ for all proper F, x_E to $1 - [\mathcal{L}_E^{\mathrm{aug}}]$, and y_e to $[\mathcal{O}_{D_{\rho_e^{\mathrm{aug}}}}]$ for all $e \in E$. Composing with the pullback along the inclusion $X_{\Sigma_M} \cong D_{\rho_\varnothing^{\mathrm{aug}}} \subset X_{\Sigma_M^{\mathrm{aug}}}$, we obtain a homomorphism $\Psi_{\mathrm{FY}} : T_M \to K(M)$ that sends x_F to $[\mathcal{O}_{D_{\rho_F}}]$ for all nonempty proper F and y_e to zero.

Remark 1.14. When M is the matroid associated with the arrangement $\mathcal{A}$, the following diagram commutes (as does its augmented analogue), thus justifying the repeated use of the notation Ψ_{FY}:

$$
\begin{array}{ccc}
T_M & \xrightarrow{\quad = \quad} & T_{\mathcal{A}} \\
\downarrow{\scriptstyle \Psi_{\mathrm{FY}}} & & \downarrow{\scriptstyle \Psi_{\mathrm{FY}}} \\
K(M) & \xrightarrow{\quad \cong \quad} & K(W_{\mathcal{A}}).
\end{array}
$$

We have the following generalization of Theorem 1.3 to matroids.

Theorem 1.15. *Let M be a loopless matroid.*

(1) *The map Ψ_{FY} is surjective with kernel $\mathcal{I}_1' + \mathcal{I}_2' + \mathcal{I}_3 + \mathcal{I}_4$.*
(2) *The map $\Psi_{\mathrm{FY}}^{\mathrm{aug}}$ is surjective with kernel $\mathcal{I}_1' + \mathcal{I}_2' + \mathcal{I}_3 + \mathcal{I}_4^{\mathrm{aug}}$.*

We also have the following extension of Theorem 1.6 to matroids.

Theorem 1.16. *There exist isomorphisms $\zeta_M \colon K(M) \to A(M)$ and $\zeta_M^{\mathrm{aug}} \colon K^{\mathrm{aug}}(M) \to A^{\mathrm{aug}}(M)$, characterized by the property that*

$$
\zeta_M^{-1}(h_F) = 1 - \prod_{F \subset G} \left(1 - \Psi_{\mathrm{FY}}(x_F)\right)^{-1} \qquad and \qquad (\zeta_M^{\mathrm{aug}})^{-1}(h_F^{\mathrm{aug}}) = 1 - \prod_{F \subset G} \left(1 - \Psi_{\mathrm{FY}}^{\mathrm{aug}}(x_F)\right)^{-1}.
$$

Motivated by the above characterization of $\zeta_{\mathcal{A}}$ and $\zeta_{\mathcal{A}}^{\mathrm{aug}}$, we define maps $\Psi_{\nabla} \colon S_M \to K(M)$ and $\Psi_{\nabla}^{\mathrm{aug}} \colon S_M \to K^{\mathrm{aug}}(M)$ by the formulas

$$
(5) \quad \Psi_{\nabla}(u_F) = 1 - \prod_{F \subset G} \left(1 - \Psi_{\mathrm{FY}}(x_G)\right)^{-1} \qquad \text{and} \qquad \Psi_{\nabla}^{\mathrm{aug}}(u_F) = 1 - \prod_{F \subset G} \left(1 - \Psi_{\mathrm{FY}}^{\mathrm{aug}}(x_G)\right)^{-1}.
$$

Remark 1.17. Let M be the matroid associated with an arrangement $\mathcal{A}$. Identifying $K(M)$ with $K(W_{\mathcal{A}})$ via the isomorphism in Proposition 1.13, we have constructed two isomorphisms $\zeta_M, \zeta_{\mathcal{A}} \colon K(M) \to A(M)$. Proposition 5.1 implies that these two isomorphisms coincide, as do the analogous pair of isomorphisms in the augmented setting. Similarly, we have constructed two maps $\Psi_{\nabla} \colon S_M \to K(M)$, one defined geometrically in Section 1.3, and the other defined algebraically in Equation (5). Proposition 5.1 also implies that these maps coincide, as do the analogous pair of maps in the augmented setting.

Theorem 1.12 and Theorem 1.16 immediately give the following presentations of $K(M)$ and $K^{\mathrm{aug}}(M)$, extending Corollary 1.9 to matroids.

Corollary 1.18. *Let M be a loopless matroid.*

(1) *The map Ψ_{∇} is surjective with kernel $\mathcal{J}_1 + \mathcal{J}_2$.*
(2) *The map $\Psi_{\nabla}^{\mathrm{aug}}$ is surjective with kernel $\mathcal{J}_1 + \mathcal{J}_2^{\mathrm{aug}}$.*

The Chow rings $A(M)$ and $A^{\mathrm{aug}}(M)$ satisfy **Poincaré duality**, meaning that there exist maps

$$
\deg_M \colon A(M) \to \mathbb{Z} \qquad \text{and} \qquad \deg_M^{\mathrm{aug}} \colon A^{\mathrm{aug}}(M) \to \mathbb{Z}
$$

such the pairing $A(M) \otimes A(M) \to \mathbb{Z}$ taking $f \otimes g$ to $\deg_M(fg)$ is perfect [AHK18, Theorem 6.19], and similarly in the augmented setting [BHM+22a, Theorem 1.3(4)]. Define maps

$$
\chi \colon K(M) \to \mathbb{Z} \qquad \text{and} \qquad \chi^{\mathrm{aug}} \colon K(M) \to \mathbb{Z}
$$

by putting

$$\chi(M,\xi) := \deg_M \left(\frac{\zeta_M(\xi)}{1 - h_E} \right) \qquad \text{and} \qquad \chi^{\mathrm{aug}}(M,\xi^{\mathrm{aug}}) := \deg_M^{\mathrm{aug}} \left(\frac{\zeta_M^{\mathrm{aug}}(\xi^{\mathrm{aug}})}{1 - h_E^{\mathrm{aug}}} \right).$$

Then the analogous pairing $K(M) \otimes K(M) \to \mathbb{Z}$ taking $\xi \otimes \eta$ to $\chi(M, \xi\eta)$ is perfect, and similarly in the augmented setting. For realizable matroids, this is a general property of the Euler pairing on the K-theory of a smooth proper linear variety [AP15, Theorem 1.3].

Remark 1.19. When M is not realizable, the maps χ and χ^{aug} still have a natural geometric interpretation as Euler characteristics on the wonderful variety (respectively augmented wonderful variety) of the Boolean arrangement. See Proposition 5.3.

Although the K-rings $K(M)$ and $K^{\mathrm{aug}}(M)$ are isomorphic to their respective Chow rings, the fact that they are K-rings endow them with several additional structures, including the structure of augmented λ-rings (see Section 6 for details). Furthermore, when M is realizable, Serre duality gives a nontrivial identity satisfied by χ and χ^{aug}. In Theorem 6.2, we show that that these identities extend to matroids.

1.6. **Applications to Euler characteristics.** Much is known about intersection theory on (augmented) wonderful varieties. The Hirzebruch–Riemann–Roch-type formulas in Theorem 1.6 allow us to transfer intersection-theoretic computations to K-theory. As applications, we give formulas for the Euler characteristic of the class of any line bundle in $K(M)$ expressed in terms of the bundles $\mathcal{L}_F$ (Theorem 7.2 and Corollary 7.5), and similarly in the augmented setting. We also give a formula for the Euler characteristic of any line bundle expressed in terms of the Feichtner–Yuzvinsky generators of $K(M)$ (Theorem 8.2), extending as a corollary Eur's formula [Eur20] for the degrees of monomials in the Feichtner–Yuzvinsky generators of $A(M)$ (Corollary 8.4). Likewise, we give formulas for the Euler characteristic of any line bundle on $\overline{\mathcal{M}}_{0,n}$ expressed in terms the line bundles $\{\mathcal{L}_S\}$ (Theorem 9.1) or $\{\mathbb{L}_i\}$ (Theorem 9.2).

1.7. **Structure of the paper.** In Sections 2 and 3, we prove Theorems 1.3 and 1.6, assuming the following technical lemma:

Lemma 1.20. *The K-rings of $W_{\mathcal{A}}$ and $W_{\mathcal{A}}^{\mathrm{aug}}$ are generated by the classes of line bundles.*

In Section 4, we discuss the presentation of $K(\overline{\mathcal{M}}_{0,n})$ and the exceptional isomorphism to $A(\overline{\mathcal{M}}_{0,n})$ in Theorem 1.10. In Section 5, we prove Theorem 1.15, and we derive Lemma 1.20 as a corollary. We have chosen to order our presentation in this way, rather than beginning with arbitrary matroids, in order better to highlight the geometric ideas underlying the proofs of our main results.

Section 6 is devoted to a description of the Adams operations in $K(M)$ and $K^{\mathrm{aug}}(M)$ along with a combinatorial Serre duality theorem (Theorem 6.2). Section 7 is dedicated to computing Euler characteristics in the simplicial generators, Section 8 to Euler characteristics in the Feichtner–Yuzvinsky generators, and Section 9 to Euler characteristics on $\overline{\mathcal{M}}_{0,n}$. Finally, we include an

appendix with a proof of the simplicial presentation for the augmented Chow ring of a matroid, which has not previously appeared in the literature.

We note that some of the proofs in Sections 6 and 8 proceed by first giving geometric proofs for realizable matroids, and then using the notion of valuativity to extend the results to all matroids. Thus, in these sections, separating the statements for hyperplane arrangements from the statements for matroids is absolutely essential.

2. FEICHTNER–YUZVINSKY PRESENTATIONS

In this section, we prove Theorem 1.3. We begin with some general lemmas about a smooth variety X, which we will also apply later in the proofs of Theorems 1.6 and 1.15.

Lemma 2.1. *If $A(X)$ is a free abelian group of finite rank p, then so is $K(X)$, and the Chern character* $\mathrm{ch} : K(X) \to A(X)_{\mathbb{Q}}$ *is injective.*

Proof. The Chern character becomes an isomorphism after tensoring with $\mathbb{Q}$ [Ful98, Example 15.2.16], so $K(X)$ has rank p. There is a surjective map from $A(X)$ to the associated graded of $K(X)$ with respect to the coniveau filtration [Ful98, Example 15.1.5]. Since $A(X)$ is free of rank p and $K(X)$ also has rank p, this implies that that $K(X)$ is free. Finally, the Chern character factors as $K(X) \to K(X)_{\mathbb{Q}} \to A(X)_{\mathbb{Q}}$, with the first map being injective by freeness of $K(X)$ and the second map being an isomorphism, so the Chern character is injective. □

Remark 2.2. Lemma 2.1 will be applied to the varieties $W_{\mathcal{A}}$, X_{Σ_M} [FY04, Corollary 1] and $\overline{\mathcal{M}}_{0,n}$ [FY04, Corollary 2], as well as to $W_{\mathcal{A}}^{\mathrm{aug}}$ and $X_{\Sigma_M^{\mathrm{aug}}}$ [AHK18, Theorem 6.19].

Lemma 2.3. *Let X be a smooth variety, and suppose that $K(X)$ is generated as a ring by the classes of line bundles. Let $D_1, \ldots, D_k$ be divisors on X. If $A^1(X)$ is generated as an abelian group by $[D_1], \ldots, [D_k]$, then $K(X)$ is generated as a ring by $[\mathcal{O}_{D_1}], \ldots, [\mathcal{O}_{D_k}]$.*

Proof. Let R be the subring of $K(X)$ generated by $[\mathcal{O}_{D_1}], \ldots, [\mathcal{O}_{D_k}]$. We need to show that the class of every line bundle is contained in R. Since $[D_1], \ldots, [D_k]$ generate $A^1(X)$ as an abelian group, the line bundles $\mathcal{O}(\pm D_1), \ldots, \mathcal{O}(\pm D_k)$ generate the Picard group of X under multiplication, so it will be sufficient to show that $[\mathcal{O}(\pm D_i)] \in R$ for all i.

For any divisor D, we have an exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0,$$

which implies that
$$[\mathcal{O}(-D)] = [\mathcal{O}] - [\mathcal{O}_D] = 1 - [\mathcal{O}_D].$$

We also have

(6) $$[\mathcal{O}(D)] = [\mathcal{O}(-D)]^{-1} = \frac{1}{1 - [\mathcal{O}_D]} = 1 + [\mathcal{O}_D] + [\mathcal{O}_D]^2 + \cdots.$$

Since $[\mathcal{O}_D]$ lives in the first piece of the coniveau filtration on $K(X)$, it is nilpotent, so the sum terminates. This allows us to conclude that both $[\mathcal{O}(-D_i)]$ and $[\mathcal{O}(D_i)]$ live in the ring R. $\qquad\square$

Proof of Theorem 1.3. We begin with statement (2). Surjectivity of the map $\Phi_{\mathrm{FY}}^{\mathrm{aug}}$ follows from Theorem 1.1, Lemma 1.20, and Lemma 2.3. Next, we show that $\mathcal{I}_1' + \mathcal{I}_2' + \mathcal{I}_3 + \mathcal{I}_4^{\mathrm{aug}}$ is contained in the kernel of $\Phi_{\mathrm{FY}}^{\mathrm{aug}}$. The ideal $\mathcal{I}_3$ is contained in the kernel because D_F and D_G are disjoint when F and G are incomparable. Similarly, $\mathcal{I}_4^{\mathrm{aug}}$ is contained in the kernel because D_e is disjoint from D_F whenever $e \notin F$. To prove that $\mathcal{I}_1'$ is contained in the kernel, we observe that its generator maps to
$$1 - [\mathcal{L}_E^{\mathrm{aug}}] \prod_{F \neq E} \left(1 - [\mathcal{O}_{D_F^{\mathrm{aug}}}] \right) = 1 - [\mathcal{L}_E^{\mathrm{aug}}] \prod_{F \neq E} [\mathcal{O}(-D_F^{\mathrm{aug}})].$$

Since $\mathcal{I}_1$ is contained in the kernel of $\Phi_{\mathrm{FY}}^{\mathrm{aug}}$, the line bundle
$$\mathcal{L}_E^{\mathrm{aug}} \otimes \bigotimes_{F \neq E} \mathcal{O}(-D_F^{\mathrm{aug}})$$

is trivial, thus the generator of $\mathcal{I}_1'$ is contained in the kernel of $\Psi_{\mathrm{FY}}^{\mathrm{aug}}$.

To prove that $\mathcal{I}_2'$ is contained in the kernel, Lemma 2.1 tells us that it is sufficient to prove that, for all $e \in E$, we have
$$\mathrm{ch}\left(1 - [\mathcal{O}_{D_e^{\mathrm{aug}}}] \right) = \prod_{e \notin F} \mathrm{ch}\left(1 - [\mathcal{O}_{D_F^{\mathrm{aug}}}] \right).$$

We have
$$\mathrm{ch}\left(1 - [\mathcal{O}_{D_e^{\mathrm{aug}}}] \right) = \mathrm{ch}([\mathcal{O}(-D_e^{\mathrm{aug}})]) = \exp\left(-[D_e^{\mathrm{aug}}] \right)$$

and
$$\prod_{e \notin F} \mathrm{ch}\left(1 - [\mathcal{O}_{D_F^{\mathrm{aug}}}] \right) = \prod_{e \notin F} \exp\left(-[D_F^{\mathrm{aug}}] \right) = \exp\left(-\sum_{e \notin F} [D_F^{\mathrm{aug}}] \right).$$

The fact that these are equal follows from the fact that $\mathcal{I}_2$ is contained in the kernel of $\Phi_{\mathrm{FY}}^{\mathrm{aug}}$.

Let $R := T_{\mathcal{A}} / \mathcal{I}_1' + \mathcal{I}_2' + \mathcal{I}_3 + \mathcal{I}_4^{\mathrm{aug}}$. We have now shown that R surjects onto $K(W_{\mathcal{A}}^{\mathrm{aug}})$, and we need to prove that the map is injective. Let p be the rank of the free abelian group $A(W_{\mathcal{A}}^{\mathrm{aug}})$. Consider the decreasing filtration
$$R = F_0 \supset F_1 \supset \cdots,$$

where F_i is the span of all monomials of total degree $\geq i$. Since the leading terms of the generators $\mathcal{I}_1'$ and $\mathcal{I}_2'$ coincide with the generators of $\mathcal{I}_1$ and $\mathcal{I}_2$, we have a surjection
$$A(W_{\mathcal{A}}^{\mathrm{aug}}) \cong T_{\mathcal{A}} / \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4^{\mathrm{aug}} \to \mathrm{gr}\, R.$$

In particular, this implies that the abelian group $\operatorname{gr} R$ can be generated by p elements, and so the same is true of R. Lemma 2.1 tells us that $K(W_{\mathcal{A}}^{\mathrm{aug}})$ is also free abelian of rank p, so any surjection from R to $K(W_{\mathcal{A}}^{\mathrm{aug}})$ must be an isomorphism.

The proof of statement (1) is nearly identical. The only extra ingredient is the argument that $\mathcal{I}_4$ is contained in the kernel of the map Ψ_{FY}, which follows from the fact that D_e^{aug} is disjoint from $D_{\varnothing}^{\mathrm{aug}} \cong W_{\mathcal{A}}$ inside of $W_{\mathcal{A}}^{\mathrm{aug}}$. $\qquad\square$

3. Simplicial presentations

The purpose of this section is to prove Theorem 1.6. We begin by observing that we have an inclusion

$$W_{\mathcal{A}} \quad \subset \quad \prod_{\substack{F \subset E \\ \text{a nonempty flat}}} \mathbb{P}(L^F) \quad \subset \quad \prod_{\substack{S \subset E \\ \text{a nonempty subset}}} \mathbb{P}(L^S).$$

The first inclusion comes from the definition of $W_{\mathcal{A}}$, and the second from the diagonal embedding of $\mathbb{P}(L^F)$ into the product of $\mathbb{P}(L^S)$ for all S such that $L^S = L^F$.[1] Similarly, we have

$$W_{\mathcal{A}}^{\mathrm{aug}} \quad \subset \quad \prod_{\substack{F \subset E \\ \text{a nonempty flat}}} \mathbb{P}(L^F \oplus \mathbb{F}) \quad \subset \quad \prod_{\substack{S \subset E \\ \text{a nonempty subset}}} \mathbb{P}(L^S \oplus \mathbb{F}).$$

Suppose that $L' \subset L$ is a linear subspace. Let $E' := \{e \in E \mid L' \not\subset H_e\}$, and define a new hyperplane arrangement

$$\mathcal{A}_{L'} := \{H_e \cap L' \mid e \in E'\}.$$

If $E' = E$, then we have inclusions $W_{\mathcal{A}_{L'}} \subset W_{\mathcal{A}}$ and $W_{\mathcal{A}_{L'}}^{\mathrm{aug}} \subset W_{\mathcal{A}}^{\mathrm{aug}}$, each inside of the corresponding product of projective spaces indexed by subsets of E. If L' is contained in one or more hyperplane, then we still have an inclusion $W_{\mathcal{A}_{L'}}^{\mathrm{aug}} \subset W_{\mathcal{A}}^{\mathrm{aug}}$, where $W_{\mathcal{A}_{L'}}^{\mathrm{aug}}$ sits inside the product indexed by subsets of E by setting the S coordinate to $0 \in L^S \subset \mathbb{P}(L^S \oplus \mathbb{F})$ unless $S \subset E'$.

For any flat F of $\mathcal{A}$, let $F' := \{e \in E' \mid L_F \cap L' \subset H_e\}$ be the corresponding flat of L'. Then the line bundle $\mathcal{L}_F^{\mathrm{aug}}$ on $W_{\mathcal{A}}^{\mathrm{aug}}$ restricts to the line bundle $\mathcal{L}_{F'}$ on $W_{\mathcal{A}_{L'}}^{\mathrm{aug}}$. If $E = E'$, then the line bundle $\mathcal{L}_F$ on $W_{\mathcal{A}}$ restricts to the line bundle $\mathcal{L}_{F'}$ on $W_{\mathcal{A}_{L'}}$.

Lemma 3.1. *Fix a flat F, and suppose that $H \subset L$ is a hyperplane with the property that $L_F \subset H$ and $L_G \not\subset H$ for all flats $G \subsetneq F$. Then*

$$h_F^{\mathrm{aug}} = [W_{\mathcal{A}_H}^{\mathrm{aug}}] \qquad and \qquad \eta_F^{\mathrm{aug}} = [\mathcal{O}_{W_{\mathcal{A}_H}^{\mathrm{aug}}}].$$

If the rank of F is greater than 1, then

$$h_F = [W_{\mathcal{A}_H}] \qquad and \qquad \eta_F = \left[\mathcal{O}_{W_{\mathcal{A}_H}}\right].$$

Proof. We prove only the first statement; the second is similar. Let g be a section of $\mathcal{O}_{\mathbb{P}(L^F \oplus \mathbb{F})}(1)$ with vanishing locus equal to $\mathbb{P}(H/L_F \oplus \mathbb{F}) \subset \mathbb{P}(L^F \oplus \mathbb{F})$. Then $(\pi_F^{\mathrm{aug}})^* g$ is a regular section of $\mathcal{L}_F^{\mathrm{aug}}$ with vanishing locus $W_{\mathcal{A}_H}^{\mathrm{aug}}$, which shows that $h_F^{\mathrm{aug}} = c_1(\mathcal{L}_F^{\mathrm{aug}}) = [W_{\mathcal{A}_H}^{\mathrm{aug}}]$. We interpret $(\pi_F^{\mathrm{aug}})^* g$

---

[1]In matroid theoretic language, this is the collection of subsets S whose closure is F.

as an element of $\operatorname{Hom}(\mathcal{O}_{W_\mathcal{A}^{\mathrm{aug}}}, \mathcal{L}_F^{\mathrm{aug}}) \cong \operatorname{Hom}\left((\mathcal{L}_F^{\mathrm{aug}})^{-1}, \mathcal{O}_{W_\mathcal{A}^{\mathrm{aug}}}\right)$ to obtain an exact sequence

$$0 \to (\mathcal{L}_F^{\mathrm{aug}})^{-1} \xrightarrow{(\pi_F^{\mathrm{aug}})^* g} \mathcal{O}_{W_\mathcal{A}^{\mathrm{aug}}} \to \mathcal{O}_{W_{\mathcal{A}_H}^{\mathrm{aug}}} \to 0,$$

which shows that $\eta_F^{\mathrm{aug}} = 1 - [(\mathcal{L}_F^{\mathrm{aug}})^{-1}] = [\mathcal{O}_{W_\mathcal{A}^{\mathrm{aug}}}] - [(\mathcal{L}_F^{\mathrm{aug}})^{-1}] = [\mathcal{O}_{W_{\mathcal{A}_H}^{\mathrm{aug}}}].$ $\qquad\square$

For any tuple of natural numbers $\mathbf{m} = (m_F \mid F \text{ a nonempty flat})$, we define the monomials

$$h^{\mathbf{m}} := \prod_F h_F^{m_F} \in A(W_\mathcal{A}) \qquad (h^{\mathrm{aug}})^{\mathbf{m}} := \prod_F (h_F^{\mathrm{aug}})^{m_F} \in A(W_\mathcal{A}^{\mathrm{aug}})$$

$$\eta^{\mathbf{m}} := \prod_F \eta_F^{m_F} \in K(W_\mathcal{A}) \qquad (\eta^{\mathrm{aug}})^{\mathbf{m}} := \prod_F (\eta_F^{\mathrm{aug}})^{m_F} \in K(W_\mathcal{A}^{\mathrm{aug}}).$$

Lemma 3.2. *Suppose that $\mathbb{F}$ is infinite. For any $\mathbf{m}$, one of the following two statements holds:*

- $\eta^{\mathbf{m}} = 0$ *and* $h^{\mathbf{m}} = 0$.
- *There exists a linear subspace $L' \subset L$, not contained in any of the hyperplanes in $\mathcal{A}$, such that $\eta^{\mathbf{m}} = [\mathcal{O}_{W_{\mathcal{A}_{L'}}}]$ and $h^{\mathbf{m}} = [W_{\mathcal{A}_{L'}}]$.*

In addition, one of the following two statements holds:

- $(\eta^{\mathrm{aug}})^{\mathbf{m}} = 0$ *and* $(h^{\mathrm{aug}})^{\mathbf{m}} = 0$.
- *There exists a linear subspace $L' \subset L$ such that $(\eta^{\mathrm{aug}})^{\mathbf{m}} = [\mathcal{O}_{W_{\mathcal{A}_{L'}}^{\mathrm{aug}}}]$ and $(h^{\mathrm{aug}})^{\mathbf{m}} = [W_{\mathcal{A}_{L'}}^{\mathrm{aug}}].$*

Proof. We will prove only the first claim; the augmented case is similar. We proceed by induction on $\sum m_F$. When $\mathbf{m} = 0$, we can take $L' = L$. For the inductive step, assume that the second statement holds for $\mathbf{m}'$, and take $\mathbf{m}$ such that $m_F = m'_F + 1$ and $\mathbf{m}$ agrees with $\mathbf{m}'$ in every other coordinate. Then

$$\eta^{\mathbf{m}} = \eta_F \eta^{\mathbf{m}'} = \eta_F [\mathcal{O}_{W_{\mathcal{A}_{L'}}}] = \left(1 - [\mathcal{L}_F^{-1}]\right)[\mathcal{O}_{W_{\mathcal{A}_{L'}}}] = [\mathcal{O}_{W_{\mathcal{A}_{L'}}}] - [\mathcal{L}_{F'}^{-1}],$$

where $F' = \{e \mid L_F \cap L' \subset H_e\}$ and $\mathcal{L}_{F'}$ is the corresponding line bundle on $W_{\mathcal{A}_{L'}}$. If F' has rank 1, then $\mathcal{L}_{F'}$ is trivial, and we get zero. If F' has rank greater than 1, then we may choose a hyperplane[2] $H' \subset L'$ such that $H' \notin \mathcal{A}'$, $L_{F'} \subset H'$, and $L_{G'} \not\subset H'$ for all flats $G' \subsetneq F'$. Then by Lemma 3.1, $\eta^{\mathbf{m}} = [\mathcal{O}_{W_{\mathcal{A}_{L'}}}] - [\mathcal{L}_{F'}^{-1}] = [\mathcal{O}_{W_{\mathcal{A}_{H'}}}]$. Similarly, we have

$$h^{\mathbf{m}} = h_F h^{\mathbf{m}'} = h_F [W_{\mathcal{A}_{L'}}] = c_1(\mathcal{L}_F)[W_{\mathcal{A}_{L'}}].$$

By the projection formula, this is equal to the pushforward of $c_1(\mathcal{L}_{F'})$ from $W_{\mathcal{A}_{L'}}$ to $W_\mathcal{A}$. If F' has rank 1, then $\mathcal{L}_{F'}$ is trivial, and we get zero. If F' has rank greater than 1, then we may choose H' as above, and we get $[W_{\mathcal{A}_{H'}}]$ by Lemma 3.1. $\qquad\square$

Remark 3.3. It is useful to think about the special case when $F = E$ in the proof of Lemma 3.2. In this case, $F' = E$, and the rank of F' is equal to the dimension of L'. This means that, if $h^{\mathbf{m}} = [W_{\mathcal{A}_{L'}}]$ for some L' of dimension greater than 1, then $h^{\mathbf{m}} h_E = [W_{\mathcal{A}_{H'}}]$ for some generic hyperplane $H' \subset L'$. Iterating this observation, we see that, if $k = \dim L - 1 - \sum m_F$, then

---

[2] This is where we use the fact that $\mathbb{F}$ is infinite.

$h^{\mathbf{m}} h_E^k \in A^{\dim L-1}(W_{\mathcal{A}})$ is the class of a point. Similarly, $(h^{\mathrm{aug}})^{\mathbf{m}} (h_E^{\mathrm{aug}})^{k+1} \in A^{\dim L}(W_{\mathcal{A}}^{\mathrm{aug}})$ is the class of a point.

Proposition 3.4. *We have*

$$(7) \qquad \chi(W_{\mathcal{A}}, \eta^{\mathbf{m}}) = \deg_{W_{\mathcal{A}}}\left(\frac{h^{\mathbf{m}}}{1 - h_E}\right) \quad and \quad \chi(W_{\mathcal{A}}^{\mathrm{aug}}, (\eta^{\mathrm{aug}})^{\mathbf{m}}) = \deg_{W_{\mathcal{A}}^{\mathrm{aug}}}\left(\frac{(h^{\mathrm{aug}})^{\mathbf{m}}}{1 - h_E^{\mathrm{aug}}}\right).$$

Proof. We will prove only the non-augmented case; the augmented case is identical. The Euler characteristic can be computed after extension of scalars, so we can assume that $\mathbb{F}$ is infinite. By Lemma 3.2, either $\eta^{\mathbf{m}} = 0$ and $h^{\mathbf{m}} = 0$, in which case Equation (7) holds trivially, or there exists some $L' \subset L$ such that $\eta^{\mathbf{m}} = [\mathcal{O}_{W_{\mathcal{A}_{L'}}}]$ and $h^{\mathbf{m}} = [W_{\mathcal{A}_{L'}}]$. Then $\chi(W_{\mathcal{A}}, \eta^{\mathbf{m}}) = 1$ because $W_{\mathcal{A}_{L'}}$ is a smooth iterated blow-up of projective space, and $\deg_{W_{\mathcal{A}}}\left(\frac{h^{\mathbf{m}}}{1-h_E}\right) = 1$ by Remark 3.3. $\square$

We are now ready to prove the existence of exceptional isomorphisms from K-rings to Chow rings for wonderful and augmented wonderful varieties.

Proof of Theorem 1.6. Again, we prove only the non-augmented case. Note that, once we prove the first sentence in the theorem, the second sentence will follow from Proposition 3.4.

For each nonempty flat F, we have an isomorphism

$$\zeta_F : K(\mathbb{P}(L^F)) \to A(\mathbb{P}(L^F))$$

sending the structure sheaf of a hyperplane (which we will denote by σ_F) to the Chow class of a hyperplane (which we will denote by s_F). This isomorphism has the property that, for any $\xi_F \in K(\mathbb{P}(L^F))$,

$$\chi(\mathbb{P}(L^F), \xi_F) = \deg_{\mathbb{P}(L^F)}\left(\frac{\zeta_F(\xi_F)}{1 - s_F}\right).$$

By the Künneth formula [AP15, Proposition 6.4], we have an isomorphism

$$\zeta = \otimes_F \zeta_F : K\left(\prod_F \mathbb{P}(L^F)\right) \to A\left(\prod_F \mathbb{P}(L^F)\right)$$

that takes σ_F to s_F for every F, and has the property that, for any $\xi \in K\left(\prod_F \mathbb{P}(L^F)\right)$,

$$(8) \qquad \chi\left(\prod_F \mathbb{P}(L^F), \xi\right) = \deg_{\prod_F \mathbb{P}(L^F)}\left(\frac{\zeta(\xi)}{\prod_F (1 - s_F)}\right).$$

By Theorem 1.8, the restriction map

$$A\left(\prod_F \mathbb{P}(L^F)\right) \to A(W_{\mathcal{A}}) \qquad \text{given by} \qquad s_F \mapsto h_F$$

is surjective. By Poincaré duality, its kernel is equal to the annihilator of $[W_{\mathcal{A}}]$. By Lemma 1.20 and Lemma 2.3, the analogous restriction map

$$K\left(\prod_F \mathbb{P}(L^F)\right) \to K(W_{\mathcal{A}}) \qquad \text{given by} \qquad \sigma_F \mapsto \eta_F$$

is also surjective. By the nondegeneracy of the Euler pairing [AP15, Proposition 6.3], the kernel is equal to the annihilator of $[\mathcal{O}_{W_{\mathcal{A}}}]$. We will prove that

$$(9) \qquad \zeta([\mathcal{O}_{W_{\mathcal{A}}}]) = \prod_{F \neq E} (1 - s_F) \cdot [W_{\mathcal{A}}].$$

Since s_F is nilpotent, the above product is a unit, which implies that ζ takes the annihilator of $[\mathcal{O}_{W_{\mathcal{A}}}]$ to the annihilator of $[W_{\mathcal{A}}]$. This in turn shows that ζ descends to an isomorphism $\zeta_{\mathcal{A}} : K(W_{\mathcal{A}}) \to A(W_{\mathcal{A}})$.

We now prove Equation (9). For any $\mathbf{m} = (m_F \mid F$ a nonempty flat), let

$$s^{\mathbf{m}} := \prod_F s_F^{m_F} \qquad \text{and} \qquad \sigma^{\mathbf{m}} := \prod_F \sigma_F^{m_F},$$

so that $s^{\mathbf{m}} \mapsto h^{\mathbf{m}}$ and $\sigma^{\mathbf{m}} \mapsto \eta^{\mathbf{m}}$ under restriction. We have

$$
\begin{aligned}
\deg_{\prod_F \mathbb{P}(L^F)} \left(s^{\mathbf{m}} \cdot \zeta([\mathcal{O}_{W_{\mathcal{A}}}]) \right) &= \chi \left(\prod_F \mathbb{P}(L^F), \sigma^{\mathbf{m}} \cdot \prod_F (1 - \sigma_F) \cdot [\mathcal{O}_{W_{\mathcal{A}}}] \right) && \text{by Equation (8)} \\
&= \chi \left(W_{\mathcal{A}}, \eta^{\mathbf{m}} \cdot \prod_F (1 - \eta_F) \right) \\
&= \deg_{W_{\mathcal{A}}} \left(h^{\mathbf{m}} \cdot \prod_{F \neq E} (1 - h_F) \right) && \text{by Proposition (3.4)} \\
&= \deg_{\prod_F \mathbb{P}(L^F)} \left(s^{\mathbf{m}} \cdot \prod_{F \neq E} (1 - s_F) \cdot [W_{\mathcal{A}}] \right).
\end{aligned}
$$

Equation (9) then follows from Poincaré duality. $\qquad \square$

4. The K-ring of $\overline{\mathcal{M}}_{0,n}$

We now apply our results to study the K-ring of $\overline{\mathcal{M}}_{0,n}$, and in particular prove the exceptional isomorphism of Theorem 1.10. From Section 1.4, we observe that the following diagram commutes:

$$
\begin{array}{ccc}
W_{\mathcal{B}_{n-1}} & \lhook\joinrel\longrightarrow & \prod_F \mathbb{P}(L^F) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle pr} \\
\overline{\mathcal{M}}_{0,n} & \lhook\joinrel\longrightarrow & \prod_{S \subset [n-1]} \mathbb{P}(L^{F_S}),
\end{array}
$$

where pr is the projection onto the factors indexed by flats of the form F_S, and p is the restriction of pr. By [FM05, Theorem 4.2], p is a composition of blow-ups at smooth centers. By [Ful98, Proposition 6.7(b)], the pullback maps

$$p^* : A(\overline{\mathcal{M}}_{0,n}) \to A(W_{\mathcal{B}_{n-1}}) \qquad \text{given by} \qquad c_S \mapsto h_{F_S}, \qquad \text{and}$$

$$p^* : K(\overline{\mathcal{M}}_{0,n}) \to K(W_{\mathcal{B}_{n-1}}) \qquad \text{give by} \qquad 1 - [\mathcal{L}_S^{-1}] \mapsto \eta_{F_S}$$

are both injective.

Consider the ring $R_n := \mathbb{Z}[u_S \mid S \subset [n-1], |S| \geq 2] \subset S_{\mathcal{B}_{n-1}}$, where the inclusion sends u_S to u_{F_S}. Then the map $S_{\mathcal{B}_{n-1}} \to A(W_{\mathcal{B}_{n-1}})$ of Theorem 1.8 restricts to a map $R_n \to A(\overline{\mathcal{M}}_{0,n})$. Consider the following ideals in R_n:

$$\mathcal{K}_1 := \langle (u_S - u_{S \cup T})(u_T - u_{S \cup T}) \mid S \cap T \neq \varnothing \rangle$$

$$\mathcal{K}_2 := \langle u_S \mid |S| = 2 \rangle.$$

Theorem 4.1. [Sin04] [EHKR10, Theorem 5.5] *The map* $R_n \to A(\overline{\mathcal{M}}_{0,n})$ *is surjective with kernel* $\mathcal{K}_1 + \mathcal{K}_2$.

The surjectivity statement in Theorem 4.1 allows us to prove Theorem 1.10.

Proof of Theorem 1.10. It is known to experts that $K(\overline{\mathcal{M}}_{0,n})$ is generated by line bundles; see the discussion in the introduction of [CT20]. This can also be proved in the same way as Lemma 1.20. Therefore, Theorem 4.1 and Lemma 2.3 imply that $K(\overline{\mathcal{M}}_{0,n})$ is generated by $1 - [\mathcal{L}_S^{-1}]$ for all subsets $S \subset [n-1]$ of cardinality at least 2. By Theorems 1.6 and 4.1, restriction of the isomorphism $\zeta_{\mathcal{B}_{n-1}} : K(W_{\mathcal{B}_{n-1}}) \to A(W_{\mathcal{B}_{n-1}})$ to

$$K(\overline{\mathcal{M}}_{0,n}) \subset K(W_{\mathcal{B}_{n-1}})$$

takes $K(\overline{\mathcal{M}}_{0,n})$ isomorphically to $A(\overline{\mathcal{M}}_{0,n}) \subset A(W_{\mathcal{B}_{n-1}})$. $\square$

Remark 4.2. In [CT21], Castravet and Tevelev studied $K(\overline{\mathcal{M}}_{0,n})$ as a representation of the symmetric group S_n; in particular, they show that the S_n-action on $K(\overline{\mathcal{M}}_{0,n})$ is a permutation representation over $\mathbb{Z}$. The action of S_n on $A(\overline{\mathcal{M}}_{0,n})$ has also been studied, beginning with [Get95]. Our isomorphism in Theorem 1.10 is not S_n-equivariant, but it is equivariant with respect to the action of the subgroup $S_{n-1} \subset S_n$ that fixes n. Note that $K(\overline{\mathcal{M}}_{0,n}) \otimes \mathbb{Q}$ and $A(\overline{\mathcal{M}}_{0,n}) \otimes \mathbb{Q}$ are S_n-equivariantly isomorphic via the Chern character, but one can check that $A(\overline{\mathcal{M}}_{0,5})$ is not a permutation representation of S_5 over $\mathbb{Z}$, and therefore cannot be S_5-equivariantly isomorphic to $K(\overline{\mathcal{M}}_{0,5})$.

Theorems 1.10 and 4.1 combine to give us the following corollary.

Corollary 4.3. *The homomorphism* $R_n \to K(\overline{\mathcal{M}}_{0,n})$ *taking* u_S *to* $1 - [\mathcal{L}_S^{-1}]$ *is surjective with kernel* $\mathcal{K}_1 + \mathcal{K}_2$.

Remark 4.4. The variety $\overline{\mathcal{M}}_{0,n}$ is a special case of a wonderful variety with a *building set*. The proof of Theorem 1.10 generalizes to any wonderful variety with a building set, as the surjectivity part of Theorem 4.1 can be proved for any wonderful variety with a building set along of the lines of [BES21, Section 3.2]. Therefore, the application in this section can be extended to the Hassett compactification of *heavy/light* weighted rational stable curves [Has03]. It is also a wonderful variety with a certain building set [CHMR16], and its Chow ring has been studied in [KKL21].

5. MATROID K-RINGS

In this section, we prove Proposition 1.13, Theorem 1.15, Theorem 1.16, Proposition 5.1, and Lemma 1.20. We also prove Proposition 5.3, which provides a geometric interpretation of the

Euler characteristic map on the K-ring and augmented K-ring of an arbitrary matroid. We begin by introducing the following pieces of notation

$$t_F := \Phi_{\mathrm{FY}}(x_F) \in A(M) \qquad t_F^{\mathrm{aug}} := \Phi_{\mathrm{FY}}^{\mathrm{aug}}(x_F) \in A^{\mathrm{aug}}(M)$$

$$\tau_F := \Psi_{\mathrm{FY}}(x_F) \in K(M) \qquad \tau_F^{\mathrm{aug}} := \Psi_{\mathrm{FY}}^{\mathrm{aug}}(x_F) \in K^{\mathrm{aug}}(M)$$

By Equation (4), we have

(10)
$$h_F = -\sum_{F \subset G} t_G \qquad \text{and} \qquad h_F^{\mathrm{aug}} = -\sum_{F \subset G} t_G^{\mathrm{aug}}.$$

Proof of Proposition 1.13. Since $A(M) = A(X_{\Sigma_M})$ is torsion-free, it is isomorphic to the associated graded of $K(M) = A(X_{\Sigma_M})$ with respect to the coniveau filtration [Ful93, Example 15.2.16]. Since we know that the associated graded map $A(M) \to A(W_\mathcal{A})$ is an isomorphism, the filtered map $K(M) \to K(W_\mathcal{A})$ is also an isomorphism [Wei94, Theorem 5.2.12]. The augmented case is identical. $\qquad\square$

Proof of Lemma 1.20. By Proposition 1.13, this is equivalent to the statement that the rings $K(M)$ and $K^{\mathrm{aug}}(M)$ are generated by line bundles on the toric varieties X_{Σ_M} and $X_{\Sigma_M^{\mathrm{aug}}}$. This is a general property of smooth toric varieties, see [AP15, Lemma 2.2]. $\qquad\square$

Proof of Theorem 1.15. The proof of statements (1) and (2) is essentially identical to the proof of Theorem 1.3, except that we use Theorem 1.12 in place of Theorem 1.1, and we use the fact that the K-ring of a smooth toric variety is generated by classes of line bundles (see [AP15, Lemma 2.2]) in place of Lemma 1.20. $\qquad\square$

Proof of Theorem 1.16. We prove only the augmented case; the non-augmented case is similar. Define a map $\kappa_M : S_M \to K^{\mathrm{aug}}(M)$ by putting

$$\kappa_M(u_F) = 1 - \prod_{F \subset G}(1 - \tau_G^{\mathrm{aug}})^{-1} = 1 - \prod_{F \subset G}(1 + \tau_G^{\mathrm{aug}} + (\tau_G^{\mathrm{aug}})^2 + \cdots).$$

Our first task will be to show that κ vanishes on $\mathcal{J}_1 + \mathcal{J}_2^{\mathrm{aug}}$, and therefore descends to a map from $A^{\mathrm{aug}}(M)$ to $K^{\mathrm{aug}}(M)$. We will make use of Theorem 1.15(2), which says that $\tau_F^{\mathrm{aug}} \tau_G^{\mathrm{aug}} = 0$ for any incomparable F and G,

$$\prod_F (1 - \tau_F^{\mathrm{aug}}) = 1, \qquad \text{and} \qquad \Psi_{FY}^{\mathrm{aug}}(y_e) = 1 - \prod_{e \notin G}(1 - \tau_G^{\mathrm{aug}}).$$

The later equation implies that $K^{\mathrm{aug}}(M)$ is generated by the τ_G^{aug}. Using the relations in $\mathcal{I}_4^{\mathrm{aug}}$, we get that

(11)
$$\tau_F^{\mathrm{aug}} \left(1 - \prod_{e \notin G}(1 - \tau_G^{\mathrm{aug}})\right) = 0$$

for any $e \notin F$.

We begin by checking that κ_M vanishes on a generator of $\mathcal{J}_1$. We have

$$\kappa_M\Big((u_F - u_{F\vee G})(u_G - u_{F\vee G})\Big)$$

$$= \left(\prod_{F\subset H}\left(1-\tau_H^{\mathrm{aug}}\right)^{-1} - \prod_{F\vee G\subset I}\left(1-\tau_I^{\mathrm{aug}}\right)^{-1}\right)\left(\prod_{G\subset J}\left(1-\tau_J^{\mathrm{aug}}\right)^{-1} - \prod_{F\vee G\subset I}\left(1-\tau_I^{\mathrm{aug}}\right)^{-1}\right)$$

$$= \prod_{F\vee G\subset I}\left(1-\tau_I^{\mathrm{aug}}\right)^{-2}\left(\prod_{F\subset H\subsetneq F\vee G}\left(1-\tau_H^{\mathrm{aug}}\right)^{-1} - 1\right)\left(\prod_{G\subset J\subsetneq F\vee G}\left(1-\tau_J^{\mathrm{aug}}\right)^{-1} - 1\right),$$

which vanishes because H and J are incomparable for any H appearing in the second product and J appearing in the third product.

Next, we check that κ_M vanishes on a generator of $\mathcal{J}_2^{\mathrm{aug}}$. Fix a flat F of rank 1 and an element $e \in F$, so that

$$\kappa_M(u_F) = 1 - \prod_{F\subset H}\left(1-\tau_H^{\mathrm{aug}}\right)^{-1} = 1 - \prod_{e\in H}\left(1-\tau_H^{\mathrm{aug}}\right)^{-1} = 1 - \prod_{e\notin K}\left(1-\tau_K^{\mathrm{aug}}\right).$$

Then

$$\kappa_M(u_F^2) = \left(1 - \prod_{e\in H}\left(1-\tau_H^{\mathrm{aug}}\right)^{-1}\right)\left(1 - \prod_{e\notin K}\left(1-\tau_K^{\mathrm{aug}}\right)\right),$$

which vanishes by Equation (11). Now for any flat G, we have

$$\kappa_M\Big(u_F(u_G - u_{F\vee G})\Big)$$

$$= \left(1 - \prod_{e\notin K}\left(1-\tau_K^{\mathrm{aug}}\right)\right)\prod_{F\vee G\subset I}\left(1-\tau_I^{\mathrm{aug}}\right)^{-1}\left(\prod_{G\subset J\subsetneq F\vee G}\left(1-\tau_J^{\mathrm{aug}}\right)^{-1} - 1\right),$$

which also vanishes by Equation (11) because $e \notin J$ for any J appearing in the last product.

We have now proved that κ_M descends to a homomorphism $\overline{\kappa}_M : A^{\mathrm{aug}}(M) \to K^{\mathrm{aug}}(M)$. Möbius inversion tells us that

$$1 - \tau_F^{\mathrm{aug}} = \prod_{F\subset G}\left(1 - \overline{\kappa}_M(h_G)\right)^{-\mu(F,G)},$$

where μ is the Möbius function on the lattice of flats of M. This, along with the surjectivity statement of Theorem 1.15(2), implies that $\overline{\kappa}_M$ is surjective. By Lemma 2.1 applied to the variety $X_{\Sigma_M^{\mathrm{aug}}}$, $A^{\mathrm{aug}}(M)$ and $K^{\mathrm{aug}}(M)$ are free abelian groups of the same rank, thus $\overline{\kappa}_M$ is an isomorphism. We may then take ζ_M^{aug} to be the inverse of $\overline{\kappa}_M$. $\qquad\square$

Proposition 5.1. *Let M be the matroid associated with a hyperplane arrangement $\mathcal{A}$. The following diagrams commute:*

$$\begin{array}{ccc} K(M) & \xrightarrow{\zeta_M} & A(M) \\ \downarrow & & \downarrow \\ K(W_{\mathcal{A}}) & \xrightarrow{\zeta_{\mathcal{A}}} & A(W_{\mathcal{A}}) \end{array} \qquad\qquad \begin{array}{ccc} K^{\mathrm{aug}}(M) & \xrightarrow{\zeta_M^{\mathrm{aug}}} & A^{\mathrm{aug}}(M) \\ \downarrow & & \downarrow \\ K(W_{\mathcal{A}}^{\mathrm{aug}}) & \xrightarrow{\zeta_{\mathcal{A}}^{\mathrm{aug}}} & A(W_{\mathcal{A}}^{\mathrm{aug}}). \end{array}$$

Proof. We do the non-augmented case. Since $\zeta_{\mathcal{A}}$ takes η_F to h_F and ζ_M is characterized in the statement of Theorem 1.16, we need to show that

$$\eta_F = 1 - \prod_{F \subset G} (1 - \tau_G)^{-1}.$$

By considering first Chern classes, we have

$$\mathcal{L}_F \cong \mathcal{L}_E \otimes \bigotimes_{F \subset G \subsetneq E} \mathcal{O}(-D_G)$$

of line bundles on the wonderful variety $W_{\mathcal{A}}$. Since $\tau_E = 1 - [\mathcal{L}_E]$, $\tau_G = 1 - [\mathcal{O}(-D_G)]$ for $G \neq E$, and $\eta_F = 1 - [\mathcal{L}_F^{-1}]$, the result follows. $\square$

We now discuss an analogue of the projection formula that will allow us to interpret the Euler characteristic maps on matroid K-rings using geometry. Let $\mathcal{U}_E$ denote the Boolean arrangement on the ground set E, consisting of the coordinate hyperplanes in $\mathbb{F}^E$, realizing the Boolean matroid U_E. The wonderful variety of $\mathcal{U}_E$ is a toric variety, called the **permutohedral variety**, which we denote X_E. The augmented wonderful variety is also a toric variety, called the **stellahedral variety**, which we denote X_E^{aug}. For any matroid M, there is an open inclusion $\iota \colon X_{\Sigma_M} \hookrightarrow X_E$, so we have restriction maps $\iota^* \colon K(X_E) \to K(M)$ and $\iota^* \colon A(X_E) \to A(M)$. Similarly, there is an open inclusion $\iota^{\mathrm{aug}} \colon X_{\Sigma_M^{\mathrm{aug}}} \to X_E^{\mathrm{aug}}$ and corresponding restriction maps on K-rings and Chow rings. These maps are characterized by the property that

$$\iota^* t_S = \begin{cases} t_S, & S \text{ is a flat of } M \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \iota^* \tau_S = \begin{cases} \tau_S, & S \text{ is a flat of } M \\ 0, & \text{otherwise,} \end{cases}$$

and similarly in the augmented setting.

Lemma 5.2. *For any subset $S \subset E$, let $\bar{S}$ be the closure of S in M. We have*

$$\iota^*(\eta_S) = \eta_{\bar{S}} \in K(M) \qquad (\iota^{\mathrm{aug}})^*(\eta_S^{\mathrm{aug}}) = \eta_{\bar{S}}^{\mathrm{aug}} \in K^{\mathrm{aug}}(M)$$

$$\iota^*(h_S) = h_{\bar{S}} \in A(M) \qquad (\iota^{\mathrm{aug}})^*(h_S^{\mathrm{aug}}) = h_{\bar{S}}^{\mathrm{aug}} \in A^{\mathrm{aug}}(M).$$

Proof. We have that

$$\iota^* h_S = -\iota^* \sum_{S \subset S'} t_{S'} = - \sum_{\substack{S \subset F \\ F \text{ a flat}}} t_F = h_{\bar{S}}.$$

The other statements are similar. $\square$

There exist unique Chow classes

$$\Delta_M \in A(U_E) \qquad \text{and} \qquad \Delta_M^{\mathrm{aug}} \in A^{\mathrm{aug}}(U_E),$$

called the **Bergman class** and **augmented Bergman class**, respectively, with the properties that, for any $\xi \in A(U_E)$ and $\xi^{\mathrm{aug}} \in A^{\mathrm{aug}}(U_E)$, we have

$$\deg_M(\iota^* \xi) = \deg_{U_E}(\xi \cdot \Delta_M) \qquad \text{and} \qquad \deg_M((\iota^{\mathrm{aug}})^* \xi^{\mathrm{aug}}) = \deg_{U_E}(\xi^{\mathrm{aug}} \cdot \Delta_M^{\mathrm{aug}}).$$

In the non-augmented setting, this is proved in [BES21, Theorem 4.2.1]. The same argument works in the augmented setting, using [EHL22, Theorem 1.11].

Proposition 5.3. *The following diagrams commute:*

$$
\begin{array}{ccc}
K(U_E) & \xrightarrow{\zeta_{U_E}} & A(U_E) \\
\downarrow{\iota^*} & & \downarrow{\iota^*} \\
K(M) & \xrightarrow{\zeta_M} & A(M)
\end{array}
\qquad\qquad
\begin{array}{ccc}
K^{\mathrm{aug}}(U_E) & \xrightarrow{\zeta_{U_E}^{\mathrm{aug}}} & A^{\mathrm{aug}}(U_E) \\
\downarrow{(\iota^{\mathrm{aug}})^*} & & \downarrow{(\iota^{\mathrm{aug}})^*} \\
K^{\mathrm{aug}}(M) & \xrightarrow{\zeta_M^{\mathrm{aug}}} & A^{\mathrm{aug}}(M).
\end{array}
$$

Additionally, for any $\xi \in K(U_E)$ and $\xi^{\mathrm{aug}} \in K^{\mathrm{aug}}(U_E)$, we have

$$
\chi\big(M, \iota^*(\xi)\big) = \chi\left(U_E, \xi \cdot \zeta_{U_E}^{-1}(\Delta_M)\right), \qquad and
$$

$$
\chi^{\mathrm{aug}}\big(M, (\iota^{\mathrm{aug}})^*(\xi^{\mathrm{aug}})\big) = \chi\left(U_E, \xi^{\mathrm{aug}} \cdot (\zeta_{U_E}^{\mathrm{aug}})^{-1}(\Delta_M^{\mathrm{aug}})\right).
$$

Proof of Proposition 5.3. Commutativity of the diagrams follows from Lemma 5.2. For the second statement, we do the non-augmented case. For any $\xi \in K(U_E)$, we have

$$
\begin{aligned}
\chi\big(M, \iota^*(\xi)\big) &= \deg_M\left(\frac{\zeta_M \circ \iota^*(\xi)}{1 - h_E}\right) \\
&= \deg_M\left(\frac{\iota^* \circ \zeta_{U_E}(\xi)}{1 - h_E}\right) \\
&= \deg_{U_E}\left(\frac{\zeta_{U_E}(\xi) \cdot \Delta_M}{1 - h_E}\right) \\
&= \chi\left(U_E, \xi \cdot \zeta_{U_E}^{-1}(\Delta_M)\right).
\end{aligned}
$$

The augmented case is similar. $\qquad\square$

As a corollary, we are to prove the following characterizing properties of ζ_{U_E} and $\zeta_{U_E}^{\mathrm{aug}}$, which demonstrate that they agree with the maps denoted ζ in [BEST22, Corollary 10.6] and [EHL22, Theorem 8b].

Corollary 5.4. *Let $\mathcal{A}$ be a hyperplane arrangement equipped with a choice of linear function cutting out each hyperplane, which induces embeddings $W_{\mathcal{A}} \subset X_E$ and $W_{\mathcal{A}}^{\mathrm{aug}} \subset X_E^{\mathrm{aug}}$. Then*

$$
\zeta_{U_E}\big([\mathcal{O}_{W_{\mathcal{A}}}]\big) = [W_{\mathcal{A}}] \qquad and \qquad \zeta_{U_E}^{\mathrm{aug}}\big([\mathcal{O}_{W_{\mathcal{A}}^{\mathrm{aug}}}]\big) = [W_{\mathcal{A}}^{\mathrm{aug}}].
$$

Proof. Let M be the matroid associated with $\mathcal{A}$. For any $\xi \in K(X_E) = K(U_E)$, we have

$$
\chi\left(X_E, \xi \cdot \zeta_{U_E}^{-1}(\Delta_M)\right) = \chi\left(U_E, \xi \cdot \zeta_{U_E}^{-1}(\Delta_M)\right) = \chi(M, \iota^*(\xi)) = \chi(W_{\mathcal{A}}, \iota^*(\xi)) = \chi(X_E, \xi \cdot [\mathcal{O}_{W_{\mathcal{A}}}]),
$$

where the second equality comes from Proposition 5.3 and the fourth comes from the projection formula on the permutohedral variety. The nondegeneracy of the pairing $(x, y) \mapsto \chi(X_E, xy)$ [AP15, Theorem 1.3] implies that $\zeta_{U_E}^{-1}(\Delta_M) = [\mathcal{O}_{W_{\mathcal{A}}}]$. Applying ζ_{U_E}, we find that

$$
[W_{\mathcal{A}}] = \Delta_M = \zeta_{U_E}\big([\mathcal{O}_{W_{\mathcal{A}}}]\big).
$$

The augmented case is similar. $\qquad\square$

Motivated by Corollary 5.4, we define

$$[\mathcal{O}_M] := \zeta_{U_E}^{-1}(\Delta_M) \in K(U_E) \qquad \text{and} \qquad [\mathcal{O}_M^{\mathrm{aug}}] := (\zeta_{U_E}^{\mathrm{aug}})^{-1}(\Delta_M^{\mathrm{aug}}) \in K^{\mathrm{aug}}(U_E)$$

for any loopless matroid M. Proposition 5.3 may then be interpreted as a purely combinatorial projection formula.

We conclude this section with the following lemma, which we will need in Section 8.

Lemma 5.5. *The exceptional isomorphism* $\zeta_M : K(M) \to A(M)$ *has the following properties:*

- $\zeta_M(\tau_\varnothing) = t_\varnothing$
- $\zeta_M(\tau_E) = \frac{t_E}{1+t_E}$
- *For any flat* F, $\zeta_M(\tau_F) = t_F +$ *terms of degree at least 2.*

Proof. By Proposition 5.3, it is sufficient to prove these statements when $M = U_E$ is the Boolean matroid. The first two statements for the Boolean matroid are proved in the course of the proof of [BEST22, Theorem 10.11].

For the last statement, let F be a nonempty flat. After restricting from $X_{\Sigma_M^{\mathrm{aug}}}$ to X_{Σ_M}, we have that

$$\tau_F = - \sum_{F \subset G} \mu(F, G)\eta_G + \text{terms of degree at least 2 in } \{\eta_G \mid F \subset G\}.$$

Applying ζ_M, we find that

$$\begin{aligned} \zeta_M(\tau_F) &= - \sum_{F \subset G} \mu(F, G)h_G + \text{terms of degree at least 2 in } \{h_G \mid F \subset G\} \\ &= t_F + \text{terms of degree at least 2,} \end{aligned}$$

which completes the proof. $\qquad\qquad\square$

6. Adams operations and Serre duality

We now discuss some additional structures on matroid K-rings. The fact that $K(M)$ and $K^{\mathrm{aug}}(M)$ are (by definition) K-rings of varieties endows them with the structure of **augmented λ-rings** [BGI71, Exposé V, Exemple 3.9.1]. This means that we have a rank function ϵ that takes values in $\mathbb{Z}$, and for each natural number k, we have operations λ^k and Ψ^k (the latter called **Adams operations**) characterized by the property that

$$\lambda^k([\mathcal{E}]) = [\wedge^k \mathcal{E}] \qquad \text{and} \qquad \Psi^k([\mathcal{L}]) = [\otimes^k \mathcal{L}]$$

for any vector bundle $\mathcal{E}$ and line bundle $\mathcal{L}$. Since our simplicial generators η_F are all of the form $1 - [\mathcal{L}]$ for some line bundle $\mathcal{L}$, we have $\epsilon(\eta_F) = 0$, and

$$\Psi^k(\eta_F) = \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} \eta_F^i.$$

The formula for augmented simplicial generators is identical. Note that the Adams operations are ring homomorphisms, which is not at all combinatorially obvious from the above formula. The Adams operations become simultaneously diagonalizable after tensoring with $\mathbb{Q}$, and their

eigenspaces (which are independent of $k > 1$) map isomorphically to the graded pieces of the Chow ring via the Chern character. We also have a **duality automorphism** D, characterized by the property that $D([\mathcal{E}]) = [\mathcal{E}^\vee]$ for any vector bundle $\mathcal{E}$. In terms of the simplicial generators, this takes the form

$$D(\eta_F) = \frac{-\eta_F}{1 - \eta_F} = -\eta_F - \eta_F^2 - \cdots,$$

and similarly in the augmented setting. We note that the operations ϵ, λ^k, Ψ^k, and D all commute with the maps ι^* and $(\iota^{\mathrm{aug}})^*$ introduced in Section 5 because they are compatible with functorial maps between K-rings of varieties.

On a d-dimensional smooth projective variety X with canonical bundle ω_X, Serre duality implies that, for any vector bundle $\mathcal{E}$,

$$\chi(X, \mathcal{E}) = (-1)^d \chi(X, \omega_X \otimes \mathcal{E}^\vee).$$

In particular, this holds for $X = W_{\mathcal{A}}$ or $W_{\mathcal{A}}^{\mathrm{aug}}$. We will show that a similar formula holds on $K(M)$ for any matroid M, even if M is not realizable. Our first task is to define classes $\omega_M \in K(M)$ and $\omega_M^{\mathrm{aug}} \in K^{\mathrm{aug}}(M)$ that will play the roles of the canonical bundles.

For any matroid on the ground set E, the **matroid polytope** $P(M) \subset \mathbb{R}^E$ is defined to be the convex hull of the indicator functions of bases of M, and the **independence polytope** $IP(M) \subset \mathbb{R}^E$ is defined to be the convex hull of indicator functions of independent subsets of M.

Recall that, on a smooth projective toric variety with fan Σ, there is a correspondence between torus equivariant nef line bundles and lattice polytopes whose normal fans coarsen Σ. For any matroid on the ground set E, $P(M)$ coarsens Σ_{U_E} and $IP(M)$ coarsens $\Sigma_{U_E}^{\mathrm{aug}}$, so we obtain line bundles $[P(M)]$ and $[IP(M)]$ on X_E and X_E^{aug}, respectively.

Let $M^\perp$ be the matroid dual to M, characterized by the property that the bases of $M^\perp$ are the complements of the bases of M. We define the classes

$$\omega_M := \iota^* \left(\omega_{X_E} \cdot [P(M^\perp)] \right) \in K(M)$$

and

$$\omega_M^{\mathrm{aug}} := (\iota^{\mathrm{aug}})^* \left(\omega_{X_E}^{\mathrm{aug}} \cdot [IP(M^\perp)] \right) \in K^{\mathrm{aug}}(M).$$

These definitions are motivated by the following proposition.

Proposition 6.1. *Let $\mathcal{A}$ be a hyperplane arrangement with associated matroid M. The isomorphisms $K(M) \cong K(W_{\mathcal{A}})$ and $K^{\mathrm{aug}}(M) \cong K(W_{\mathcal{A}}^{\mathrm{aug}})$ take ω_M and ω_M^{aug} to the canonical bundles of $W_{\mathcal{A}}$ and $W_{\mathcal{A}}^{\mathrm{aug}}$, respectively.*

Proof. The adjunction formula states that the canonical class of $W_{\mathcal{A}}^{\mathrm{aug}}$ is equal to the restriction of the canonical class of X_E^{aug} tensored with the determinant of the normal bundle to $W_{\mathcal{A}}^{\mathrm{aug}}$ inside of X_E^{aug}. The determinant of the normal bundle is equal to the restriction of $[IP(M^\perp)]$ by [EHL22, Proposition 4.6 and Corollary 5.4]. The non-augmented case can be prove similarly, using [BEST22, Theorem 7.10]. $\qquad\square$

The following theorem is a combinatorial version of Serre duality for K-rings of matroids.

Theorem 6.2. *For any $\xi \in K(M)$ and $\xi^{\mathrm{aug}} \in K^{\mathrm{aug}}(M)$, we have*

$$\chi(M, \xi) = (-1)^{\mathrm{rk}\, M - 1} \chi(M, \omega_M \cdot D(\xi)) \qquad \text{and} \qquad \chi^{\mathrm{aug}}(M, \xi^{\mathrm{aug}}) = (-1)^{\mathrm{rk}\, M} \chi^{\mathrm{aug}}(M, D(\xi^{\mathrm{aug}})).$$

In the realizable case, Theorem 6.2 follows from Proposition 6.1 and Serre duality for the varieties $W_{\mathcal{A}}$ and $W_{\mathcal{A}}^{\mathrm{aug}}$. Our strategy is to deduce the general case using the concept of valuativity. Given an abelian group A, a function $f\colon \{\text{matroids on } E\} \to A$ is called **valuative** if it factors through the map that takes M to the indicator function of its matroid polytope $P(M) \subset \mathbb{R}^E$. That is, for any matroids $M_1, \ldots, M_k$ and integers $a_1, \ldots, a_k$ such that $\sum a_i 1_{P(M_i)} = 0$, we require that $\sum a_i f(M_i) = 0$.

Remark 6.3. The notion of valuativity is typically defined on functions that take values on the set of all matroids on E, whereas we are only considering loopless matroids in this paper. This is not an important distinction, as the valuative group of loopless matroids on E is a direct summand of the valuative group of all matroids on E. That is, a function on the set of loopless matroids on E is valuative if and only if it extends to a valuative function on the set of all matroids on E, if and only if it extends by zero to a valuative function on the set of all matroids on E.

Lemma 6.4. *Fix a pair of classes $\xi \in K(U_E)$ and $\xi^{\mathrm{aug}} \in K^{\mathrm{aug}}(U_E)$. The following four $\mathbb{Z}$-valued functions are valuative:*

$$M \mapsto \chi(M, \iota^* \xi)$$
$$M \mapsto \chi^{\mathrm{aug}}(M, (\iota^{\mathrm{aug}})^* \xi^{\mathrm{aug}})$$
$$M \mapsto (-1)^{\mathrm{rk}\, M - 1} \chi\big(M, \omega_M \cdot D(\iota^* \xi)\big)$$
$$M \mapsto (-1)^{\mathrm{rk}\, M} \chi^{\mathrm{aug}}\big(M, \omega_M^{\mathrm{aug}} \cdot D((\iota^{\mathrm{aug}})^* \xi^{\mathrm{aug}})\big).$$

Proof. We begin with the first function. By Proposition 5.3, $\chi(M, \iota^* \xi) = \chi(X_E, [\mathcal{O}_M] \cdot \xi)$. Recall that $[\mathcal{O}_M] = \zeta_{U_E}^{-1}(\Delta_M)$, and the function $M \mapsto \Delta_M$ is valuative by [BEST22, Corollary 7.11]. Therefore the function $M \mapsto [\mathcal{O}_M]$ is valuative, and the result follows by linearity of the Euler characteristic map on $K(X_E)$. The proof of valuativity of the second function is similar, except that we now use [EHL22, Proposition 4.7] to establish the valuativity of the function $M \mapsto [\mathcal{O}_M^{\mathrm{aug}}]$.

After applying $\zeta_{U_E}^{\mathrm{aug}}$, it follows from [EHL22, Proposition 4.7 and Corollary 6.5] that the function $M \mapsto [\mathcal{O}_M^{\mathrm{aug}}] \cdot [IP(M^\perp)]$ is valuative. This implies the valuativity of the fourth function. One can argue similarly for the third function. $\qquad \square$

Proof of Theorem 6.2. We have already used Proposition 6.1 to prove the theorem for realizable matroids. We have shown that each side of both claimed equalities is valuative. The full theorem now follows from a result of Derksen and Fink [DF10] (see also [BEST22, Lemma 5.9]), which says that the matroid polytope of any matroid can be expressed as a linear combination of indicator functions of matroid polytopes of realizable matroids. $\qquad \square$

Question 6.5. Can Theorem 6.2 be applied to give interesting identities for matroids? For example, let $f(\ell) := \chi(M, \omega_M^\ell)$; then Theorem 6.2 implies that $f(\ell) = (-1)^{r-1} f(1 - \ell)$. Does this

statement admit an elementary proof, without using valuativity and Serre duality for wonderful varieties?

7. EULER CHARACTERISTICS AND SIMPLICIAL SNAPPER POLYNOMIALS

In this section, we give purely combinatorial formulas for the Euler characteristic of a monomial in the simplicial generators on $K(M)$ or $K^{\mathrm{aug}}(M)$. Given a pair of tuples

$$\mathbf{m} = (m_F \mid F \text{ a nonempty flat}) \qquad \text{and} \qquad \mathbf{m}' = (m'_F \mid F \text{ a nonempty flat}),$$

we say that $\mathbf{m}' \leq \mathbf{m}$ if $m'_F \leq m_F$ for all F. We say that $\mathbf{m}$ satisfies the **Hall–Rado condition** if, for all $\mathbf{m}' \leq \mathbf{m}$, we have

$$\mathrm{rank} \bigcup_{m'_F > 0} F \geq \sum_F m'_F.$$

If the inequality is always strict, we say that $\mathbf{m}$ satisfies the **dragon Hall–Rado condition**. The following theorem computes the degrees of the monomials $h^{\mathbf{m}}$ and $(h^{\mathrm{aug}})^{\mathbf{m}}$.

Theorem 7.1. *If $\sum m_F = \mathrm{rk}\, M - 1$, then* [BES21, Theorem 5.2.4]

$$\deg_M(h^{\boldsymbol{m}}) = \begin{cases} 1 & \text{if } \boldsymbol{m} \text{ satisfies the dragon Hall–Rado condition} \\ 0 & \text{otherwise.} \end{cases}$$

If $\sum m_F = \mathrm{rk}\, M$, then [EL, Theorem 1.3]

$$\deg_M^{\mathrm{aug}}((h^{\mathrm{aug}})^{\boldsymbol{m}}) = \begin{cases} 1 & \text{if } \boldsymbol{m} \text{ satisfies the Hall–Rado condition} \\ 0 & \text{otherwise.} \end{cases}$$

The K-theoretic analogue of Theorem 7.1 is almost exactly the same, except without the condition on $\sum m_F$.

Theorem 7.2. *For any m, we have*

$$\chi(M, \eta^{\boldsymbol{m}}) = \begin{cases} 1 & \text{if } \boldsymbol{m} \text{ satisfies the dragon Hall–Rado condition} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\chi^{\mathrm{aug}}(M, (\eta^{\mathrm{aug}})^{\boldsymbol{m}}) = \begin{cases} 1 & \text{if } \boldsymbol{m} \text{ satisfies the Hall–Rado condition} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We prove the first statement; the augmented case is similar. If $\sum m_F \geq \mathrm{rk}\, M$, then $\eta^{\mathbf{m}} = 0$ and $\mathbf{m}$ fails to satisfy the dragon Hall–Rado condition. If $\sum m_F < \mathrm{rk}\, M$, we let

$$k = \mathrm{rk}\, M - 1 - \sum m_F \geq 0,$$

and define $\tilde{\mathbf{m}}$ by putting $\tilde{m}_E = m_E + k$ and $\tilde{m}_F = m_F$ for all $F \neq E$. Then

$$\chi(M, \eta^{\mathbf{m}}) = \deg_M \left(\frac{\zeta_M(\eta^{\mathbf{m}})}{1 - h_E} \right) = \deg_M \left(\frac{h^{\mathbf{m}}}{1 - h_E} \right) = \deg_M \left(h^{\tilde{\mathbf{m}}} \right).$$

We observe that $\mathbf{m}$ satisfies dragon Hall–Rado if and only if $\tilde{\mathbf{m}}$ does, thus our formula follows from Theorem 7.1. $\qquad\square$

Given a proper variety X and a finite tuple of line bundles $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_k)$ on X, the function from $\mathbb{Z}^k$ to $\mathbb{Z}$ given by the formula

$$\mathrm{Snap}_{\mathcal{L}}(\mathbf{a}) := \chi\left(X, \mathcal{L}_1^{a_1} \otimes \cdots \otimes \mathcal{L}_k^{a_k}\right)$$

is a polynomial of degree at most $\dim X$ [Sna59, Theorem 9.1], which we call the **Snapper polynomial**. Given an integer x and a natural number d, let

$$x^{(d)} = \frac{x(x+1)\cdots(x+d-1)}{d!} = \binom{d+x-1}{d} = (-1)^d \binom{-x}{d}.$$

If $\mathbf{a} \in \mathbb{Z}^k$ and $\mathbf{d} \in \mathbb{N}^k$, let

$$\mathbf{a}^{(\mathbf{d})} := a_1^{(d_1)} \cdots a_k^{(d_k)}.$$

Lemma 7.3. *If $\sigma_i = 1 - [\mathcal{L}_i^{-1}]$ for each $i \in \{1, \ldots, k\}$, then*

$$\mathrm{Snap}_{\mathcal{L}}(\boldsymbol{a}) = \sum_{\boldsymbol{d} \in \mathbb{N}^k} \chi\left(X, \sigma_1^{d_1} \cdots \sigma_k^{d_k}\right) \boldsymbol{a}^{(\boldsymbol{d})}.$$

Proof. We have

$$\mathcal{L}_i^{a_i} = \frac{1}{(1 - \sigma_i)^{a_i}} = \sum_{d_i=0}^{\infty} a_i^{(d_i)} \sigma_i^{d_i}.$$

The lemma follows. $\qquad\square$

Let M be a matroid, and consider a tuple $\mathbf{a} = (a_F \mid F \text{ a nonempty flat})$. Let

$$\mathrm{Snap}_M(\mathbf{a}) := \sum_m \chi(M, \eta^{\mathbf{m}})\mathbf{a}^{(\mathbf{m})} \qquad \text{and} \qquad \mathrm{Snap}_M^{\mathrm{aug}}(\mathbf{a}) := \sum_m \chi^{\mathrm{aug}}(M, (\eta^{\mathrm{aug}})^{\mathbf{m}})\mathbf{a}^{(\mathbf{m})}.$$

This definition is motivated by the following observation.

Lemma 7.4. *Let $\mathcal{A}$ be a hyperplane arrangement with associated matroid M. Then Snap_M and $\mathrm{Snap}_M^{\mathrm{aug}}$ coincide with the geometrically defined Snapper polynomials for the varieties $W_{\mathcal{A}}$ and $W_{\mathcal{A}}^{\mathrm{aug}}$ with respect to the line bundles $(\mathcal{L}_F)$ and $(\mathcal{L}_F^{\mathrm{aug}})$.*

Proof. By definition, $\eta_F = 1 - [\mathcal{L}_F^{-1}]$. The fact that Snap_M coincides with the Snapper polynomial for $W_{\mathcal{A}}$ with respect to $(\mathcal{L}_F)$ then follows from Lemma 7.3. The augmented case is identical. $\qquad\square$

We can now rephrase Theorem 7.2 as a statement about Snapper polynomials.

Corollary 7.5. *We have*

$$\mathrm{Snap}_M(\boldsymbol{a}) = \sum_{\substack{\boldsymbol{m} \text{ satisfies} \\ \text{dragon Hall–Rado}}} \boldsymbol{a}^{(\boldsymbol{m})} \qquad \text{and} \qquad \mathrm{Snap}_M^{\mathrm{aug}}(\boldsymbol{a}) = \sum_{\substack{\boldsymbol{m} \text{ satisfies} \\ \text{Hall–Rado}}} \boldsymbol{a}^{(\boldsymbol{m})}.$$

Remark 7.6. In the special case where $M = U_E$ is Boolean, the first equality in Corollary 7.5 appears in [Pos09, Theorem 11.3].

8. Snapper polynomials in the Feichtner–Yuzvinsky generators

Section 7 was about the Snapper polynomial of the wonderful variety of an arrangement with respect to the line bundles $\{\mathcal{L}_F \mid F \text{ a nonempty flat}\}$, or of an arbitrary matroid with respect to the corresponding K-classes. This was fundamentally a "simplicial" construction, since the simplicial generators for K-theory were defined in terms of these line bundles. In this section, we consider the Snapper polynomial with respect to a collection of line bundles related to the Feichtner–Yuzvinsky generators. For simplicity, we work only in the non-augmented setting.

Recall that, for all flats F, we have defined

$$t_F := \Phi_{\mathrm{FY}}(x_F) \in A(M) \qquad \text{and} \qquad \tau_F := \Psi_{\mathrm{FY}}(x_F) \in K(M).$$

For any tuple of natural numbers $\mathbf{m} = (m_F \mid F \text{ a flat})$, let

$$\tau^{\mathbf{m}} := \prod_F \tau_F^{m_F}.$$

For any tuple of integers $\mathbf{a} = (a_F \mid F \text{ a flat})$, we define

$$\mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a}) := \sum_{\mathbf{m}} \chi(M, \tau^{\mathbf{m}}) \mathbf{a}^{(\mathbf{m})}.$$

If M is the matroid associated with a hyperplane arrangement $\mathcal{A}$, let $D_\emptyset$ and D_E be the divisor classes on $W_{\mathcal{A}}$ with first Chern classes $t_\emptyset$ and t_E respectively, so that we have $\tau_F = 1 - [\mathcal{O}(-D_F)]$ for all flats F. Lemma 7.3 immediately implies that $\mathrm{Snap}_M^{\mathrm{FY}}$ is the Snapper polynomial of $W_{\mathcal{A}}$ with respect to the tuple of line bundles $(\mathcal{O}(D_F) \mid F \text{ a flat})$.

We now provide an explicit formula for the polynomial $\mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a})$. For any natural number k, let Flag_k be the collection of flags of flats of the form

$$\mathcal{F} = \{\varnothing = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = E\}.$$

If F and G are incomparable flats, then $\tau_F \tau_G = 0 \in K(M)$, therefore $\mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a})$ does not contain any monomials that are multiples of $a_F a_G$. This implies that we may write

$$\mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a}) = \sum_{k \geq 1} \sum_{\mathcal{F} \in \mathrm{Flag}_k} \sum_{\mathbf{m}} c(\mathcal{F}, \mathbf{m})\, a_{F_0}^{(m_0)} a_{F_1}^{(m_1+1)} a_{F_2}^{(m_2+1)} \cdots a_{F_{k-1}}^{(m_{k-1}+1)} a_{F_k}^{(m_k)},$$

where $\mathbf{m}$ ranges over all tuples $(m_0, \ldots, m_k)$ of natural numbers. It remains only to compute the constants $c(\mathcal{F}, \mathbf{m})$.

Remark 8.1. It may be slightly confusing that the first and last exponents are (m_0) and (m_k), whereas the middle exponents are $(m_i + 1)$ for $1 \leq i < k$. The point is that a particular "monomial" appears in the summand indexed by $\mathcal{F}$ if and only if the nonempty proper flats in its support are precisely $\{F_1, \ldots, F_{k-1}\}$. Also, this convention leads to a tidier formula for $c(\mathcal{F}, \mathbf{m})$.

Given $\mathcal{F} \in \mathrm{Flag}_k$ and $i \in \{1, \ldots, k\}$, consider the matroid $M_{F_{i-1}}^{F_i}$ obtained by contracting F_{i-1} and deleting the complement of F_i, whose poset of flats is canonically isomorphic to the interval $[F_{i-1}, F_i]$ in the poset of flats of M. Let $d_i = \mathrm{rk}\, M_{F_{i-1}}^{F_i} - 1$. Let $\mu^j(M)$ denote the absolute value of the coefficient of $t^{\mathrm{rk}\, M - 1 - j}$ in the reduced characteristic polynomial of M.

Theorem 8.2. *For any $k \geq 1$, $\mathcal{F} \in \mathrm{Flag}_k$, and $\boldsymbol{m} = (m_0, \ldots, m_k)$, we have*

$$c(\mathcal{F}, \boldsymbol{m}) = \sum_{\substack{\boldsymbol{e}, \boldsymbol{f} \in \mathbb{N}^k \\ e_1 = m_0}} (-1)^{|\boldsymbol{e}| + |\boldsymbol{f}|} \prod_{i=1}^{k} \binom{d_i - e_i}{f_i} \mu^{e_i}(M_{F_{i-1}}^{F_i}) \binom{m_i}{e_{i+1} + f_i - m_i, m_i - f_i, m_i - e_{i+1}},$$

where we adopt the convention that $e_{k+1} = 0$.

Corollary 8.3. *Fix an element $\mathcal{F} \in \mathrm{Flag}_k$ and a tuple of natural numbers $\boldsymbol{m} = (m_0, \ldots, m_k)$ with $\sum m_i = \mathrm{rk}\, M - k$. Then the degree*

$$\deg_M \left(t_{F_0}^{m_0} t_{F_1}^{m_1 + 1} \cdots t_{F_{k-1}}^{m_{k-1} + 1} t_{F_k}^{m_k} \right)$$

is equal to

$$(-1)^{\mathrm{rk}\, M - k} \sum_{\substack{\boldsymbol{e} \in \mathbb{N}^k \\ e_1 = m_0}} \prod_{i=1}^{k} \mu^{e_i}(M_{F_{i-1}}^{F_i}) \binom{m_i}{e_{i+1} - e_i + d_i - m_i, m_i - d_i + e_i, m_i - e_{i+1}},$$

again with the convention that $e_{k+1} = 0$.

Proof. By the definition of χ and the third item of Lemma 5.5, we have

$$\deg_M \left(t_{F_0}^{m_0} t_{F_1}^{m_1 + 1} \cdots t_{F_{k-1}}^{m_{k-1} + 1} t_E^{m_k} \right) = \chi \left(M, \tau_\varnothing^{m_0} \tau_{F_1}^{m_1 + 1} \cdots \tau_{F_{k-1}}^{m_{k-1} + 1} \tau_{F_k}^{m_k} \right) = c(\mathcal{F}, \mathbf{m}).$$

In the formula for $c(\mathcal{F}, \mathbf{m})$ in Theorem 8.2, we see that a particular term vanishes unless $e_i + f_i \leq d_i$ for all $i \in \{1, \ldots, k\}$. Taking the sum over all i, this implies that

$$|e| + |f| \leq |d| = \mathrm{rk}\, M - k = m_0 + \cdots + m_k.$$

On the other hand, a term also vanishes unless $m_i \leq e_{i+1} + f_i$ for all $i \in \{1, \ldots, k\}$. Taking the sum over all i and including the equality $e_0 = m_0$, this implies that

$$m_0 + \cdots + m_k \leq |e| + |f|.$$

Thus the aforementioned inequalities are all equalities, and we can simplify our formula for $c(\mathcal{F}, \mathbf{m})$ by setting $f_i = d_i - e_i$ for all i. The result follows. $\qquad \square$

If we take $m_0 = 0 = m_k$ in Corollary 8.3, we recover a result of Eur [Eur20, Theorem 3.2].

Corollary 8.4. *Fix an element $\mathcal{F} \in \mathrm{Flag}_k$ and a tuple of natural numbers $(m_1, \ldots, m_{k-1})$ with $(m_1 + 1) + \cdots + (m_{k-1} + 1) = \mathrm{rk}\, M - 1$. For all $i \in \{1, \ldots, k\}$, let $e_i = (m_1 - d_1) + \cdots + (m_{i-1} - d_{i-1})$, and let $e_{k+1} = 0$. Then we have*

$$\deg_M \left(t_{F_1}^{m_1 + 1} \cdots t_{F_{k-1}}^{m_{k-1} + 1} \right) = (-1)^{\mathrm{rk}\, M - k} \prod_{i=1}^{k} \mu^{e_i}(M_{F_{i-1}}^{F_i}) \binom{m_i}{e_{i+1}}.$$

Proof. The fact that $m_0 = 0 = m_k$ implies that the only nonzero term in Corollary 8.3 is the one in which $e_i = (m_1 - d_1) + \cdots + (m_{i-1} - d_{i-1})$ for all i. $\qquad \square$

The remainder of this section is devoted to the proof of Theorem 8.2. We begin by analyzing a particular specialization of $\mathrm{Snap}_M^{\mathrm{FY}}$, where we set all variables except $a_\varnothing$ and a_E to 0.

Lemma 8.5. *We have*

$$\mathrm{Snap}_M^{\mathrm{FY}}(a_\emptyset, \mathbf{0}, a_E) = \sum_{e,f \in \mathbb{N}} (-1)^{e+f} \binom{\mathrm{rk}\, M - 1 - e}{f} \mu^e(M) a_\emptyset^{(e)} a_E^{(f)}.$$

Proof. By definition, we have

$$\mathrm{Snap}_M^{\mathrm{FY}}(a_\emptyset, \mathbf{0}, a_E) = \sum_{e,f \in \mathbb{N}} \chi(M, \tau_\emptyset^e \tau_E^f) a_\emptyset^{(e)} a_E^{(f)}.$$

We also have

$$\chi(M, \tau_\emptyset^e \tau_E^f) = \deg_M \left(\frac{\zeta_M(\tau_\emptyset^e \tau_E^f)}{1 - h_E} \right) = \deg_M \left(\frac{\zeta_M(\tau_\emptyset^e \tau_E^f)}{1 + t_E} \right).$$

By Lemma 5.5, this is equal to

$$\deg_M \left(\frac{t_\emptyset^e t_E^f}{(1 + t_E)^{f+1}} \right) = \sum_\ell (-1)^{\ell - f} \binom{\ell}{f} \deg_M(t_\emptyset^e t_E^\ell).$$

The degree of $t_\emptyset^e t_E^\ell$ vanishes unless $e + \ell = \mathrm{rk}\, M - 1$, in which case it is equal to $(-1)^{\mathrm{rk}\, M - 1} \mu^e(M)$ [AHK18, Proposition 9.5], thus

$$\chi(M, \tau_\emptyset^e \tau_E^f) = (-1)^{e+f} \binom{\mathrm{rk}\, M - 1 - e}{f} \mu^e(M).$$

This completes the proof. $\qquad\square$

For the statement and proof of the next proposition, it will be convenient to regard $\mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a})$ as a function that takes inputs a_S for all subsets $S \subset E$, with the property that the coefficient of any monomial involving a_S is zero if S is not a flat of M. It will also be convenient to allow M to have loops, with the convention that $\mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a}) = 0$ whenever M has a loop. This allows us to define the matroid M_F^G on the ground set $G \setminus F$ with respect to an arbitrary pair of sets $F \subset G$; note that this matroid is loopless if and only if F is a flat.

Fix a subset $G \subset E$, and for any $\mathbf{a} = (a_F)$, define $\mathbf{a}'$ by putting $a_G' = a_G - 1$ and $a_F' = a_F$ for all $F \neq G$. For any polynomial P in $\mathbf{a}$, define $\partial_G P$ by putting

$$\partial_G P(\mathbf{a}) := P(\mathbf{a}) - P(\mathbf{a}').$$

Note that if P does not depend on a_G, then $\partial_G P = 0$. For any pair of subsets $F \subset G$, let $\mathbf{a}|_{[F,G]}$ be the restriction of $\mathbf{a}$ to this interval, which we identify with the collection of subsets of $G \setminus F$.

Proposition 8.6. *For any proper nonempty subset $G \subset E$, we have*

$$\partial_G \mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a}) = \mathrm{Snap}_{M_\emptyset^G}^{\mathrm{FY}}(\mathbf{a}|_{[\emptyset, G]}) \, \mathrm{Snap}_{M_G^E}^{\mathrm{FY}}(\mathbf{a}|_{[G,E]}).$$

Proof. We will first prove the statement when M is the matroid associated with a hyperplane arrangement $\mathcal{A}$, and then deduce the general case from valuativity. Note that if G is not a flat of M, then $M_\emptyset^G$ has a loop and both sides vanish. We therefore assume that G is a flat.

Consider the hyperplane arrangements

$$\mathcal{A}_\emptyset^G = \{H_e/L_G \mid e \in G\} \qquad \text{and} \qquad \mathcal{A}_G^E := \{H_e \cap L_G \mid e \notin G\}.$$

We observe that $\mathcal{A}_\varnothing^G$ is an arrangement in the vector space L^G with associated matroid $M_\varnothing^G$, and $\mathcal{A}_G^E$ is an arrangement in the vector space L_G with associated matroid M_G^E.

There is a short exact sequence

$$0 \to \mathcal{O}\left((t_G - 1)D_G + \sum_{F \neq G} t_F D_F \right) \to \mathcal{O}\left(\sum t_F D_F \right) \to \mathcal{O}_{D_G}\left(\sum t_F D_F \right) \to 0.$$

Taking Euler characteristics, we find that

$$\partial_G \mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a}) = \mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a}) - \mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a}') = \chi\left(D_G, \mathcal{O}_{D_G}\left(\sum t_F D_F \right) \right).$$

By [DCP95, Page 482], the divisor D_G is isomorphic to $W_{\mathcal{A}_\varnothing^G} \times W_{\mathcal{A}_G^E}$. Furthermore, the restriction of $\mathcal{O}_{D_G}(\sum t_F D_F)$ to D_G is isomorphic to the pullback from $W_{\mathcal{A}_\varnothing^G}$ of $\mathcal{O}_{D_G}(\sum_{F \leq G} t_F D_F)$ tensored with the pullback from $W_{\mathcal{A}_G^E}$ of $\mathcal{O}_{D_G}(\sum_{G \leq F} t_F D_F)$ [BHM⁺22a, Proposition 2.20]. Then by the Künneth formula, we have

$$\chi\left(D_G, \mathcal{O}_{D_G}\left(\sum t_F D_F \right) \right) = \chi\left(W_{\mathcal{A}_\varnothing^G}, \mathcal{O}_{D_G}\left(\sum_{F \leq G} t_F D_F \right) \right) \cdot \chi\left(W_{\mathcal{A}_G^E}, \mathcal{O}_{D_G}\left(\sum_{G \leq F} t_F D_F \right) \right)$$

$$= \mathrm{Snap}_{M_\varnothing^G}^{\mathrm{FY}}(\mathbf{a}|_{[\varnothing,G]}) \, \mathrm{Snap}_{M_G^E}^{\mathrm{FY}}(\mathbf{a}|_{[G,E]}).$$

This completes the proof in the realizable case. For the general case, it will suffice to show that both sides of the equation in the statement of the proposition are valuative invariants of M. It follows from Lemma 6.4 that $\mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a})$ is valutive, and therefore that $\partial_G \mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a})$ is valuative. The valuativity of $\mathrm{Snap}_{M_\varnothing^G}^{\mathrm{FY}}(\mathbf{a}|_{[\varnothing,G]}) \, \mathrm{Snap}_{M_G^E}^{\mathrm{FY}}(\mathbf{a}|_{[G,E]})$ follows from the valuativity of $\mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a})$ and general properties of valuativity [McM09, Theorem 4.6] (see also [AS21, Theorem A]). $\square$

Remark 8.7. Our reason for regarding $\mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a})$ as a polynomial with variables indexed by arbitrary subsets rather than flats, and for allowing matroids with loops, is that it would not otherwise make sense to assert that the two sides of the equation in the statement of Proposition 8.6 are valuative invariants of the matroid M. When considering all matroids at once, we cannot know in advance whether or not G is a flat.

Iterating Proposition 8.6, we obtain the following corollary.

Corollary 8.8. *For any $\mathcal{F} \in \mathrm{Flag}_k$, we have*

$$\partial_{F_1} \cdots \partial_{F_{k-1}} \mathrm{Snap}_M^{\mathrm{FY}}(\boldsymbol{a}) = \prod_{i=1}^k \mathrm{Snap}_{M_{F_{i-1}}^{F_i}}^{\mathrm{FY}} \left(\boldsymbol{a}|_{[F_{i-1}, F_i]} \right).$$

Proof of Theorem 8.2. We use the fact that

$$\partial_{F_1} \cdots \partial_{F_{k-1}} \left(a_{F_0}^{(m_0)} a_{F_1}^{(m_1+1)} a_{F_2}^{(m_2+1)} \cdots a_{F_{k-1}}^{(m_{k-1}+1)} a_{F_k}^{(m_k)} \right) = a_{F_0}^{(m_0)} \cdots a_{F_k}^{(m_k)}$$

to interpret $c(\mathcal{F}, \mathbf{m})$ as the coefficient of $a_{F_0}^{(m_0)} \cdots a_{F_k}^{(m_k)}$ in the polynomial $\partial_{F_1} \cdots \partial_{F_{k-1}} \mathrm{Snap}_M^{\mathrm{FY}}(\mathbf{a})$. Corollary 8.8 provides us with a formula for this polynomial. Setting $a_F = 0$ for all F not

appearing in $\mathcal{F}$, we obtain the polynomial

$$\prod_{i=1}^{k} \mathrm{Snap}^{\mathrm{FY}}_{M^{F_i}_{F_{i-1}}}(a_{F_{i-1}}, \mathbf{0}, a_{F_i}).$$

By Lemma 8.5, we may rewrite this as

$$(12) \qquad \sum_{\mathbf{e},\mathbf{f}\in\mathbb{N}^k} (-1)^{|\mathbf{e}|+|\mathbf{f}|} \prod_{i=1}^{k} \binom{d_i - e_i}{f_i} \mu^{e_i}(M^{F_i}_{F_{i-1}}) a^{(e_i)}_{F_{i-1}} a^{(f_i)}_{F_i}.$$

To compute the coefficient of $a^{(m_0)}_{F_0} \cdots a^{(m_k)}_{F_k}$, we make use of the identity

$$x^{(m)} x^{(n)} = \sum_{\ell=0}^{m} \binom{m+n-\ell}{\ell, m-\ell, n-\ell} x^{(m+n-\ell)}$$

to rewrite (12) as

$$\sum_{\mathbf{e},\mathbf{f}\in\mathbb{N}^k} (-1)^{|\mathbf{e}|+|\mathbf{f}|} a^{(e_1)}_{\varnothing} \prod_{i=1}^{k} \binom{d_i - e_i}{f_i} \mu^{e_i}(M^{F_i}_{F_{i-1}}) \sum_{\ell_i=0}^{e_{i+1}} \binom{e_{i+1} + f_i - \ell_i}{\ell_i, e_{i+1} - \ell_i, f_i - \ell_i} a^{(e_{i+1}+f_i-\ell_i)}_{F_i},$$

with the convention that $e_{k+1} = 0$. The coefficient of $a^{(m_0)}_{F_0} \cdots a^{(m_k)}_{F_k}$ consists of those terms for which $e_1 = m_0$ and $e_{i+1} + f_i - \ell_i = m_i$ for all $0 \le i \le k$. Thus

$$c(\mathcal{F}, \mathbf{m}) = \sum_{\substack{\mathbf{e},\mathbf{f}\in\mathbb{N}^k \\ e_1 = m_0}} (-1)^{|\mathbf{e}|+|\mathbf{f}|} \prod_{i=1}^{k} \binom{d_i - e_i}{f_i} \mu^{e_i}(M^{F_i}_{F_{i-1}}) \binom{m_i}{e_{i+1} + f_i - m_i, m_i - f_i, m_i - e_{i+1}}. \qquad \square$$

9. EULER CHARACTERISTICS AND SNAPPER POLYNOMIALS FOR $\overline{\mathcal{M}}_{0,n}$

Last, we turn our attention to the moduli space $\overline{\mathcal{M}}_{0,n}$. Recall that we have defined a tuple of line bundles

$$\mathcal{L} := (\mathcal{L}_S \mid S \subset [n-1], |S| \ge 3).$$

We begin this section by computing the Snapper polynomial $\mathrm{Snap}_{\mathcal{L}}(\mathbf{a})$ with respect to these line bundles. Given a tuple of natural numbers $\mathbf{m} = (m_S \mid S \subset [n-1] \mid |S| \ge 3)$, we say that $\mathbf{m}$ satisfies the **Cerberus condition** if, for every $\mathbf{m}' \le \mathbf{m}$, we have

$$\left| \bigcup_{m'_S > 0} S \cup \{n\} \right| - 3 \ge \sum_S m'_S.$$

Theorem 9.1. *We have*

$$\mathrm{Snap}_{\mathcal{L}}(\mathbf{a}) = \sum_{\substack{\mathbf{m} \text{ satisfies} \\ \text{Cerberus}}} \mathbf{a}^{(\mathbf{m})}.$$

Proof. Let $\mathbf{m}$ be given, and define

$$\tilde{\mathbf{m}} = (\tilde{m}_F \mid F \text{ a nonempty flat of } \mathcal{B}_{n-1})$$

by putting $\tilde{m}_{F_S} = m_S$ and $\tilde{m}_F = 0$ for all flats F not of the form F_S. Because the map $p : W_{\mathcal{B}_{n-1}} \to \overline{\mathcal{M}}_{0,n}$ is a composition of blowups at smooth centers, the Euler characteristic of a line bundle on $\overline{\mathcal{M}}_{0,n}$ is equal to the Euler characteristic of its pullback to $W_{\mathcal{B}_{n-1}}$. Thus, if we can show that the Cerberus condition for $\mathbf{m}$ is equivalent to the dragon Hall–Rado condition for $\tilde{\mathbf{m}}$, our result will follow from Theorem 7.2. Note that $\tilde{\mathbf{m}}$ fails the dragon Hall–Rado condition if and only if there is $\tilde{\mathbf{m}}' \leq \tilde{\mathbf{m}}$ with

$$\text{rank} \bigcup_{\tilde{m}'(F)>0} F \geq \sum_F \tilde{m}'(F) \text{ and } \bigcup_{\tilde{m}'(F)>0} F \text{ connected.}$$

The remainder of the proof follows by direct calculation, using the facts that $\text{rk}\, F_S = |S| - 1$ and $\text{rk}\, F_S \cup F_T = \text{rk}\, F_{S\cup T}$ if S and T are not disjoint. $\qquad\square$

We conclude by computing the Snapper polynomial with respect to a different tuple of line bundles on $\overline{\mathcal{M}}_{0,n}$. Recall that $\mathbb{L}_i$ is the ith cotangent line bundle, whose first Chern class is equal to ψ_i, and let $\mathbb{L} := (\mathbb{L}_1, \ldots, \mathbb{L}_n)$. In [Pan97], Pandharipande showed that if $a_1, \ldots, a_n \geq 0$, then $H^i(\overline{M}_{0,n}, \mathbb{L}_1^{\otimes a_1} \otimes \cdots \otimes \mathbb{L}_n^{\otimes a_n}) = 0$ for $i > 0$, and so

$$\chi(\overline{M}_{0,n}, \mathbb{L}_1^{\otimes a_1} \otimes \cdots \otimes \mathbb{L}_n^{\otimes a_n}) = h^0(\overline{M}_{0,n}, \mathbb{L}_1^{\otimes a_1} \otimes \cdots \otimes \mathbb{L}_n^{\otimes a_n}).$$

In [Lee97], Lee gave an expression for the generating function of this Euler characteristic, which is equivalent to the following theorem.

Theorem 9.2. [Lee97] *We have*

$$\text{Snap}_{\mathbb{L}}(\boldsymbol{a}) = \sum_{|\boldsymbol{d}| \leq n-3} \binom{a_1}{d_1} \cdots \binom{a_n}{d_n} \binom{n-3}{n-3-|\boldsymbol{d}|, d_1, \ldots, d_n}.$$

We give a new proof of Theorem 9.2, beginning with the following lemma. Recall the exceptional isomorphism $\zeta_n : K(\overline{\mathcal{M}}_{0,n}) \to A(\overline{\mathcal{M}}_{0,n})$ of Theorem 1.10. Let $z_i := 1 - [\mathbb{L}_i] \in K(\overline{\mathcal{M}}_{0,n})$.

Lemma 9.3. *We have $\zeta_n(z_i) = -\psi_i$ for all $i < n$, and $\zeta_n(z_n) = -\psi_n/(1 - \psi_n)$.*

Proof. We will consider a family of closely related moduli spaces of curves, called the Losev–Manin spaces [LM00]. The Losev–Manin space $\overline{\mathcal{M}}_{0,w_{k,\ell}}$ is a moduli space of stable rational curves with n weighted marked points [Has03], with weights 1 for the the kth and ℓ^{th} points and $\varepsilon \in \mathbb{Q} \cap (0, \frac{1}{n-2})$ for the remaining $n-2$ points. Reduction of weights from $(1, \ldots, 1)$ to $w_{k,\ell}$ induces a morphism [Has03, Theorem 4.1] $\rho_{k,\ell} : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,w_{k,\ell}}$. Furthermore, $\overline{\mathcal{M}}_{0,w_{k,\ell}}$ is isomorphic to the permutohedral variety $X_{A_{[n]\smallsetminus\{k,\ell\}}}$ [LM00], so we have an exceptional isomorphism

$$\zeta_{[n]\smallsetminus\{k,\ell\}} : K(\overline{\mathcal{M}}_{0,w_{k,\ell}}) \to A(\overline{\mathcal{M}}_{0,w_{k,\ell}}).$$

For $i \neq n$, consider the classes $t_E, t_\emptyset \in A(\overline{\mathcal{M}}_{0,w_{i,n}}) \cong A(X_{A_{[n]\smallsetminus\{i,n\}}})$ and $\tau_E, \tau_\emptyset \in K(\overline{\mathcal{M}}_{0,w_{i,n}}) \cong K(X_{A_{[n]\smallsetminus\{i,n\}}})$. Then we have the following identities:

$$\rho_{n,i}^* t_E = -\psi_n \qquad \rho_{n,i}^* \tau_E = 1 - [\mathbb{L}_n]$$
$$\rho_{n,i}^* t_\emptyset = -\psi_i \qquad \rho_{n,i}^* \tau_\emptyset = 1 - [\mathbb{L}_i],$$

see, for example, [DR22, Section 2]. For each $i \neq n$, we have the following diagram

$$
\begin{array}{ccc}
K(\overline{\mathcal{M}}_{0,w_{n,i}}) & \xrightarrow{\zeta_{[n]\smallsetminus\{n,i\}}} & A(\overline{\mathcal{M}}_{0,w_{n,i}}) \\
\downarrow{\scriptstyle\rho^*_{n,i}} & & \downarrow{\scriptstyle\rho^*_{n,i}} \\
K(\overline{\mathcal{M}}_{0,n}) & \xrightarrow{\quad\zeta_n\quad} & A(\overline{\mathcal{M}}_{0,n}),
\end{array}
$$

whose commutativity can be checked by using that $\rho^*_{n,i}\tau_E = 1 - [\mathbb{L}_n^{-1}]$, and that similar formulas hold for pullbacks of the nth cotangent line bundle under forgetful maps.

The Lemma 5.5 gives that

$$
\zeta_n(z_i) = \zeta_n(1 - [\mathbb{L}_i]) = \zeta_n \circ \rho^*_{n,i}(\tau_\emptyset) = \rho^*_{n,i} \circ \zeta_{[n]\smallsetminus\{n,i\}}(\tau_\emptyset) = \rho^*_{n,i}(t_\emptyset) = -\psi_i,
$$

and similarly $\zeta_n(z_n) = \rho^*_{n,i}\left(\frac{t_E}{1+t_E}\right) = \frac{-\psi_n}{1-\psi_n}$. $\square$

Proof of Theorem 9.2. By Lemma 7.3, we have

$$
\mathrm{Snap}_{\mathbb{L}}(-\mathbf{a}) = \sum_{\mathbf{d}\in\mathbb{N}^n} \chi\left(\overline{\mathcal{M}}_{0,n},\, z_1^{d_1}\cdots z_n^{d_n}\right) \mathbf{a}^{(\mathbf{d})}.
$$

(Note that the minus sign comes from the fact that $z_i = 1 - [\mathbb{L}_i]$ rather than $1 - [\mathbb{L}_i^{-1}]$.) By Theorem 1.10 and Lemma 9.3, we have

$$
\chi\left(\overline{\mathcal{M}}_{0,n},\, z_1^{d_1}\cdots z_n^{d_n}\right) = \deg\left(\frac{\zeta_n(z_1^{d_1}\cdots z_n^{d_n})}{1 - c_{[n-1]}}\right) = (-1)^{|\mathbf{d}|}\deg\left(\frac{\psi_1^{d_1}\cdots\psi_n^{d_n}}{(1-\psi_n)^{d_n+1}}\right).
$$

Next, we observe that

$$
\frac{\psi_1^{d_1}\cdots\psi_n^{d_n}}{(1-\psi_n)^{d_n+1}} = \psi_1^{d_1}\cdots\psi_n^{d_n}\sum_{k=0}^{\infty}(d_n+1)^{(k)}\psi_n^k,
$$

and therefore the part of this sum in degree $n-3$ is equal to

$$
\psi_1^{d_1}\cdots\psi_{n-1}^{d_{n-1}}\psi_n^{n-3-d_1-\cdots-d_{n-1}}(d_n+1)^{(n-3-|\mathbf{d}|)}
$$

if $|\mathbf{d}| \leq n-3$, and zero otherwise. By work of Witten [Wit91], the degree of this term is equal to

$$
\binom{n-3}{d_1,\ldots,d_{n-1},n-3-d_1-\cdots-d_{n-1}}(d_n+1)^{(n-3-|\mathbf{d}|)}.
$$

Putting it all together, we have

$$
\begin{aligned}
\mathrm{Snap}_{\mathbb{L}}(-\mathbf{a}) \;&=\; \sum_{d\in\mathbb{N}^n} \chi\left(\overline{\mathcal{M}}_{0,n},\, z_1^{d_1}\cdots z_n^{d_n}\right)\mathbf{a}^{(\mathbf{d})}\\[2mm]
&=\; \sum_{|\mathbf{d}|\le n-3} (-1)^{|\mathbf{d}|}\mathbf{a}^{(\mathbf{d})} \deg\left(\frac{\psi_1^{d_1}\cdots\psi_n^{d_n}}{(1-\psi_n)^{d_n+1}}\right)\\[2mm]
&=\; \sum_{|\mathbf{d}|\le n-3} (-1)^{|\mathbf{d}|}\mathbf{a}^{(\mathbf{d})}(d_n+1)^{(n-3-|\mathbf{d}|)}\binom{n-3}{d_1,\ldots,d_{n-1},n-3-d_1-\cdots-d_{n-1}}\\[2mm]
&=\; \sum_{|\mathbf{d}|\le n-3} (-1)^{|\mathbf{d}|}\mathbf{a}^{(\mathbf{d})}\binom{d_n+n-3-|\mathbf{d}|}{n-3-|\mathbf{d}|}\binom{n-3}{d_1,\ldots,d_{n-1},n-3-d_1-\cdots-d_{n-1}}\\[2mm]
&=\; \sum_{|\mathbf{d}|\le n-3} (-1)^{|\mathbf{d}|}\mathbf{a}^{(\mathbf{d})}\binom{n-3}{d_1,\ldots,d_n,n-3-|\mathbf{d}|}.
\end{aligned}
$$

Finally, this implies that

$$
\mathrm{Snap}_{\mathbb{L}}(\mathbf{a}) = \sum_{|\mathbf{d}|\le n-3} \binom{a_1}{d_1}\cdots\binom{a_n}{d_n}\binom{n-3}{d_1,\ldots,d_n,n-3-|\mathbf{d}|},
$$

which completes the proof. $\square$

Appendix A. The simplicial presentation of Chow rings

This appendix is devoted to proving the simplicial presentation for the (augmented) Chow ring of a matroid. The simplicial generators of $A(M)$ were extensively studied in [BES21], and the surjectivity of Φ_∇ was proved there. This immediately generalizes to $\Phi_\nabla^{\mathrm{aug}}$. That work did not give a simple description of the kernel of Φ_∇.

For any pair of flats F and G, let

$$
z_{F,G} := \sum_{\substack{F\subset F'\subsetneq F\vee G\\ G\subset G'\subsetneq F\vee G}} x_{F'}x_{G'} \in T_M.
$$

For any element $e\in E$ and flat $F\neq\emptyset$, let

$$
w_{e,F} := \sum_{F\subset G\not\ni e} y_e x_G \in T_M.
$$

Recall the definitions of the ideals $\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3,\mathcal{I}_4^{\mathrm{aug}}\subset T_M$ in Equation (1).

Lemma A.1. *We have the following equalities of ideals in T_M:*

$$
\begin{aligned}
\mathcal{I}_3 \;&=\; \langle z_{F,G} \mid F,G \text{ arbitrary}\rangle\\
\mathcal{I}_2+\mathcal{I}_4^{\mathrm{aug}} \;&=\; \mathcal{I}_2+\langle y_e^2 \mid e\in E\rangle+\langle w_{e,F} \mid e\in E, F\neq\varnothing\rangle.
\end{aligned}
$$

Proof. We begin with the first statement. Every term in $z_{F,G}$ is a product of incomparable flats, thus $z_{F,G}$ is contained in $\mathcal{I}_3$. The opposite inclusion follows by induction on sum of the coranks

of F and G, using the observation that, whenever F and G are incomparable,

$$z_{F,G} = x_F x_G + \text{terms with lower sum of coranks.}$$

For the second statement, we write

$$\mathcal{I}_4^{\mathrm{aug}} = \langle y_e x_\varnothing \mid e \in E \rangle + \langle y_e x_F \mid e \notin F \neq \varnothing \rangle.$$

A similar argument to the one above shows that

$$\langle y_e x_F \mid e \notin F \neq \varnothing \rangle = \langle w_{e,F} \mid e \in E, F \neq \varnothing \rangle,$$

so it remains only to show that $y_e x_\varnothing$ is congruent to y_e^2 modulo the ideal $\mathcal{I}_2 + \langle w_{e,F} \mid e \in E, F \neq \varnothing \rangle$. Indeed, we have

$$y_e^2 - x_\varnothing y_e = y_e \left(y_e - \sum_{e \notin F} x_F \right) + \sum_{e \notin F \neq \varnothing} y_e x_F,$$

which completes the proof. $\qquad\square$

Recall that we have defined the homomorphism $\Phi_\nabla^{\mathrm{aug}} : S_M \to A^{\mathrm{aug}}(M)$ by the formula

$$\Phi_\nabla^{\mathrm{aug}}(u_F) := - \sum_{F \subset G} \Phi_{\mathrm{FY}}(x_G).$$

It is easy to see that the kernel of $A^{\mathrm{aug}}(M) \to A(M)$ is generated by $\langle h_F : \mathrm{rk}\, F = 1 \rangle$, so the follow theorem implies Theorem 1.8.

Theorem A.2. *The map $\Phi_\nabla^{\mathrm{aug}}$ is surjective with kernel $\mathcal{J}_1 + \mathcal{J}_2^{\mathrm{aug}}$.*

Proof. Consider the homomorphism $\Theta : S_M \to T_M$ defined by the formula

$$\Theta(u_F) := - \sum_{F \subset G} x_G$$

for all nonempty flats F, so that $\Phi_\nabla^{\mathrm{aug}} = \Phi_{\mathrm{FY}}^{\mathrm{aug}} \circ \Theta$. The image of Θ is equal to the subring $\mathbb{Z}[x_F \mid F \text{ a nonempty flat}] \subset T_M$. In particular, the composition

$$\bar{\Theta} : S_M \to T_M \to T_M / (\mathcal{I}_1 + \mathcal{I}_2)$$

is an isomorphism. This implies that $\Phi_\nabla^{\mathrm{aug}}$ is surjective, and its kernel is equal to

$$\Theta^{-1}(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4^{\mathrm{aug}}).$$

For the remainder of the proof, we put a bar over an element of T_M or an ideal in T_M to denote its image in $T_M / (\mathcal{I}_1 + \mathcal{I}_2)$. Then we need to compute $\bar{\Theta}^{-1}(\bar{\mathcal{I}}_3 + \bar{\mathcal{I}}_4^{\mathrm{aug}})$. By Lemma A.1, we may rewrite this as

$$\bar{\Theta}^{-1}\Big(\langle \bar{z}_{F,G} \mid F, G \text{ arbitrary} \rangle + \langle \bar{y}_e^2 \mid e \in E \rangle + \langle w_{e,F} \mid e \in E, F \neq \varnothing \rangle \Big).$$

We have the following equalities:

$$
\begin{aligned}
\bar{z}_{F,G} &= \bar{\Theta}\Big((u_F - u_{F\vee G})(u_G - u_{F\vee G})\Big), \\
\bar{y}_e^2 &= \bar{\Theta}(u_{\bar{e}}^2), \\
\bar{w}_{e,F} &= \bar{\Theta}\Big(u_{\bar{e}}(u_{\bar{e}\vee F} - u_F)\Big),
\end{aligned}
$$

which together imply that the kernel of $\Phi_\nabla^{\mathrm{aug}}$ is equal to

$$
\mathcal{J}_1 + \langle u_{\bar{e}}^2 \mid e \in E \rangle + \langle u_{\bar{e}}(u_{\bar{e}\vee F} - u_F) \mid e \in E, F \neq \varnothing \rangle.
$$

The theorem now follows from the fact that the second and third summand above add to $\mathcal{J}_2^{\mathrm{aug}}$. $\square$

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