

ON SOME ASPECTS OF K-THEORETIC POSITIVITY

MATT LARSON

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In these notes, we will discuss some positivity phenomena arising from the K -theory of projective varieties. This can be summarized as showing that some integer is non-negative (or non-positive) by realizing it as the Euler characteristic of a vector bundle on some variety, and then showing that the cohomology of that vector bundle is concentrated in a particular degree.

This technique has a number of combinatorial applications. The first applications of this idea, in some form, are due to Stanley [Sta75, Sta80]. Our main example will be a paper of Speyer [Spe09], which bounds the complexity of matroid polytope subdivisions, as well as subsequent developments. Throughout we will mostly focus on the case of realizable matroids, and we will only sketch how to extend results to the non-realizable case.

We will work over a field k , which we will often need to assume has characteristic 0. We will assume that it is algebraically closed for convenience. The reader will lose little by assuming that $k = \mathbb{C}$. Let $[n] = \{1, \dots, n\}$, and let $\binom{[n]}{r}$ denote the set of subsets of $[n]$ of size r . A “variety” means a separated scheme of finite type over k , which is not assumed to be integral. All toric varieties are normal.

We summarize the content of these notes. In Section 1, we introduce the Grassmannian and study torus-orbit closures in it. This leads to the definition of matroids, and we discuss some basic examples and properties. No background in matroid theory is assumed.

In Section 2, we study matroid polytope subdivisions. We state Speyer’s f -vector conjecture (Conjecture 2.11), a conjectural bound on the complexity of matroid polytope subdivisions. Our main goal in these notes is to explain the mathematics surrounding this conjecture.

In Section 3, we introduce K -theory of spaces. This is a geometric construction which can be viewed as a type of intersection theory. Throughout these notes, we will give applications of positivity properties in K -theory to several combinatorial problems.

In Section 4, we describe the K -theory of Grassmannians. We discuss Brion’s positivity results [Bri02] for the K -theory of Grassmannians and the key role that the Kawamata–Viehweg vanishing theorem (Theorem 4.6) plays in its proof. We discuss how the K -theory of Grassmannians can be used to construct some matroid invariants, although these invariants are not strong enough to prove Conjecture 2.11.

In Section 5, we introduce the g -polynomial of a matroid. This invariant, which was defined in [Spe09], is the key to proving the known cases of Conjecture 2.11. We explain Speyer’s proof, using the Kawamata–Viehweg vanishing theorem, that the g -polynomial of a matroid which is realizable over a field of characteristic 0 has non-negative coefficients. We also explain how to extend the definition of the g -polynomial to non-realizable matroids, and we establish its basic properties.

Besides Speyer’s proof of the non-negativity of the g -polynomial for matroids realizable over a field of characteristic 0, the best result towards Conjecture 2.11 is that the last non-trivial coefficient of the g -polynomial is non-negative. This coefficient, which is called the omega invariant of a matroid, was proved to be non-negative by Berget and Fink in [BF24]; see Theorem 5.38. A different proof was given in [EFL25]. The rest of these notes are aimed at explaining the proofs of this result.

Starting in Section 6, an increasing amount of algebro-geometric background is assumed. We discuss Cohen–Macaulayness of graded rings. As originally observed by Stanley, Cohen–Macaulayness can be used to deduce positivity properties of Hilbert series of graded rings. We explain the connection between Cohen–Macaulayness and certain cohomology vanishing properties.

In Section 7, we discuss some additional properties of the g -polynomial. We prove some formulas for the g -polynomial from [FS12] and [BEST23], and we give some examples which illustrate the difficulties in some possible approaches to proving the non-negativity of the coefficients of the g -polynomial of a matroid.

In Section 8, we study an important family of subschemes of a product of projective lines. These subschemes, which were dubbed *kindred* subschemes in [EFL25], satisfy many remarkable properties, and they are very useful in the proof of Theorem 5.38.

In Section 9, we discuss the proof of Theorem 5.38. We sketch three different approaches, focusing on the case of realizable matroids. Each of these three approaches proves different cohomology vanishing theorems which can be used to prove the non-negativity of the omega invariant of a realizable matroid.

In Section 10, we discuss some connections between the ideas in the previous section and graph rigidity.

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1. THE GRASSMANNIAN AND MATROIDS

1.1. The Grassmannian. For a natural number n and $0 \leq r \leq n$, the *Grassmannian* $\text{Gr}(r, n)$ is the space of r -dimensional subspaces of k^n . This can be given the structure of a projective variety, as follows. For an r -dimensional subspace L of k^n , choose an $r \times n$ matrix M whose row span is L . If M' is another such matrix, then there is $g \in GL_r(k)$ such that $M' = gM$.

The *Plücker coordinates* of L are the $\binom{n}{r}$ determinants of the maximal minors of M . Up to scaling by a non-zero constant, they are independent of the choice of M : if we replace M by gM for some $g \in GL_r(k)$, then all of the Plücker coordinates are scaled by $\det g$. We therefore obtain a well-defined element $[L]$ of projective space $\mathbb{P}^{\binom{n}{r}-1}$. For a subset S of $\binom{[n]}{r}$, we denote the corresponding Plücker coordinate by $p_S([L])$ or simply by p_S .

It turns out that the subset $\{[L] : L \in \text{Gr}(r, n)\}$ is a Zariski-closed subset of $\mathbb{P}^{\binom{n}{r}-1}$, i.e., it is the vanishing locus of a set of polynomials. The ideal of these polynomials is generated by a very explicit set of polynomials called *Plücker relations*. These equations have the following form. Given subsets I, J of $[n]$ of size $r-1$ and $r+1$, respectively, the following equation holds on $\text{Gr}(r, n)$:

$$(1) \quad \sum_{\ell \in J \setminus I} (-1)^{S_\ell} p_{I \cup \ell} p_{J \setminus \ell} = 0,$$

where $S_\ell = |\{i \in I : i > \ell\}| + |\{j \in J : j > \ell\}|$. As we vary over all choices of subsets, these equations cut out $\text{Gr}(r, n)$. One can also show that the linear subspace L can be recovered from its Plücker coordinates, and we can identify $\text{Gr}(r, n)$ with this locus in $\mathbb{P}^{\binom{n}{r}-1}$. For a proof of the above statements, see [Ful97].

Example 1.1. The Grassmannian $\text{Gr}(1, n)$ is \mathbb{P}^{n-1} .

Example 1.2. The Grassmannian $\text{Gr}(2, 4)$ is a hypersurface in \mathbb{P}^5 . If the coordinates on \mathbb{P}^5 are $p_{12}, p_{13}, p_{14}, p_{23}, p_{24}$, and p_{34} , then $\text{Gr}(2, 4)$ is the vanishing locus of $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}$.

Example 1.3. There is an isomorphism from $\text{Gr}(r, n)$ to $\text{Gr}(n-r, n)$, given by taking $[L] \in \text{Gr}(r, n)$ to the subspace $L^\perp = \{v \in k^n : \langle v, w \rangle = 0 \text{ for all } w \in L\}$, where $\langle -, - \rangle$ is the standard inner product on k^n .

The Grassmannian admits a transitive action of $GL_n(k)$. In terms of the $r \times n$ matrix M whose row span is L , an element $g \in GL_n(k)$ replaces L by the row span of Mg . This transitive action implies that $\text{Gr}(r, n)$ is smooth. Because GL_n is connected, this action also implies that $\text{Gr}(r, n)$ is irreducible. One can show that the dimension of $\text{Gr}(r, n)$ is $r(n - r)$.

Inside of GL_n , there are two particularly important subgroups. The *Borel* subgroup B is the space of upper triangular matrices. The *torus* T is the space of diagonal matrices. If L is the row span of an $r \times n$ matrix M and $t = (t_1, \dots, t_n) \in T$, then the point $t \cdot [L]$ represents the row span of the matrix obtained by scaling the n columns of M . This torus is isomorphic to \mathbb{G}_m^n , i.e., the n th power of the multiplicative group. Much of these notes will focus on the geometry and combinatorics of torus-orbits on Grassmannians.

The action of $GL_n(k)$ on $\text{Gr}(r, n)$ extends to an action on $\mathbb{P}^{\binom{n}{r}-1}$. This is particularly easy to see for the torus: an element $t = (t_1, \dots, t_n)$ scales the coordinate labeled by a subset S of $\{1, \dots, n\}$ of size r by $\prod_{i \in S} t_i$.

Remark 1.4. The Grassmannian $\text{Gr}(r, n)$ can also be viewed as a moduli space of (essential) arrangements of n hyperplanes in an r -dimensional subspace. Given a subspace L of k^n , we obtain n hyperplanes by intersecting L with the coordinate hyperplanes (at least if L is not contained in any coordinate hyperplane). If we have an arrangement of n hyperplanes in L whose intersection is 0, then we can obtain an embedding of L into k^n by writing each hyperplane as the vanishing locus of a linear form. Different choices of the linear forms yield points of the Grassmannian which are in the same torus-orbit.

1.2. Torus-orbits on the Grassmannian. We will now begin a detailed study of the torus-orbits on the Grassmannian, using ideas introduced in [GS87]. The first step is to analyze the stabilizers of points in the Grassmannian. Note that the subgroup of T that consists of diagonal multiples of the identity acts trivially on $\text{Gr}(r, n)$, so every point has a stabilizer.

Proposition 1.5. *For each subspace L of k^n , there is a unique finest partition $[n] = S_1 \sqcup \dots \sqcup S_c$ such that*

$$L = \bigoplus_{i=1}^c L \cap k^{S_i}.$$

The stabilizer of $[L] \in \text{Gr}(r, n)$ in the torus T is \mathbb{G}_m^c , with the i th factor diagonally scaling k^{S_i} .

Proof. The *support* of a linear form $\ell = \sum a_i x_i$ which vanishes on L is the set $\{i : a_i \neq 0\}$. A linear form which vanishes on L is said to have *minimal support* if it is non-zero, and there is no non-zero linear form whose support is strictly contained in the support of ℓ . If two linear forms of minimal support have the same support, then they are constant multiples of each other. Every linear form vanishing on L can be written as a linear combination of linear forms of minimal support which vanish on L .

Consider the equivalence relation generated by setting $i \sim j$ if i and j are both contained in the support of some linear form vanishing on L with minimal support. Let S_1, \dots, S_c be the equivalence classes. Then the ideal of L is generated by linear forms whose support is contained in some S_i , implying that there is a direct sum decomposition

$$L = \bigoplus_{i=1}^c L \cap k^{S_i}.$$

For any direct sum decomposition like this, the ideal of L is generated by linear forms whose support is contained in the blocks, implying that this is the finest partition which induces a direct sum decomposition.

Let $t = (t_1, \dots, t_n)$ be an element of the stabilizer of L , i.e., $tv \in L$ for all $v \in L$. Then for each linear form $\ell = \sum a_i x_i$ vanishing on L , the linear form $\sum t_i a_i x_i$ also vanishes on L . If ℓ has minimal support, then this must be a constant multiple of ℓ . We deduce that $t_i = t_j$ for all i, j in the support of ℓ . The description of the S_i in terms of the equivalence relation \sim then implies the result. \square

We say that the partition $[n] = S_1 \sqcup \cdots \sqcup S_c$ is the partition into *connected components*. If $0 < r < n$, then a general point of $\text{Gr}(r, n)$ has a 1-dimensional stabilizer in T , i.e., there is only a single connected component.

The key tool that we will use to study torus-orbits in $\text{Gr}(r, n)$ will be the theory of *moment polytopes*. We first recall the general theory; see [Ful93, Chapter 3], [Sot03, Section 8], or [EFLS24, Section 6.1]. Let H be a torus with character lattice M . Let V be a representation of H . Then V has a unique decomposition $V \simeq \bigoplus_{i=1}^N V_i$ into H -eigenspaces, where V_i is the subspace of V where H acts by some character $a_i \in M$.

There is an action of H on $\mathbb{P}V$. Given a point $x \in \mathbb{P}V$ with representative $v \in V$, let

$$\mathcal{A}_x = \{a_i : v_i \neq 0 \text{ in the expression } v = \sum_{i=1}^N v_i\}.$$

Then the normalization of the torus-orbit closure $\overline{H \cdot x}$ is the toric variety corresponding to the normal fan of $\text{Conv}(\mathcal{A}_x) \subset M \otimes_{\mathbb{Z}} \mathbb{R}$, with respect to the affine lattice generated by \mathcal{A}_x , i.e., the translation of the sublattice of M generated by $\{a_i - a_j : a_i, a_j \in \mathcal{A}_x\}$ so that it contains \mathcal{A}_x . The polytope $\text{Conv}(\mathcal{A}_x) \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ is called the *moment polytope* of $\overline{H \cdot x}$. In particular, by [CLS11, Corollary 3.A.6], there is a bijection between k -dimensional faces of the moment polytope and k -dimensional H -orbits in $\overline{H \cdot x}$.

We apply this to points of the Grassmannian in its Plücker embedding. The character lattice of the torus T is identified with \mathbb{Z}^n , and the eigenspaces of T acting on the ambient space of the Plücker embedding are the coordinate lines. The character corresponding to a subset $S \in \binom{[n]}{r}$ is $e_S := \sum_{i \in S} e_i$. The torus-fixed points of $\mathbb{P}^{\binom{[n]}{r}-1}$ are in bijection with $\binom{[n]}{r}$.

Given $[L] \in \text{Gr}(r, n)$, we see that the set $\overline{\mathcal{A}_{[L]}}$ is $\{e_S : p_S([L]) \neq 0\}$, where p_S is the Plücker coordinate corresponding to S . The moment polytope of $\overline{T \cdot [L]}$ is

$$\text{Conv}(e_S : p_S([L]) \neq 0).$$

However, $\overline{T \cdot [L]}$ is contained in $\text{Gr}(r, n)$. Although there is a unique T -fixed curve between any two distinct T -fixed points in $\mathbb{P}^{\binom{[n]}{r}-1}$, most of these curves do not lie in $\text{Gr}(r, n)$.

Proposition 1.6. *The T -fixed curve in $\mathbb{P}^{\binom{[n]}{r}-1}$ between the T -fixed points corresponding to distinct subsets S_1, S_2 in $\binom{[n]}{r}$ lies in $\text{Gr}(r, n)$ if and only if $S_2 = (S_1 \setminus i) \cup j$ for some $i \in S_1, j \in S_2$.*

Proof. Suppose that $S_2 = (S_1 \setminus i) \cup j$ for some $i \in S_1, j \in S_2 \setminus S_1$. Let $S_1 \cap S_2 = \{a_1, \dots, a_{r-1}\}$. For $t \in k^\times$, set $[L_t]$ to be the point of $\text{Gr}(r, n)$ corresponding to the span of $e_{a_1}, \dots, e_{a_{r-1}}$ and $e_i + te_j$. Then the closure of $\{[L_t] : t \in k^\times\}$ is the T -fixed curve between the T -fixed points corresponding to S_1 and S_2 .

If we don't have $S_2 = (S_1 \setminus i) \cup j$ for some $i \in S_1, j \in S_2$, then we claim that the corresponding T -fixed curve, i.e., the locus in $\mathbb{P}^{\binom{[n]}{r}-1}$ where all Plücker coordinates are 0 except for those corresponding to S_1 and S_2 , does not lie in $\text{Gr}(r, n)$. Indeed, let $S_1 = \{i_1, \dots, i_r\}$, and let $S_2 = \{j_1, \dots, j_r\}$. Assume that $i_r \notin S_2$. Then using the sets $\{i_1, \dots, i_{r-1}\}$ and $\{j_1, \dots, j_r, i_r\}$ in (1), there is exactly one term which is non-zero for any linear subspace which has $p_{S_1}, p_{S_2} \neq 0$ and $p_T = 0$ for all other $T \in \binom{[n]}{r}$, which is impossible. \square

Remark 1.7. There are finitely many 2-dimensional T -fixed subvarieties of $\text{Gr}(r, n)$. However, $\text{Gr}(2, 4)$ has infinitely many 3-dimensional T -fixed subvarieties.

In particular, every edge in the moment polytope of $\overline{T \cdot [L]}$ must connect two vertices e_{S_1} and e_{S_2} with $S_2 = (S_1 \setminus i) \cup j$. This happens if and only if every edge is parallel to $e_i - e_j$, i.e., it is parallel to a root of type A_{n-1} . This proves the following.

Theorem 1.8. [GS87] *All edges of the moment polytope of a torus-orbit closure in $\text{Gr}(r, n)$ are parallel to a vector of the form $e_i - e_j$.*

Motivated by this, we define *matroids* as a generalization of moment polytopes of torus-orbits in Grassmannians. Matroids are certain subsets of $\binom{[n]}{r}$ that model the set of non-vanishing Plücker coordinates of a linear subspace.

Definition 1.9. A subset \mathcal{B} of $\binom{[n]}{r}$ is a matroid of rank r on ground set $[n]$ if the polytope

$$\text{Conv}(e_S : S \in \mathcal{B})$$

has all edges parallel to a vector of the form $e_i - e_j$.

A matroid is usually called M , and we say that the set \mathcal{B} appearing in Definition 1.9 is its set of *bases*.

For a matroid M , the polytope $\text{Conv}(e_S : S \in \mathcal{B})$ is called the *matroid polytope* of M . It is denoted $P(M)$. From Theorem 1.8, we deduce that each linear subspace $L \subseteq k^n$ of dimension r gives rise to a matroid on ground set $[n]$ of rank r . A matroid arising in this way is called *realizable* (over k).

Example 1.10. For $0 \leq r \leq n$, the *uniform matroid* $U_{r,n}$ is the matroid whose bases are all subsets of $[n]$ of size r . A general linear subspace realizes $U_{r,n}$.

Example 1.11. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

If k has characteristic 2, then the row span of A in k^7 realizes a matroid called the *Fano matroid* F_7 . If k has characteristic different from 2, then it realizes a matroid called the *non-Fano matroid* F_7^+ , which is the same as the Fano matroid except that $\{4, 5, 6\}$ is a basis. The Fano matroid is realizable exactly over fields of characteristic 2, and the non-Fano matroid is realizable exactly over fields of characteristic different from 2.

Example 1.12. Given two matroids M_1 and M_2 of ranks r_1, r_2 on ground sets $\{1, \dots, n\}$ and $\{n+1, \dots, m\}$, their *direct sum* $M_1 \oplus M_2$ is the matroid of rank $r_1 + r_2$ whose bases are $\{B_1 \cup B_2 : B_i \text{ basis of } M_i\}$. The matroid $F_7 \oplus F_7^+$ is not realizable over any field.

We say that a matroid is *connected* if it cannot be written as a direct sum of two matroids on non-empty ground sets. For a matroid M on $[n]$, the dimension of $P(M)$ is $n - c(M)$, where $c(M)$ is the number of connected components of M . In particular, M is connected if and only if its polytope $P(M)$ has dimension $n - 1$.

Example 1.13. A face of a matroid polytope is a matroid polytope.

Example 1.14. Given a matroid M on $[n]$, the *dual matroid* M^\perp is the matroid with bases $\{[n] \setminus B : B \text{ basis of } M\}$. If M is realized by $L \subseteq k^n$, then M^\perp is realized by $L^\perp \subseteq k^n$, i.e., the image of the point $[L]$ under the isomorphism $\text{Gr}(r, n) \simeq \text{Gr}(n-r, n)$.

Remark 1.15. The Plücker equations (1) imply another property of the set of non-vanishing Plücker coordinates of a linear subspace. Suppose that B and B' are sets with $p_B([L])$ and $p_{B'}([L])$ both non-zero. Then, for each $i \in B \setminus B'$, (1) gives a relation of the form

$$p_B([L])p_{B'}([L]) = \sum_{j \in B', j \neq i} \pm p_{B \setminus i \cup j}([L])p_{B' \setminus j \cup i}([L]).$$

As the left-hand side is non-zero, there must be some term on the right-hand side which is non-zero. This implies that the set $S = \{B : p_B([L]) \neq 0\}$ satisfies the following property:

for all $B, B' \in S$ and $i \in B \setminus B'$, there is some $j \in B' \setminus B$ such that $B \setminus i \cup j$ and $B' \setminus j \cup i$ lie in S .

Surprisingly, this is equivalent to S being the bases of a matroid. See [Oxl11, Exercise 2.1.19].

1.3. Normality of torus-orbit closures. Proposition 1.5 describes the dense torus inside a torus-orbit closure in the Grassmannian, and Theorem 1.8 describes the normal fan. In order to use this to identify the torus-orbit closure (up to isomorphism), we need to know that torus-orbit closures are normal. This follows from the following highly non-obvious property of matroid polytopes. For a polytope P , let aP denote its a th dilate.

Theorem 1.16. [Wel76, Chapter 18.6, Theorem 3] *For any positive integer a , every lattice point in $aP(M)$ is a sum of a lattice points in $P(M)$.*

To prove the normality of torus-orbit closures, we discuss semigroup algebras of lattice polytopes.

Definition 1.17. *Let P be a lattice polytope in \mathbb{R}^n . The semigroup algebra R_P of P is the graded vector space $\bigoplus_{k \geq 0} \bigoplus_{p \in kP \cap \mathbb{Z}^n} k \cdot p$, equipped with the multiplication induced by $p \cdot q = p + q$.*

Proposition 1.18. [BH93, Theorem 6.1.4] *The semigroup algebra R_P of a lattice polytope is normal.*

If L is a realization of M , then the homogeneous coordinate ring of $\overline{T \cdot [L]}$ is the subalgebra of the semigroup algebra which is generated in degree 1. As Theorem 1.16 implies that the semigroup algebra is generated in degree 1, this implies that the homogeneous coordinate ring is equal to the semigroup algebra. As such a semigroup algebra is normal, this implies the following result.

Corollary 1.19. [Spe09, Proposition A.1] *Each torus-orbit closure $\overline{T \cdot [L]}$ in $\text{Gr}(r, n)$ is projectively normal in $\mathbb{P}^{\binom{n}{r}-1}$, i.e., the cone over it is normal in $\mathbb{A}^{\binom{n}{r}}$.*

In particular, torus-orbit closures in Grassmannians are normal, and so if $[L_1]$ and $[L_2]$ are points of the Grassmannian over k which realize the same matroid, then there is a toric isomorphism from $\overline{T \cdot [L_1]}$ to $\overline{T \cdot [L_2]}$.

Remark 1.20. Torus-orbit closures in homogeneous spaces of other types do not enjoy the nice properties of torus-orbit closures in the Grassmannian. Outside of type A , torus-orbit closures frequently fail to be normal. Except in a few other cases (such as maximal orthogonal Grassmannians), the vanishing or non-vanishing of the generalized Plücker coordinates of a point does not determine the stabilizer, i.e., there is no description of the stabilizer like in Proposition 1.5. See [ELS25, Example 2.5] for an example of two points in the Lagrangian Grassmannian $\text{LGr}(2, 4)$ whose torus-orbit closures have the same moment polytope, but different stabilizers. In that example, there is an interior lattice point in the moment polytope which may or may not be contained in \mathcal{A} . The generalized Plücker coordinates correspond to the vertices of the moment polytope, and knowing the vanishing or non-vanishing of generalized Plücker coordinates does not determine the lattice affinely generated by \mathcal{A} .

2. MATROID POLYTOPE SUBDIVISIONS

We will be interested in subdivisions of matroid polytopes into unions of matroid polytopes. It turns out that there is a natural source of these subdivisions arising from the geometry of the Grassmannian. This seems to have first been noticed by Kapranov [Kap93]. The properties of these subdivisions have taken on an increasingly important role in matroid theory, thanks to the development of a technique which allows one to use matroid polytope subdivisions to reduce statements to the case of realizable matroids. In particular, this allows one to use tools from algebraic geometry to prove results about non-realizable matroids. We will discuss this technique in Section 4.3.

Definition 2.1. *Let P be a polytope in \mathbb{R}^n . A polyhedral subdivision of P is a collection of polytopes Q_1, Q_2, \dots, Q_k which are contained in P and of the same dimension as P , such that $P = \bigcup_{i=1}^k Q_i$, the relative interiors of the Q_i are disjoint, and the intersection of any two of the Q_i is a face of both.*

An *interior face* of a polyhedral subdivision is a face of some Q_i which is contained in the relative interior of P . In particular, each of the Q_i is an interior face.

The easiest subdivisions to construct are *regular subdivisions*. These are subdivisions that are constructed by choosing a finite subset S of P which includes all of the vertices, and choosing a height function $h: S \rightarrow \mathbb{R}$, forming the polytope \hat{P} in $\mathbb{R}^n \times \mathbb{R}$ as the convex hull of $(p, h(p))$ for $p \in S$, and then projecting the lower faces.

Definition 2.2. A matroid polytope subdivision of a matroid polytope $P(M)$ is a polyhedral subdivision of $P(M)$ such that all of the polytopes appearing in the subdivision are matroid polytopes.

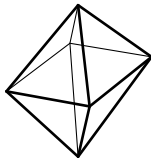


FIGURE 1. The matroid polytope $P(U_{2,4})$.

Example 2.3. There is a subdivision of the polytope $P(U_{2,4})$ with two full-dimensional cells, corresponding to the top half and the bottom half of the bipyramid in Figure 1. See Figure 2.

Because a face of a matroid polytope is a matroid polytope by Example 1.13, to check that a subdivision is a matroid polytope subdivision, it suffices to check that the top-dimensional cells are matroid polytopes.

We will now construct a large number of matroid polytope subdivisions. Let R be a discrete valuation ring, with fraction field K , valuation $\nu: K^\times \rightarrow \mathbb{Z}$, and residue field κ . Let L be a saturated R -submodule of R^n of rank r . This is the same thing as a map $\text{Spec } R \rightarrow \text{Gr}(r, n)$, i.e., the germ of a curve in $\text{Gr}(r, n)$. We have a matroid M induced by the subspace $L \otimes_R K$ of K^n .

For each basis B of M , we obtain an integer $\nu(p_B)$ by taking the valuation of the Plücker coordinate corresponding to B . The Plücker coordinates are defined up to scaling, and so these integers are defined up to translation by a global constant. We obtain a regular polyhedral subdivision of $P(M)$ by using the $\nu(p_B)$ as a height function on the vertices of $P(M)$. This subdivision does not change if we translate the heights by a global constant, so it does not depend on the choice of scaling of the Plücker coordinates.

Theorem 2.4. [Spe09, Proposition A.2] *The subdivision of $P(M)$ described above is a matroid polytope subdivision.*

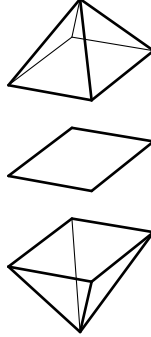
Example 2.5. Let R be the discrete valuation ring $k[t]_{(t)}$, and let L be the R -submodule of R^4 generated by the rows of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & t+1 \end{pmatrix}.$$

We have $M = U_{2,4}$. Using the matrix A to compute the Plücker coordinates of $L \otimes_R K$, we have $\nu(p_{34}) = 1$ and $\nu(p_B) = 0$ for all other bases. Then the induced matroid polytope subdivision is the subdivision in Example 2.3.

There are two ways to prove Theorem 2.4. The first is to observe that (1) implies some constraints on the valuation ν . Let B, B' be bases of M . Then, for each $i \in B \setminus B'$, we have

$$\nu(p_B) + \nu(p_{B'}) = \nu \left(\sum_{j \in B', j \neq i} \pm p_{B \setminus i \cup j} p_{B' \setminus j \cup i} \right).$$

FIGURE 2. A subdivision of $P(U_{2,4})$

By the definition of a valuation, this implies that there must be some $j \in B' \setminus B$ with

$$\nu(p_B) + \nu(p_{B'}) \geq \nu(p_{B \setminus i \cup j}) + \nu(p_{B' \setminus j \cup i}).$$

This means that ν is a *valuated matroid*, in the sense of the following definition.

Definition 2.6. Let M be a matroid. A function ν from the bases of M to \mathbb{R} is a valuated matroid if, for all bases B, B' of M and $i \in B \setminus B'$, there is $j \in B' \setminus B$ with

$$\nu(B) + \nu(B') \geq \nu(B \setminus i \cup j) + \nu(B' \setminus j \cup i).$$

The following result then immediately implies Theorem 2.4.

Proposition 2.7. [Spe08, Proposition 2.2] (see also [OPS19, Corollary 13]) A height function ν on the vertices of $P(M)$ induces a matroid polytope subdivision if and only if ν is a valuated matroid.

Example 2.8. Let M be a matroid of rank r on $[n]$, and let $\text{rk}_M: 2^{[n]} \rightarrow \mathbb{Z}$ be the rank function of M , given by $\text{rk}_M(S) = \max_{B \text{ basis}} |B \cap S|$. Then the regular subdivision of $U_{r,n}$ induced by the height function $h(S) = -\text{rk}_M(S)$ on the vertices of $P(U_{r,n})$ is a matroid polytope subdivision, because it can be checked that $-h$ is a valuated matroid. If M is connected, then $P(M)$ intersects the relative interior of $P(U_{r,n})$, and so it is a face of this subdivision.

The second approach to Theorem 2.4, which can be generalized without difficulty to torus-orbit closures in any homogeneous space, is to use a general description of projective toric varieties over a discrete valuation ring [Smi96, Section 2]. This is the proof in [Spe09, Proposition A.2], which shows the following. Inside of $\mathbb{P}_R^{\binom{n}{r}-1}$, projective space over $\text{Spec } R$, take the closure of $T \cdot [L \otimes_R K]$. Then the special fiber in $\mathbb{P}_\kappa^{\binom{n}{r}-1}$ is a reduced union of torus-orbit closures, which are in inclusion-preserving bijection with faces of the regular subdivision of $P(M)$ induced by ν .

Example 2.9. The example in Example 2.5 can be computed very explicitly. Inside of \mathbb{P}_R^5 , $\text{Gr}(2,4)$ is the hypersurface defined by the equation $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$. The torus-orbit closure of $[L]$ is 3-dimensional, and it is defined by the additional equation $(1+t)p_{12}p_{34} = -tp_{14}p_{23}$. When we set $t = 0$, the equations defining the torus-orbit closure become $p_{12}p_{34} = 0$ and $p_{14}p_{23} = p_{13}p_{24}$. This has two components, each of which is the toric variety of a square pyramid.

Remark 2.10. Outside of type A , the description of a degeneration of a torus-orbit closure in terms of a regular subdivision of the moment polytope is still valid, but the components may not be reduced.

The main purpose of these notes is to discuss the mathematics surrounding the following conjecture.

Conjecture 2.11. *A matroid polytope subdivision of $P(U_{r,n})$ has at most $\frac{(n-c-1)!}{(r-c)!(n-r-c)!(c-1)!}$ interior faces of dimension $n-c$ for $c \leq \min\{r, n-r\}$, and it has no interior faces of dimension less than $\max\{r, n-r\}$.*

Remarkably, there are many non-isomorphic subdivisions of $P(U_{r,n})$ which achieve equality. In [Spe08, Proposition 2.11], Speyer shows that Conjecture 2.11 implies sharp bounds on the total number of faces in a matroid polytope subdivision of $P(U_{r,n})$. Conjecture 2.11 was formulated in [Spe08], and it is known as Speyer’s f -vector conjecture. Major progress was made in [Spe09], where Conjecture 2.11 was reduced to verifying the non-negativity of a certain invariant of matroids. In [Spe09], Speyer proved this non-negativity for matroids realizable over a field of characteristic 0. See Section 5.7 for a summary of the known results.

We now outline the strategy introduced in [Spe09]. For each n , define the *valuative group* Val_n to be the subgroup of functions from \mathbb{R}^n to \mathbb{R} generated by the indicator functions of matroid polytopes on $[n]$. There is a direct sum decomposition $\text{Val}_n = \bigoplus_{r=0}^n \text{Val}_{r,n}$, where $\text{Val}_{r,n}$ is the subgroup generated by indicator functions of matroid polytopes of rank r . A function f from the set of matroids on $[n]$ to an abelian group A is said to be *valuative* if it factors through Val_n .

There are many easy examples of valutive invariants, such as the number of bases of a matroid, the volume of $P(M)$ (with respect to a volume form on the hyperplane $\sum x_i = r$), and the Ehrhart polynomial of $P(M)$. Surprisingly, most natural invariants of a matroid turn out to be valutive, such as the Tutte polynomial [AFR10] and most “algebro-geometric” invariants such as the Bergman class [BEST23] and various K -theoretic invariants [LLPP24].

If we have a matroid polytope decomposition of $P(M)$, then, by a version of inclusion-exclusion [AFR10, Theorem 3.5], we have an equality

$$(2) \quad (-1)^{c(M)+1} \mathbf{1}_{P(M)} = \sum_{Q \text{ interior face, } \dim Q = n-c} (-1)^{c+1} \mathbf{1}_Q.$$

Here $\mathbf{1}_P$ is the indicator function of P . Suppose that $f: \{\text{matroids on } [n]\} \rightarrow \mathbb{Z}$ is a valutive invariant such that $(-1)^{c+1} f(M) \geq 0$ for each matroid M with c connected components. Then, for any matroid polytope subdivision of $P(U_{r,n})$, we have

$$f(U_{r,n}) = \sum_{Q \text{ interior face, } \dim Q = n-c} (-1)^{c+1} f(Q).$$

Each term on the right-hand side is non-negative, and so this gives a bound on the complexity of the subdivision.

It is easy to construct valutive invariants which are non-negative on all matroids. However, there are fewer obvious ways to construct a valutive invariant with the sign property that is needed to use (2) to bound the complexity of a matroid polytope subdivision. As we will explain, such invariants arise naturally from K -theoretic constructions. Here are some elementary examples, although they are still, in some way, related to K -theory.

Example 2.12. The volume of the matroid polytope $P(M)$ is a valutive invariant which is 0 on all disconnected matroids. The (normalized) volume of $P(U_{r,n})$ is known to be the Eulerian number $A(n-1, r-1)$, the number of permutations in S_{n-1} with $r-1$ descents. A matroid polytope is a lattice polytope, so its volume is at least 1. This implies that a subdivision of $P(U_{r,n})$ has at most $A(n-1, r-1)$ top-dimensional interior faces. This is much weaker than Conjecture 2.11, which states there are at most $\binom{n-2}{r-1}$ top-dimensional interior faces.

Example 2.13. The Ehrhart polynomial $\text{ehr}_{P(M)}(t)$ is a valutive invariant of matroids. For a positive integer a , $\text{ehr}_{P(M)}(-a)$ is equal to $(-1)^{n-c}$ times the number of lattice points in the interior of the dilate $aP(M)$. In particular, this gives a bound on the complexity of a matroid polytope subdivision. It seems that these bounds are not strong enough to prove Conjecture 2.11.

To approach Conjecture 2.11, we will need more powerful tools to construct matroid invariants. Checking that they have the desired sign property will be even more difficult.

Remark 2.14. Derksen and Fink showed that Val_n is the quotient of the free abelian group with a basis labeled by matroids on $[n]$ by the subgroup generated by relations of the form (2) [DF10, Appendix A]. See [EHL23, Appendix A] for a discussion of related questions.

3. K-GROUPS OF SPACES

We now discuss the K -theory of schemes; see [Ful98, Section 15.1]. K -theory is a type of intersection theory, and, as is usual, it comes in two flavors, a homological flavor and a cohomological flavor. When the scheme is smooth, these flavors can be identified.

Let X be a scheme which is of finite type over k . Let $K_\circ(X)$ be the Grothendieck group of coherent sheaves on X , i.e., $K_\circ(X)$ is the quotient of the free abelian group generated by classes of coherent sheaves on X by the subgroup generated by relations corresponding to short exact sequences. In other words, if $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ is a short exact sequence of coherent sheaves on X , then $[\mathcal{F}_2] = [\mathcal{F}_1] + [\mathcal{F}_3]$ in $K_\circ(X)$. If there is an exact sequence

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \cdots \rightarrow \mathcal{F}_k \rightarrow 0,$$

then we have $\sum_{i=1}^k (-1)^i [\mathcal{F}_i] = 0$ in $K_\circ(X)$ by breaking this exact sequence into short exact sequences.

Let $K^\circ(X)$ be the Grothendieck group of vector bundles on X . If $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ is a short exact sequence of coherent sheaves on X and \mathcal{E} is a vector bundle, then $0 \rightarrow \mathcal{E} \otimes \mathcal{F}_1 \rightarrow \mathcal{E} \otimes \mathcal{F}_2 \rightarrow \mathcal{E} \otimes \mathcal{F}_3 \rightarrow 0$ is a short exact sequence. It follows that $K^\circ(X)$ is a ring, with multiplication given by tensor product, and $K_\circ(X)$ is a module over $K^\circ(X)$.

There is a map $K^\circ(X) \rightarrow K_\circ(X)$, obtained by viewing a vector bundle as a coherent sheaf. In general, this map is neither injective nor surjective, but if X is smooth then it is an isomorphism [Ful98, Appendix B.8], essentially because every coherent sheaf has a finite resolution by vector bundles. In this case, we denote both $K^\circ(X)$ and $K_\circ(X)$ by $K(X)$.

Example 3.1. There is an isomorphism $K(\mathbb{A}^n) \cong \mathbb{Z}$ sending a coherent sheaf \mathcal{F} on \mathbb{A}^n to its rank, i.e., the dimension of its stalk at the generic point. In other words, a coherent sheaf on \mathbb{A}^n is the same thing as a finitely generated module M over $k[x_1, \dots, x_n]$, and the rank is the dimension of $M \otimes_{k[x_1, \dots, x_n]} k(x_1, \dots, x_n)$. This map is injective because every finitely generated module over $k[x_1, \dots, x_n]$ has a finite free resolution [BH93, Corollary 2.2.14], and so is equal in $K(\mathbb{A}^n)$ to a multiple of the structure sheaf.

We list some properties of K -groups. Let $\pi: X \rightarrow Y$ be a map of schemes which are of finite type over k .

- (1) For any variety X , $K_\circ(X)$ is generated (as an abelian group) by classes of structure sheaves of (integral) subvarieties of X . This follows from [Sta25, Lemma 01YF].
 - (2) If π is proper, then there is a pushforward map $\pi_*: K_\circ(X) \rightarrow K_\circ(Y)$, which is defined by $\pi_*[\mathcal{F}] = \sum_{i \geq 0} (-1)^i [R^i \pi_* \mathcal{F}]$. That this is well-defined follows from the long exact sequence on derived pushforwards.
 - (3) There is a pullback map $\pi^*: K^\circ(Y) \rightarrow K^\circ(X)$ defined by $\pi^*[\mathcal{E}] = [\pi^* \mathcal{E}]$, because exact sequences of locally free sheaves remain exact when pulled back.
 - (4) If π is proper, $a \in K^\circ(Y)$, and $x \in K_\circ(X)$, then
- (3)
$$\pi_*(\pi^* a \cdot x) = a \cdot \pi_* x.$$

This follows from the usual projection formula: if \mathcal{F} is a sheaf on X and \mathcal{E} is a locally free sheaf on Y , then $R^i \pi_* \mathcal{F} \otimes \mathcal{E} \simeq R^i \pi_* (\mathcal{F} \otimes \pi^* \mathcal{E})$.

The property (3) is known as the *projection formula*.

Remark 3.2. Note that if X and Y are smooth, then there is a pullback map $\pi^*: K_\circ(Y) \rightarrow K_\circ(X)$, using the isomorphism between K° and K_\circ . Given a coherent sheaf \mathcal{F} on Y , $\pi^*[\mathcal{F}]$ is usually not equal to $[\pi^*\mathcal{F}]$. Rather, we resolve \mathcal{F} by vector bundles and pull those back.

Particularly important is the case when X is a projective variety and Y is a point. Then $K(Y) = \mathbb{Z}$, and the pushforward map $K_\circ(X) \rightarrow \mathbb{Z}$ is equal to the Euler characteristic. That is, the pushforward of $[\mathcal{F}]$ to a point is equal to

$$\chi(X, \mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{F}).$$

There are two practical ways to compute the Euler characteristic, which we now describe.

If \mathcal{L} is a line bundle, then the function $a \mapsto \chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes a})$ is a polynomial in a . If \mathcal{L} is ample, then for a sufficiently large, Serre vanishing implies that $\chi(X, \mathcal{F} \otimes \mathcal{L}^{\otimes a})$ agrees with $\dim H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes a})$. In this case, we can compute $\chi(X, \mathcal{F})$ by forming the graded module $\bigoplus_{a \geq 0} H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes a})$, finding the polynomial $p(a)$ which agrees with $\dim H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes a})$ for a sufficiently large, and then setting $a = 0$.

For example, we compute the Euler characteristic of the line bundle $\mathcal{O}(a)$ on \mathbb{P}^N . The corresponding graded module is $k[x_0, \dots, x_N][a]$, i.e., the ring $k[x_0, \dots, x_N]$, with 1 placed in degree $-a$, viewed as a module over $k[x_0, \dots, x_N]$. For $i \geq -a$, the dimension of the i th graded piece is $\binom{N+a+i}{N}$. Setting $i = 0$, we see that

$$(4) \quad \chi(\mathbb{P}^N, \mathcal{O}(a)) = \binom{N+a}{N}.$$

If X is embedded into \mathbb{P}^N for some N , then a coherent sheaf on X can be pushed forward to a coherent sheaf on \mathbb{P}^N . A coherent sheaf \mathcal{F} on \mathbb{P}^N corresponds to a graded module $M = \bigoplus_{i \in \mathbb{Z}} H^0(\mathbb{P}^N, \mathcal{F}(i))$ over the polynomial ring $k[x_0, \dots, x_N]$. This correspondence is a bit subtle in general, but an exact sequence of graded modules induces an exact sequence of sheaves on \mathbb{P}^N . By the Hilbert syzygy theorem [BH93, Corollary 2.2.14], any graded module M over $k[x_0, \dots, x_N]$ has a graded free resolution

$$0 \rightarrow k[x_0, \dots, x_N] \otimes V_N \rightarrow k[x_0, \dots, x_N] \otimes V_{N-1} \rightarrow \dots \rightarrow k[x_0, \dots, x_N] \otimes V_0 \rightarrow M \rightarrow 0,$$

where each V_i is a graded vector space. This induces an exact sequence of sheaves on \mathbb{P}^N involving \mathcal{F} , the sheafification of M , and a bunch of direct sums of line bundles. This writes $[\mathcal{F}]$ as a linear combination of classes of line bundles, and one can then use (4) to compute the Euler characteristic.

We now describe the K -theory of \mathbb{P}^N . This will be done more generally for Grassmannians later.

As shown above, $K(\mathbb{P}^N)$ is spanned (as an abelian group) by classes of line bundles. Repeatedly using the short exact sequence $0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_H \rightarrow 0$, where H is a hyperplane, and inducting on the dimension, we see that $K(\mathbb{P}^N)$ is generated as an abelian group by powers of $[\mathcal{O}_H]$. Using the Koszul complex, i.e., the exact sequence

$$0 \rightarrow \mathcal{O}(-(N+1)) \rightarrow \mathcal{O}(-N)^{\oplus N+1} \rightarrow \dots \rightarrow \mathcal{O}(-1)^{\oplus N+1} \rightarrow \mathcal{O} \rightarrow 0,$$

we see that $[\mathcal{O}_H]^{N+1} = ([\mathcal{O}] - [\mathcal{O}(-1)])^{N+1} = 0$, and so $K(\mathbb{P}^N)$ is generated as an abelian group by $[\mathcal{O}], [\mathcal{O}_H], [\mathcal{O}_H]^2, \dots, [\mathcal{O}_H]^N$.

We claim that these classes form a basis for $K(\mathbb{P}^N)$. There is a pairing $K(\mathbb{P}^N) \times K(\mathbb{P}^N) \rightarrow \mathbb{Z}$ given by $(a, b) \mapsto \chi(\mathbb{P}^N, ab)$. We have

$$\chi(\mathbb{P}^N, [\mathcal{O}_H]^i [\mathcal{O}_H]^j) = \begin{cases} 1, & \text{if } i+j \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix whose (i, j) th entry is $\chi(\mathbb{P}^N, [\mathcal{O}_H]^i [\mathcal{O}_H]^j)$ is nondegenerate. This means that this pairing does not descend to any quotient of the span of $[\mathcal{O}], [\mathcal{O}_H], \dots, [\mathcal{O}_H]^N$, and so these classes must be linearly

independent in $K(\mathbb{P}^N)$. We conclude that

$$K(\mathbb{P}^N) \simeq \mathbb{Z}[x]/(x^{N+1}), \quad x = [\mathcal{O}_H],$$

where H is a hyperplane. Note that, for instance by Proposition 3.4 below, $[\mathcal{O}_H]^k$ is the class of a linear subspace of \mathbb{P}^N of codimension k for each $k \leq N$.

The nondegeneracy of this pairing means that we can compute the class of a sheaf $[\mathcal{F}]$ in $K(\mathbb{P}^N)$ in terms of the Hilbert polynomial of \mathcal{F} . That is, we have $[\mathcal{F}] = \sum a_i [\mathcal{O}_H]^{N-i}$, where

$$(5) \quad \chi(\mathbb{P}^N, \mathcal{F} \otimes \mathcal{O}(a)) = \sum_{q=0}^N a_q \binom{a+q}{q}.$$

We see that $K_o(\mathbb{P}^N)$ can be identified with the subspace of polynomial functions on \mathbb{Z} which is spanned by the functions $a \mapsto \binom{a+q}{q}$, for $q = 0, \dots, N$. This is exactly the space of *numerical polynomials* of degree at most N , i.e., polynomials $p \in \mathbb{Q}[t]$ such that $p(a)$ is an integer for every $a \in \mathbb{Z}$; see [BH93, Lemma 4.1.4].

3.1. Connections with intersection theory. We now explain the sense in which K -theory is a form of intersection theory. First we describe the product on $K_o(X)$ when X is smooth. A coherent sheaf \mathcal{F} has a finite resolution $0 \rightarrow \mathcal{E}_k \rightarrow \dots \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0$ by vector bundles [Ful98, Appendix B.8], so we can write $[\mathcal{F}] = \sum_{i=0}^k (-1)^i [\mathcal{E}_i]$ in $K_o(X)$. Given a sheaf \mathcal{G} , define the i th *Tor sheaf* $\mathcal{T}or_i(\mathcal{F}, \mathcal{G})$ as the homology at $\mathcal{E}_i \otimes \mathcal{G}$ of the complex

$$(6) \quad 0 \rightarrow \mathcal{E}_k \otimes \mathcal{G} \rightarrow \dots \rightarrow \mathcal{E}_0 \otimes \mathcal{G} \rightarrow 0.$$

These sheaves behave like usual Tor groups; see [Sta25, Section 06Y7]. In particular, $\mathcal{T}or_0(\mathcal{F}, \mathcal{G}) = \mathcal{F} \otimes \mathcal{G}$, and if $0 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G}_2 \rightarrow \mathcal{G}_3 \rightarrow 0$ is a short exact sequence of sheaves, then the Tor sheaves form a long exact sequence

$$\dots \rightarrow \mathcal{T}or_2(\mathcal{F}, \mathcal{G}_3) \rightarrow \mathcal{T}or_1(\mathcal{F}, \mathcal{G}_1) \rightarrow \mathcal{T}or_1(\mathcal{F}, \mathcal{G}_2) \rightarrow \mathcal{T}or_1(\mathcal{F}, \mathcal{G}_3) \rightarrow \mathcal{F} \otimes \mathcal{G}_1 \rightarrow \mathcal{F} \otimes \mathcal{G}_2 \rightarrow \mathcal{F} \otimes \mathcal{G}_3 \rightarrow 0.$$

Because \mathcal{F} has a finite resolution by vector bundles, this long exact sequence is finite. By breaking up the complex (6) into short exact sequences, we see that

$$[\mathcal{F}] \cdot [\mathcal{G}] = \sum_{i \geq 0} (-1)^i [\mathcal{T}or_i(\mathcal{F}, \mathcal{G})] = [\mathcal{F} \otimes \mathcal{G}] + \sum_{i \geq 1} (-1)^i [\mathcal{T}or_i(\mathcal{F}, \mathcal{G})].$$

This is a finite sum because \mathcal{F} has a finite resolution by vector bundles. This describes the product in $K_o(X)$ of the classes of two coherent sheaves \mathcal{F} and \mathcal{G} as a “main term” $[\mathcal{F} \otimes \mathcal{G}]$ and “error terms” given by Tor sheaves. Note, however, that this “main term” depends on the sheaves \mathcal{F} and \mathcal{G} , not merely on their classes in $K_o(X)$.

Importantly, Tor sheaves can be analyzed locally: the stalk of $\mathcal{T}or_i(\mathcal{F}, \mathcal{G})$ at a point $x \in X$ is the Tor group $\mathrm{Tor}_i^{\mathcal{O}_{x,x}}(\mathcal{F}_x, \mathcal{G}_x)$ between the stalks over the local ring at x .

Example 3.3. Let p and q be distinct points on \mathbb{P}^1 . We have $[\mathcal{O}_p] = [\mathcal{O}_q] = [\mathcal{O}] - [\mathcal{O}(-1)]$ in $K(\mathbb{P}^1)$. We have $\mathcal{T}or_i(\mathcal{O}_p, \mathcal{O}_q) = 0$, so $[\mathcal{O}_p] \cdot [\mathcal{O}_q] = 0$. On the other hand, by tensoring the resolution $0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow 0$ of \mathcal{O}_p with \mathcal{O}_p , we see that $\mathcal{T}or_0(\mathcal{O}_p, \mathcal{O}_p) = \mathcal{T}or_1(\mathcal{O}_p, \mathcal{O}_p) = \mathcal{O}_p$, and $\mathcal{T}or_i(\mathcal{O}_p, \mathcal{O}_p) = 0$ for $i > 1$.

If Y and Z are smooth subvarieties of a smooth ambient variety X , then we say that Y and Z are *transverse* at a point $x \in Y \cap Z$ if the tangent spaces $T_x Y$, $T_x Z$ to Y and Z at x are transverse inside of $T_x X$, i.e., $\mathrm{codim} T_x Y \cap T_x Z = \mathrm{codim} T_x Y + \mathrm{codim} T_x Z$, where codim is the codimension inside of $T_x X$.

Proposition 3.4. *If Y and Z are smooth subvarieties of a smooth variety X which meet transversely, then $[\mathcal{O}_Y] \cdot [\mathcal{O}_Z] = [\mathcal{O}_{Y \cap Z}]$.*

Proof. It suffices to show that $\mathcal{T}or_i(\mathcal{O}_Y, \mathcal{O}_Z) = 0$ for all $i > 0$. In order to prove this, it suffices to show that for every closed point $x \in X$, the Tor group in the regular local ring $\mathcal{O}_{X,x}$ vanishes. Because Y and Z are smooth, the ideal of Y in $\mathcal{O}_{X,x}$ is generated by a regular sequence y_1, \dots, y_c , and the ideal of Z is generated by a regular sequence z_1, \dots, z_d . The transversality condition guarantees that $y_1, \dots, y_c, z_1, \dots, z_d$ is a regular sequence. Using the Koszul complex, one can then compute that $\mathcal{T}or_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(y_1, \dots, y_c), \mathcal{O}_{X,x}/(z_1, \dots, z_d)) = 0$ for $i > 0$. See [Bri02, Lemma 1] for a strengthening. \square

Usually one does intersection theory using a graded ring like singular cohomology or the Chow ring. K -theory captures some refined information, as illustrated by the following examples.

Example 3.5. Let Y, Z be subvarieties of a variety X . Then there is an exact sequence

$$0 \rightarrow \mathcal{O}_{Y \cup Z} \rightarrow \mathcal{O}_Y \oplus \mathcal{O}_Z \rightarrow \mathcal{O}_{Y \cap Z} \rightarrow 0,$$

so $[\mathcal{O}_{Y \cup Z}] = [\mathcal{O}_Y] + [\mathcal{O}_Z] - [\mathcal{O}_{Y \cap Z}]$. Note that, if Y and Z are the same dimension, then we have $[Y \cup Z] = [Y] + [Z]$ in Chow or in cohomology, so K -theory records some information about how Y and Z meet.

Example 3.6. Let Y be the union of two skew lines in \mathbb{P}^3 . Then $[\mathcal{O}_Y] = 2[\mathcal{O}_H]^2$ in $K(\mathbb{P}^3)$, as $[\mathcal{O}_H]^2$ is the class of a line. Let Z be the union of two lines in \mathbb{P}^3 that meet at a point. We have $\chi(Z, \mathcal{O}(a)) = 2a + 1$, so $[\mathcal{O}_Z] = 2[\mathcal{O}_H]^2 - [\mathcal{O}_H]^3$ by (5). Note that Y and Z have the same class in the cohomology ring $H^*(\mathbb{P}^3)$.

Furthermore, it is often the case that if Z_1 and Z_2 are subvarieties of a smooth scheme X which can be deformed to each other, then $[\mathcal{O}_{Z_1}] = [\mathcal{O}_{Z_2}] \in K(X)$, as in the following examples.

Example 3.7. Let X be a smooth projective variety such that $K(X)$ is a finitely generated torsion-free abelian group and the pairing $K(X) \times K(X) \rightarrow \mathbb{Z}$ given by $(a, b) \mapsto \chi(X, ab)$ is nondegenerate. For example, this holds for Grassmannians or smooth projective toric varieties. Let C be a connected curve, and suppose that \mathcal{Z} is a subscheme of $X \times C$ which is flat over C . Let p be the map from $X \times C$ to C . Then for any vector bundle \mathcal{E} on X and points $q_1, q_2 \in C$, we have $\chi(p^{-1}(q_1) \cap \mathcal{Z}, \mathcal{E}|_{p^{-1}(q_1)}) = \chi(p^{-1}(q_2) \cap \mathcal{Z}, \mathcal{E}|_{p^{-1}(q_2)})$ because the Euler characteristic is locally constant in proper flat families [Vak25, Theorem 24.7.1]. This implies that $[\mathcal{O}_{p^{-1}(q_1) \cap \mathcal{Z}}] = [\mathcal{O}_{p^{-1}(q_2) \cap \mathcal{Z}}]$ in $K(X)$.

Example 3.8. Let X be a variety, and let \mathcal{F} be a sheaf on $X \times \mathbb{A}^1$ which is flat over \mathbb{A}^1 . Then for any point $q \in \mathbb{A}^1$, we obtain a class in $K_o(X)$ given by the restriction of \mathcal{F} to the fiber over q . This class is independent of the choice of fiber. To see this, note that pulling back along the flat map $X \times \mathbb{A}^1 \rightarrow X$ induces an isomorphism $K_o(X) \cong K_o(X \times \mathbb{A}^1)$ [Wei13, Theorem II.6.5.1]. Also, because the inclusion of a fiber $i: X \hookrightarrow X \times \mathbb{A}^1$ is a regular embedding, there is a ‘‘Gysin pullback’’ $i^!: K_o(X \times \mathbb{A}^1) \rightarrow K_o(X)$, which sends the class of a coherent sheaf \mathcal{F} to $[\mathcal{T}or_0(\mathcal{O}_X, \mathcal{F})] - [\mathcal{T}or_1(\mathcal{O}_X, \mathcal{F})]$; see [KS04, Section 2.2]. This map is the inverse to the isomorphism $K_o(X) \rightarrow K_o(X \times \mathbb{A}^1)$. If \mathcal{F} is flat over \mathbb{A}^1 , then $i^![\mathcal{F}]$ is the class of the restriction of \mathcal{F} to the fiber. As this is the inverse to the isomorphism $K_o(X) \rightarrow K_o(X \times \mathbb{A}^1)$, this implies that the class of the restriction is independent of the choice of fiber.

However, in general, subschemes or sheaves which can be deformed to each other are not equal in $K(X)$. For example, the units group $K(X)^\times$ contains a copy of the Picard group $\text{Pic}(X)$.

Example 3.9. Let C be a smooth projective connected curve over k . There is a map $\det: K(C) \rightarrow \text{Pic}(C)$ which sends the class of a vector bundle to the class of its determinant, and a map $\text{rk}: K(C) \rightarrow \mathbb{Z}$ which sends the class of a vector bundle to its rank. These maps induce an isomorphism $K(C) \cong \mathbb{Z} \oplus \text{Pic}(C)$ [Wei13, Proposition II.8.2.1].

If C is a smooth projective geometrically connected curve of genus $g > 0$, then $\text{Pic}(C)$ is the k -points of a group scheme with a positive-dimensional identity component. The classes of two non-isomorphic line

bundles in the identity component of the Picard group will not be equal in $K(C)$, even though they can be deformed into each other. Furthermore, if k is uncountable, then $K(C)$ will be uncountable.

Example 3.10. Let X be a variety which has an action of \mathbb{G}_m , and let Z be a subscheme of X . We can degenerate Z to a subscheme of X which is fixed by the \mathbb{G}_m action, as follows. Let \mathcal{Z} be the subscheme $\{(t \cdot z, t) : z \in Z, t \in \mathbb{G}_m\}$ of $X \times \mathbb{G}_m$, and $\overline{\mathcal{Z}}$ be its closure in $X \times \mathbb{A}^1$. Then $\mathcal{O}_{\overline{\mathcal{Z}}}$ is flat over \mathbb{A}^1 . Let Z_0 be the (scheme-theoretic) fiber of $\overline{\mathcal{Z}}$ over 0, which is fixed by \mathbb{G}_m . Then Example 3.8 implies that $[\mathcal{O}_Z] = [\mathcal{O}_{Z_0}]$ in $K(X)$. This type of degeneration is often called a *Gröbner degeneration*.

Remark 3.11. For any manifold M , the *topological K-theory* $K^{\text{top}}(M)$ is the Grothendieck group of complex topological vector bundles on M . It has a ring structure given by tensor product. If M is a compact manifold with an almost complex structure, then there is a map $\chi: K^{\text{top}}(M) \rightarrow \mathbb{Z}$ constructed using the Thom isomorphism; see [BFM79, Section 3.2]. It can in fact be constructed somewhat more generally, for even-dimensional manifolds with a spin^c structure; see [ABS64]. It is known that $K^{\text{top}}(M)$ is a finitely generated abelian group, and that the pairing $K^{\text{top}}(M)/\text{tors} \otimes K^{\text{top}}(M)/\text{tors} \rightarrow \mathbb{Q}$ given by $(a, b) \mapsto \chi(ab)$ is unimodular, where tors is the torsion subgroup of $K^{\text{top}}(M)$. See [BFM79, Section 3.4]. If X is a smooth complex projective variety, then there is a map from $K(X)$ to $K^{\text{top}}(X)$, given by considering the underlying topological vector bundle of a locally free sheaf, and the map χ coincides with the sheaf Euler characteristic $\chi(X, -): K(X) \rightarrow \mathbb{Z}$ [BFM79, Section 4]. The map $K(X) \rightarrow K^{\text{top}}(X)$ is in general neither injective nor surjective, but it is an isomorphism if X is a toric variety, a Grassmannian, or, more generally, has an affine paving in the sense of Proposition 4.2. When this holds, the hypothesis of Example 3.7 will hold.

3.2. Connections with Chow. There is a direct connection between K -groups and Chow groups, which are an algebraic version of homology. If X is a scheme which is of finite type over a field, $K_{\circ}(X)$ is equipped with a decreasing filtration $F_0 \supseteq F_1 \supseteq \cdots \supseteq F_{\dim X} \supseteq 0$ called the *coniveau filtration*. Here F_i is the subgroup of $K_{\circ}(X)$ generated by classes of coherent sheaves whose support has codimension at least i . There is a surjective map from the Chow groups $A_{\bullet}(X)$ to the associated graded $\text{gr} K_{\circ}(X)$, obtained by sending the class of a subvariety $[Z]$ to $[\mathcal{O}_Z]$, and this map becomes an isomorphism after tensoring with \mathbb{Q} [Ful98, Example 15.1.5].

When X is smooth, the Chow groups have a ring structure which is graded by codimension. This ring is contravariant with respect to maps between smooth schemes. We denote this ring by $A^{\bullet}(X)$. If \mathcal{E} is a vector bundle on X , then there are *Chern classes* $c_i(\mathcal{E}) \in A^i(X)$ for $i = 0, 1, \dots, \text{rk}(\mathcal{E})$. For example, $c_0(\mathcal{E}) = 1$ and $c_1(\mathcal{E})$ is represented by the divisor of a rational section of $\det \mathcal{E}$, so if \mathcal{E} is a line bundle, then \mathcal{E} is determined by its first Chern class.

If \mathcal{E} is globally generated, then its Chern classes can be computed as the class of the locus where an appropriate number of sections fail to be linearly independent. For instance, if \mathcal{E} is globally generated, then $c_{\text{rk}(\mathcal{E})}(\mathcal{E})$ is the class of the vanishing locus of a general section of \mathcal{E} .

We define the *total Chern class* of a vector bundle \mathcal{E} as $c(\mathcal{E}) := 1 + c_1(\mathcal{E}) + \cdots \in A^{\bullet}(X)$. If $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ is a short exact sequence of vector bundles, then $c(\mathcal{E}) \cdot c(\mathcal{G}) = c(\mathcal{F})$. From this, one can deduce that there is a map $c: K(X) \rightarrow A^{\bullet}(X)^{\times}$ which turns addition into multiplication.

The Chow ring of X is related to $K(X)$ via a ring map called the *Chern character* $\text{ch}: K(X) \rightarrow A^{\bullet}(X)_{\mathbb{Q}}$. If \mathcal{L} is a line bundle on X , then it defines a divisor class $c_1(\mathcal{L}) \in A^1(X)$. We have

$$(7) \quad \text{ch}([\mathcal{L}]) = e^{c_1(\mathcal{L})} := 1 + c_1(\mathcal{L}) + \frac{c_1(\mathcal{L})^2}{2!} + \cdots,$$

where the sum is finite because $A^i(X) = 0$ for $i > \dim X$. The Chern character commutes with pullback: if $f: Y \rightarrow X$ is a map between smooth schemes, then $f^* \text{ch}(a) = \text{ch}(f^*a)$ for all $a \in K(X)$. It turns out that this property and (7) characterize the Chern character: if \mathcal{E} is any vector bundle on X , then there is a map $f: Z \rightarrow X$ from a smooth scheme Z such that the pullback maps $K(X) \rightarrow K(Z)$ and $A^{\bullet}(X) \rightarrow A^{\bullet}(Z)$ are

injective, and $[f^*\mathcal{E}]$ is a sum of classes of line bundles in $K(Z)$. One can take Z to be an iterated projective bundle over X . If V is a subscheme of X , then

$$(8) \quad \text{ch}([\mathcal{O}_V]) = [V] + \text{higher codimension terms},$$

see [Ful98, Example 15.1.5 and 15.2.16]. After tensoring $K(X)$ with \mathbb{Q} , the Chern character becomes an isomorphism $K(X)_{\mathbb{Q}} \rightarrow A^\bullet(X)_{\mathbb{Q}}$.

If X is smooth and projective of dimension d , there is a map $\text{deg}: A^d(X) \rightarrow \mathbb{Z}$ which sends the class of any point to 1 (or more generally, if k is not algebraically closed, the dimension of its residue field over the ground field). This map is related to the Euler characteristic by the *Hirzebruch–Riemann–Roch theorem*.

Theorem 3.12. *Let X be smooth and projective. Then there is a class $\text{td}(X)$ in $A^\bullet(X)$, called the Todd class, such that, for any $a \in K(X)$, we have*

$$\chi(X, a) = \text{deg}(\text{ch}(a) \cdot \text{td}(X)).$$

See [Ful98, Corollary 15.2.1]. The Todd class can be explicitly computed in terms of the tangent bundle of X . When X is a curve, Theorem 3.12 reduces to the usual Riemann–Roch theorem. Theorem 3.12 can be generalized to describe the connection between the proper pushforward map in K -theory and in Chow; see [Ful98, Theorem 15.2]. This generalization is called *Grothendieck–Riemann–Roch*.

3.3. Additional structures. Although the ring $K^\circ(X)$ is not graded, it does have some additional structure. Namely, it is a λ -ring. The definition of a λ -ring is a bit technical-looking, so we just describe what we will use this structure for. The operation which sends a vector bundle \mathcal{E} to its i th exterior power $\wedge^i \mathcal{E}$ is well-defined on $K^\circ(X)$, i.e., the class of $\wedge^i \mathcal{E}$ in $K^\circ(X)$ only depends on the class of \mathcal{E} in $K^\circ(X)$. For each i , there is a map $\wedge^i: K^\circ(X) \rightarrow K^\circ(X)$ which satisfies $\wedge^i[\mathcal{E}] = [\wedge^i \mathcal{E}]$ for every vector bundle \mathcal{E} . This operation \wedge^i is not linear, but it satisfies some natural compatibilities with the ring structure. For example, $\wedge^i(a+b) = \sum_{s+t=i} \wedge^s(a) \wedge^t(b)$, mirroring the formula $\wedge^i(\mathcal{E}_1 \oplus \mathcal{E}_2) \simeq \bigoplus_{s+t=i} \wedge^s \mathcal{E}_1 \otimes \wedge^t \mathcal{E}_2$. See [Gro71, Exposé V, Exemple 3.9.1].

It turns out that the operations \wedge^i are enough to define a host of other operations on any λ -ring, such as $K^\circ(X)$. For instance, for each $j \geq 0$ there is an operation $\text{Sym}^j: K^\circ(X) \rightarrow K^\circ(X)$, such that $\text{Sym}^j[\mathcal{E}] = [\text{Sym}^j \mathcal{E}]$ for any vector bundle \mathcal{E} .

Furthermore, there is a ring involution $(-)^{\vee}: K^\circ(X) \rightarrow K^\circ(X)$, defined by sending the class of a vector bundle \mathcal{E} to the class of the dual vector bundle \mathcal{E}^{\vee} . This descends to $K^\circ(X)$ because the dual of a short exact sequence of vector bundles is a short exact sequence of vector bundles.

4. K-THEORY OF THE GRASSMANNIAN

We will now describe the K -theory of the Grassmannian. Recall that GL_n acts on $\text{Gr}(r, n)$, and so B , the Borel subgroup of upper triangular matrices in GL_n , acts on $\text{Gr}(r, n)$.

Proposition 4.1. [Ful97, Section 9.4] *The action of B on $\text{Gr}(r, n)$ has finitely many orbits, one for each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ with $n - r \geq \lambda_1 \geq \dots \geq \lambda_r \geq 0$. The orbit corresponding to λ is isomorphic to $\mathbb{A}^{\lambda_1 + \dots + \lambda_r}$.*

Let Ω_λ denote the closure of the B -orbit corresponding to λ . This is called the *Schubert variety* associated to λ . Set $|\lambda| = \lambda_1 + \dots + \lambda_r$, so $\dim \Omega_\lambda = |\lambda|$.

We say that a variety X has an *affine paving* if there is a filtration of X by closed subschemes $\emptyset \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq X_k = X$ such that $X_i \setminus X_{i-1}$ is isomorphic to a disjoint union of affine spaces, possibly of different dimensions. We say that these affine spaces, viewed as subsets of X , are the *cells* of X .

Proposition 4.2. *Let X be a variety with an affine paving. Then the classes of the structure sheaves of the closures of the cells of X form a basis for $K_\circ(X)$.*

The analogue of Proposition 4.2 is well-known for Chow groups; see [Ful97, Lemma B.3.6] for the case when $k = \mathbb{C}$. When X is smooth, Proposition 4.2 can easily be deduced from the version for Chow group by applying the Chern character, but proving it in general requires more machinery. First, note that if X is a variety, then the natural map $K_{\circ}(X^{\text{red}}) \rightarrow K_{\circ}(X)$ is an isomorphism [Wei13, Corollary II.6.3.2]. Let U be an open subset of X , and let Z be the complementary closed subset; note that $K_{\circ}(Z)$ is independent of the choice of scheme structure on Z . Then there is a localization exact sequence

$$K_{\circ}(Z) \rightarrow K_{\circ}(X) \rightarrow K_{\circ}(U) \rightarrow 0,$$

where the first map is pushforward under the inclusion of Z into X , and the second map is the pullback under the flat map $U \hookrightarrow X$; see [Wei13, Application II.6.4.2].

To prove Proposition 4.2, we will use a sophisticated tool. For each variety X , there is a sequence of groups $\{G_i(X)\}_{i \geq 0}$, with $G_0(X)$ identified with $K_{\circ}(X)$. See [Wei13, Chapter IV]. These groups have similar functoriality properties to K_{\circ} . Importantly, if X is a variety, U is an open subset, and $Z = X \setminus U$, then there is an exact sequence

$$\cdots \rightarrow G_2(U) \rightarrow G_1(Z) \rightarrow G_1(X) \rightarrow G_1(U) \rightarrow G_0(Z) \rightarrow G_0(X) \rightarrow G_0(U) \rightarrow 0,$$

extending the localization sequence to a long exact sequence. See [Wei13, Example V.6.11].

Proof of Proposition 4.2. Inducting on the length of the filtration in the definition of an affine paving, we may assume that X contains an open set U which is a disjoint union of affine spaces, and $Z := X \setminus U$ has an affine paving. By Example 3.1, $K_{\circ}(U)$ is generated by the structure sheaves of the components, and the image of the class of the structure sheaf of a cell closure in X under the map $K_{\circ}(X) \rightarrow K_{\circ}(U)$ is the class of the structure sheaf of the cell. By induction, $K_{\circ}(Z)$ is generated by classes of structure sheaves of closures of cells.

As U is a disjoint union of affine spaces and k is algebraically closed, it follows from [Wei13, Theorem V.6.3] and [Wei13, Corollary VI.1.3.1 and Theorem VI.1.6] that $G_1(U)$ is a divisible group. As $K_{\circ}(Z)$ is a free abelian group of finite rank, the map $G_1(U) \rightarrow K_{\circ}(Z)$ is 0. Therefore we have a short exact sequence

$$0 \rightarrow K_{\circ}(Z) \rightarrow K_{\circ}(X) \rightarrow K_{\circ}(U) \rightarrow 0.$$

The result then follows from the five lemma. \square

Corollary 4.3. *The classes of B -orbit closures $\{[\mathcal{O}_{\Omega_{\lambda}}]\}$ form a basis for $K(\text{Gr}(r, n))$.*

Note that, when $r = 1$, this recovers the basis for $K(\mathbb{P}^{n-1})$ described previously.

4.1. K-theoretic positivity on the Grassmannian. As a consequence of Corollary 4.3, given any subscheme X of $\text{Gr}(r, n)$, we can write $[\mathcal{O}_X] = \sum_{\lambda} a_{\lambda} [\mathcal{O}_{\Omega_{\lambda}}]$ for some integers a_{λ} . We will show that, in good circumstances, these integers a_{λ} have certain positivity properties.

The first observation is that $a_{\lambda} = 0$ if $|\lambda| > \dim X$. This can be seen by using the relationship between $K_{\circ}(\text{Gr}(r, n))$ and the Chow groups $A_{\bullet}(\text{Gr}(r, n))$. This connection also implies that if $|\lambda| = \dim X$, then a_{λ} is the coefficient of $[\Omega_{\lambda}]$ in the expansion of the fundamental class of X in $A_{\dim X}(\text{Gr}(r, n))$ in terms of the classes of Schubert varieties. It follows from the Kleiman–Bertini theorem [Ful98, Lemma B.9.2] that the coefficients of the expansion of $[X]$ in terms of classes of Schubert varieties can be computed as the length of the dimensionally transverse intersection of X with a subvariety of $\text{Gr}(r, n)$, and so the a_{λ} are non-negative in this case.

When $|\lambda| < \dim X$, then the a_{λ} capture more refined information about X . For example, they can be used to distinguish two skew lines in \mathbb{P}^3 from two lines that meet at a point.

We say that an integral scheme X over a field of characteristic 0 has *rational singularities* if a resolution $\pi: \tilde{X} \rightarrow X$, i.e., a proper birational map from a smooth scheme \tilde{X} , has $\pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and $R^i \pi_* \mathcal{O}_{\tilde{X}} = 0$ for

$i > 0$. Importantly, this holds for smooth schemes. In fact, if $\pi: \tilde{X} \rightarrow X$ is a proper and birational map between smooth schemes over a field of any characteristic, then $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and $R^i\pi_*\mathcal{O}_{\tilde{X}} = 0$ for $i > 0$ [CR15].

This implies that, if the vanishing in the definition of rational singularities holds for one resolution, then it holds for any resolution, because, over a field of characteristic 0, it is a consequence of Hironaka's resolution of singularities that any two resolutions can be dominated by a third. In particular, if $\pi: \tilde{X} \rightarrow X$ is a resolution of a variety with rational singularities, then $\pi_*[\mathcal{O}_{\tilde{X}}] = [\mathcal{O}_X]$.

Example 4.4. If X is a (normal) toric variety [Ful93, pg. 76] or a Schubert variety [BK05, Section 3.4], then it has rational singularities.

If X is proper and has rational singularities, then for any $a \in K^\circ(X)$ and any resolution $\pi: \tilde{X} \rightarrow X$, we have $\chi(X, a) = \chi(\tilde{X}, \pi^*a)$ by the projection formula. In fact, the Leray spectral sequence and the projection formula imply that, for any vector bundle \mathcal{E} , the natural map from $H^i(X, \mathcal{E})$ to $H^i(\tilde{X}, \pi^*\mathcal{E})$ is an isomorphism for all i .

Theorem 4.5. [Bri02] *Let X be a closed subvariety of a Grassmannian over a field of characteristic 0, and assume that X has rational singularities. Write $[\mathcal{O}_X] = \sum_\lambda a_\lambda[\mathcal{O}_{\Omega_\lambda}]$. Then $(-1)^{\dim X - |\lambda|}a_\lambda \geq 0$.*

That is, the coefficients which are used to express $[\mathcal{O}_X]$ in the basis given by structure sheaves of Schubert varieties alternate in sign. When $|\lambda| = \dim X$, one has $a_\lambda \geq 0$ without any restriction on the singularities or characteristic. In the case of $\text{Gr}(1, n) = \mathbb{P}^{n-1}$, Theorem 4.5 predicts the signs occurring in the expansion of the Hilbert polynomial of X using the basis $\left\{\binom{a+q}{q}\right\}_{q=0, \dots, n-1}$ for the space of polynomials in a of degree at most $n-1$, as in (5). Indeed, the Schubert varieties in \mathbb{P}^{n-1} are coordinate subspaces, and the coefficients used to expand $[\mathcal{O}_X]$ in terms of the classes of structure sheaves of coordinate subspaces are the same as the coefficients used to expand the Hilbert polynomial of X in this basis.

The key tool to prove K -theoretic positivity results is the *Kawamata–Viehweg* vanishing theorem. This controls the cohomology of inverses of *nef and big* line bundles on a smooth projective variety. A line bundle \mathcal{L} on a projective variety X is nef if its restriction to every integral curve in X has non-negative degree. We say that \mathcal{L} is big if the sections of its tensor powers have the maximal possible growth rate, i.e., if

$$\limsup_{a \rightarrow \infty} \frac{\dim H^0(X, \mathcal{L}^{\otimes a})}{a^{\dim X}} > 0.$$

If \mathcal{L} is nef, then \mathcal{L} is big if and only if its top self-intersection number $\int_X c_1(\mathcal{L})^{\dim X}$ is positive. See [Laz04, Section 2.2] for proofs and more about big line bundles.

We can obtain nef and big line bundles as follows: we take a surjective map $X \rightarrow Y$ with $\dim X = \dim Y$ and an embedding of Y into projective space \mathbb{P}^n , and then we pull back the hyperplane bundle $\mathcal{O}(1)$ to X .

Theorem 4.6 (Kawamata–Viehweg vanishing theorem). *Let X be a (connected) smooth projective variety over a field of characteristic 0, and let \mathcal{L} be a nef and big line bundle on X . Then $H^i(X, \mathcal{L}^{-1}) = 0$ for $i < \dim X$.*

By Serre duality, $H^i(X, \mathcal{L}^{-1})$ is dual to $H^{\dim X - i}(X, \omega_X \otimes \mathcal{L})$, where ω_X is the canonical bundle, so Kawamata–Viehweg vanishing is often phrased as saying that if \mathcal{L} is nef and big, then $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ for all $i > 0$.

Proof of Theorem 4.6. We assume that $k = \mathbb{C}$ and that \mathcal{L} is very ample; the general case is obtained by an intricate reduction to this case. Let d be the dimension of X . By Bertini's theorem, we can find a smooth divisor H in the linear series of \mathcal{L} . The Lefschetz hyperplane theorem states that the restriction map in

singular cohomology $H_{\text{sing}}^i(X, \mathbb{C}) \rightarrow H_{\text{sing}}^i(H, \mathbb{C})$ is an isomorphism for $i < d - 1$ and is injective if $i = d - 1$. Hodge theory produces a direct sum decomposition

$$H_{\text{sing}}^i(X, \mathbb{C}) \cong \bigoplus_{p+q=i} H^{p,q}(X),$$

where $H^{p,q}(X) = H^q(X, \Omega_X^p)$. There is a similar direct sum decomposition for the cohomology of H , and the restriction map respects this decomposition. Taking the case $p = 0$, this tells us that the restriction map $H^q(X, \mathcal{O}_X) \rightarrow H^q(H, \mathcal{O}_H)$ is an isomorphism for $q < d - 1$ and is injective when $q = d - 1$. The long exact sequence on cohomology associated to the short exact sequence $0 \rightarrow \mathcal{L}^{-1} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_H \rightarrow 0$ implies the result. \square

Corollary 4.7. *Let X be a projective variety with rational singularities over a field of characteristic 0, and let \mathcal{L} be a nef and big line bundle on X . Then $(-1)^{\dim X} \chi(X, \mathcal{L}^{-1}) \geq 0$.*

Proof. Let $\pi: \tilde{X} \rightarrow X$ be a resolution of X , with \tilde{X} projective. Then $\pi^*\mathcal{L}$ is a nef and big line bundle on \tilde{X} , so, by Theorem 4.6, we have

$$\chi(\tilde{X}, \pi^*\mathcal{L}^{-1}) = \sum_{i \geq 0} (-1)^i \dim H^i(\tilde{X}, \pi^*\mathcal{L}^{-1}) = (-1)^{\dim X} \dim H^{\dim X}(\tilde{X}, \pi^*\mathcal{L}^{-1}).$$

The result follows, as $\chi(X, \mathcal{L}^{-1}) = \chi(\tilde{X}, \pi^*\mathcal{L}^{-1})$ because X has rational singularities. \square

Note that Theorem 4.6 and even Corollary 4.7 can fail in positive characteristic. Indeed, in [Tot19] Totaro constructed a purely inseparable cover of a partial flag variety which has a very ample line bundle \mathcal{L} that does not satisfy the conclusion of Corollary 4.7. Corollary 4.7 can also fail without the assumption that X has rational singularities:

Example 4.8. Choose some integers $d > n \geq 3$. By generically projecting the rational normal curve in \mathbb{P}^d , construct a smooth degree d rational curve in \mathbb{P}^n . Let Y be the cone over this curve, and let \tilde{Y} be the blow-up of Y at the cone point, so the map $\pi: \tilde{Y} \rightarrow Y$ is a resolution of singularities. The variety \tilde{Y} is isomorphic to the d th Hirzebruch surface, and, using toric geometry, one can show that $H^i(\tilde{Y}, \pi^*\mathcal{O}(-1)) = 0$ for all i . But Y is not normal: there is a $(d - n)$ -dimensional vector space of sections of $\mathcal{O}(1)$ on the rational curve which are not restricted from \mathbb{P}^n . This implies that the sheaf $\pi_*\mathcal{O}_{\tilde{Y}}/\mathcal{O}_Y$ has length at least $d - n$. Because π is an isomorphism outside of the origin, this sheaf is concentrated at the origin. Also, $R^1\pi_*\mathcal{O}_{\tilde{Y}} = R^2\pi_*\mathcal{O}_{\tilde{Y}} = 0$ because the curve is rational, so we have

$$\pi_*[\mathcal{O}_{\tilde{Y}}] = [\mathcal{O}_Y] + m[\mathcal{O}_p], \quad m \geq d - n.$$

By the projection formula (3), we have $\chi(Y, \mathcal{O}(-1)) = -m < 0$.

The example in [Tot19] shows that the extension of Theorem 4.5 to fields of characteristic p does not hold, even in the case of projective space.

We now prove Theorem 4.5 in the case $r = 1$, i.e., for subschemes of projective space. The general case is similar, except that it requires a much more sophisticated vanishing theorem.

Proof of Theorem 4.5. We will use the observation that the pairing $K(\mathbb{P}^{n-1}) \times K(\mathbb{P}^{n-1}) \rightarrow \mathbb{Z}$ given by $(a, b) \mapsto \chi(\mathbb{P}^{n-1}, ab)$ is nondegenerate. Under this pairing, the dual basis to $\{[\mathcal{O}], [\mathcal{O}_H], \dots, [\mathcal{O}_H]^{n-1}\}$ is $\{[\mathcal{O}_H]^{n-1}, [\mathcal{O}_H]^{n-2}[\mathcal{O}(-1)], [\mathcal{O}_H]^{n-3}[\mathcal{O}(-1)], \dots, [\mathcal{O}(-1)]\}$. That is, we have

$$[\mathcal{O}_X] = \sum a_{n-1-i}[\mathcal{O}_H]^i, \quad \text{where } a_{n-1-i} = \chi(\mathbb{P}^{n-1}, [\mathcal{O}_X] \cdot [\mathcal{O}_H]^{n-1-i}[\mathcal{O}(-1)]).$$

Let $\pi: \tilde{X} \rightarrow X$ be a resolution of singularities of X . By Proposition 3.4, $\pi^*[\mathcal{O}_H]^{n-1-i}$ is represented by the structure sheaf of the preimage Z_i in \tilde{X} of the intersection of X with $n - 1 - i$ transverse hyperplanes. By

an appropriate version of Bertini's theorem [Vak25, Exercise 21.6.I], Z_i will be smooth if the hyperplanes are sufficiently general. Because X has rational singularities, we have

$$\chi(\mathbb{P}^{n-1}, [\mathcal{O}_X] \cdot [\mathcal{O}_H]^{n-1-i}[\mathcal{O}(-1)]) = \chi(\tilde{X}, \pi^*([\mathcal{O}_H]^{n-i-1} \cdot [\mathcal{O}(-1)])) = \chi(Z_i, \pi^*\mathcal{O}(-1)).$$

Note that the restriction of $\pi^*\mathcal{O}(1)$ to Z_i remains big because the intersection number $\int_{Z_i} c_1(\pi^*\mathcal{O}(1))^{\dim Z_i}$ is equal to $\int_X c_1(\mathcal{O}(1))^{\dim X}$ which is positive. As Z_i has dimension $\dim X - (n - 1 - i)$ and is smooth, it follows from Corollary 4.7 that

$$(-1)^{\dim X - (n-1-i)} \chi(Z_i, \pi^*\mathcal{O}(-1)) = (-1)^{\dim X - (n-1-i)} a_{n-1-i} \geq 0. \quad \square$$

Remark 4.9. If one replaces $[\mathcal{O}_X]$ with $\pi_*[\mathcal{O}_{\tilde{X}}]$, where $\pi: \tilde{X} \rightarrow X$ is a resolution of X , then Theorem 4.5 holds without any assumption on the singularities.

4.2. Bounds on matroid polytope subdivisions. Using Theorem 4.5, we can construct valuative invariants of matroids that, at least for matroids realizable over a field of characteristic 0, have the sign property that is needed to bound the complexity of a matroid polytope subdivision.

Given a linear subspace $L \subseteq k^n$, we may expand the class of the structure sheaf of the torus-orbit closure $[\mathcal{O}_{\overline{T \cdot [L]}}]$ in terms of structure sheaves of Schubert varieties:

$$[\mathcal{O}_{\overline{T \cdot [L]}}] = \sum_{\lambda} a_{\lambda} [\mathcal{O}_{\Omega_{\lambda}}].$$

Let M be the matroid of L , and let c be the number of components of M , so $\dim \overline{T \cdot [L]} = n - c$. The torus-orbit closure $\overline{T \cdot [L]}$ is a normal toric variety, and normal toric varieties have rational singularities by Example 4.4. Then, if k has characteristic 0, Theorem 4.5 gives that $(-1)^{n-|\lambda|-c} a_{\lambda} \geq 0$.

It is reasonable to expect that the class $[\mathcal{O}_{\overline{T \cdot [L]}}] \in K(\text{Gr}(r, n))$ depends only on the matroid of L : often, though not always, it is possible to deform any other realization L' of M to L . Such a deformation induces a deformation of the corresponding torus-orbit closures, which implies that their K -classes are equal by Example 3.7.

In situations like this, when one has an “algebraic-geometric” invariant of realizable matroids, it is almost always possible to extend the definition of this invariant to all matroids. One finds a formula for the invariant, and then this formula typically makes sense for an arbitrary matroid.

Example 2.5, together with Example 3.5, suggests that the class $[\mathcal{O}_{\overline{T \cdot [L]}}]$ behaves valuatively with respect to matroid polytope subdivisions. See [Spe09, Proposition A.3]. It is reasonable to hope that this behavior extends to non-realizable matroids.

These properties were established in [FS12], although none of them are obvious. That is, for each matroid M of rank r on $[n]$, the authors construct a class $[\mathcal{O}_M] \in K(\text{Gr}(r, n))$ that agrees with the class of the structure sheaf of the torus-orbit closure of a realization, if M is realizable. The function $M \mapsto [\mathcal{O}_M]$ is valuative. We explain the construction in Section 4.4 below.

We can construct more valuative invariants by writing

$$[\mathcal{O}_M] = \sum a_{\lambda}(M) [\mathcal{O}_{\Omega_{\lambda}}] \text{ in } K(\text{Gr}(r, n)).$$

As explained previously, if M is realizable over a field of characteristic 0, then $(-1)^{n-|\lambda|-c(M)} a_{\lambda}(M) \geq 0$, i.e., the valuative function $M \mapsto (-1)^{n-|\lambda|-c(M)} a_{\lambda}(M)$ has the sign pattern needed to give bounds on the complexity of matroid polytope subdivisions, at least when all of the pieces are realizable over a field of characteristic 0.

It is conjectured, but not known, that $(-1)^{n-|\lambda|-c(M)} a_{\lambda}(M) \geq 0$ for all matroids M [BF22, Conjecture 9.8]. However, it seems that the bounds one obtains using this strategy are not enough to prove the sharp bounds in Conjecture 2.11 (except in the case of faces of dimension $n - 1$; see Proposition 5.30).

4.3. Generators for the valuative group. We now describe a basis for the valuative group $\text{Val}_{r,n}$. The matroids in this basis are realizable over any infinite field. This implies that if two valuative functions are equal on realizable matroids (or matroids realizable over \mathbb{C}), then they are equal.

Given a sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ with $n - r \geq \lambda_1 \geq \dots \geq \lambda_r \geq 0$, we can consider the corresponding Schubert variety Ω_λ in $\text{Gr}(r, n)$. This Schubert variety is irreducible, so a general point of Ω_λ realizes some particular matroid M_λ . As a dense open subset of Ω_λ is isomorphic to $\mathbb{A}^{|\lambda|}$, the matroid M_λ is realizable over any infinite field.

A *Schubert matroid* is a matroid which is isomorphic to M_λ for some λ , i.e., there is some $w \in S_n$ such that the bases of the matroid are $\{w \cdot B : B \text{ basis of } M_\lambda\}$. There are $n!$ Borel subgroups containing the diagonal torus in GL_n ; Schubert matroids are exactly the matroids realized by general points of orbits of one of these Borels on a Grassmannian.

Theorem 4.10. [DF10, Theorem 5.4] *The valuative group $\text{Val}_{r,n}$ has a basis given by indicator functions of Schubert matroids of rank r on $[n]$.*

A different proof was given in [EHL23], which proceeds by identifying Val_n with the cohomology of a certain toric variety, and then using algebraic techniques to find a basis for that ring. The description of Schubert matroids that is used there is that they are particular cotransversal matroids. See [Ham17] for a related result.

Corollary 4.11. *Let f and g be valuative invariants of matroids of rank r on $[n]$. If f and g agree on Schubert matroids, then they are equal.*

We now describe a different family of matroids which also span the valuative group. A matroid is *series-parallel* if it has a ground set of size 1, or it is the matroid associated to the row span of a matrix which can be obtained from the matrix $\begin{pmatrix} 1 & 1 \end{pmatrix}$ by doubling columns or by taking the orthogonal complement of the row span. For example, the matroid associated to the row span of the matrix

$$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

is series-parallel, as it is obtained from $\begin{pmatrix} 1 & 1 \end{pmatrix}$ by doubling a column, taking the orthogonal complement, and then doubling a column again. A matroid is *quasi series-parallel* if it is a direct sum of series-parallel matroids. If M is quasi series-parallel, then the matroid corresponding to any face of $P(M)$ is quasi series-parallel. See [FL24, Section 2] for more on quasi series-parallel matroids.

Proposition 4.12. *The valuative group $\text{Val}_{r,n}$ is generated by indicator functions of quasi series-parallel matroids.*

Proof. It is shown in [CRA11] that the matroid polytope of every Schubert matroid can be subdivided into matroid polytopes of quasi series-parallel matroids. See [FS24, Appendix A] for an exposition. \square

4.4. Equivariant K-theory. In order to extend the definition of some invariants to non-realizable matroids, it is useful to consider *equivariant K-theory*. We will focus on the case of a torus action, and we will make some rather restrictive assumptions on the scheme. See [VV03] for a much more general setting.

Let X be a variety over k with an action of a torus T . A *linearization* of a vector bundle \mathcal{E} on X is a lift of the action of T on X to the total space of \mathcal{E} . That is, let $\text{Tot}(\mathcal{E})$ denote the total space of \mathcal{E} , which is equipped with a map $p: \text{Tot}(\mathcal{E}) \rightarrow X$. A linearization is an action of T on $\text{Tot}(\mathcal{E})$ such that p is T -equivariant, and for every $t \in T$, the map from the fiber over $x \in X$ to the fiber over $t \cdot x$ given by this action is a linear map. A map between linearized vector bundles is said to be T -equivariant if the induced map between their total spaces is T -equivariant.

The T -equivariant K -theory of X , denoted $K_T^\circ(X)$, is the Grothendieck group of T -linearized vector bundles. It has a ring structure, with multiplication given by tensor product. It has similar formal properties to the Grothendieck ring of vector bundles: if $X \rightarrow Y$ is a T -equivariant map, then there is a pullback map $K_T^\circ(Y) \rightarrow K_T^\circ(X)$.

There is also an equivariant version of K_\circ , denoted $K_\circ^T(X)$. This is the Grothendieck group of T -equivariant coherent sheaves; see [Sta25, 03LF] for the definition of an equivariant coherent sheaf, which is somewhat subtle. The functoriality of $K_\circ^T(X)$ is similar to that of $K_\circ(X)$. There is a map $K_T^\circ(X) \rightarrow K_\circ^T(X)$, given by viewing an equivariant vector bundle as an equivariant coherent sheaf. As with ordinary K -theory, if X is smooth, then the map $K_T^\circ(X) \rightarrow K_\circ^T(X)$ is an isomorphism [Nie74, Proposition 2.1]. We will use $K_T(X)$ to denote $K_T^\circ(X)$ when X is smooth.

If $X \rightarrow Y$ is proper, then there is a pushforward map $K_\circ^T(X) \rightarrow K_\circ^T(Y)$, and these maps satisfy a version of the projection formula (3). In particular, if X is proper, then we can pushforward classes in $K_\circ^T(X)$ along the map from X to a point, where the T -action on the point is trivial. A T -equivariant vector bundle on a point is just a representation of T , so $K_T(\text{pt})$ is the Grothendieck ring of representations of T . In any characteristic, every (algebraic) representation of T is a direct sum of 1-dimensional representations of T , and if $T \cong \mathbb{G}_m^n$, then the 1-dimensional T -representations are indexed by \mathbb{Z}^n , with the representation corresponding to $(a_1, \dots, a_n) \in \mathbb{Z}^n$ being the map $T \rightarrow GL_1(k)$ given by $(t_1, \dots, t_n) \mapsto t_1^{a_1} \cdots t_n^{a_n}$. A 1-dimensional representation of T is called a *character*, and the group of characters under tensor product is called the *character lattice* of T . We therefore have

$$K_T(\text{pt}) \cong \mathbb{Z}[T_1^{\pm 1}, \dots, T_n^{\pm 1}].$$

Via pullback from the map $X \rightarrow \text{pt}$, $K_T^\circ(X)$ is a module over $K_T(\text{pt})$.

The map $K_\circ^T(X) \rightarrow K_T(\text{pt})$ amounts to taking the *equivariant Euler characteristic*: if \mathcal{F} is a linearized coherent sheaf on X , then each cohomology group $H^i(X, \mathcal{F})$ is a representation of T , and we can consider the class $[H^0(X, \mathcal{F})] - [H^1(X, \mathcal{F})] + [H^2(X, \mathcal{F})] - \cdots$ in the Grothendieck ring of representations of T .

There is a forgetful map $K_T^\circ(X) \rightarrow K^\circ(X)$ obtained by forgetting the linearization. This map is often surjective. For example, if X is smooth and X^T is finite, then this map is surjective. By [BB73, Theorem 4.4] and [BB76], X has an affine paving where the cells are unions of torus-orbits, so $K(X)$ is generated by the classes of structure sheaves of closures of cells by Proposition 4.2. The closure of each cell is fixed by T , and so its structure sheaf is in the image of the map $K_T(X) \rightarrow K(X)$.

One feature of T -equivariant K -theory which is not present in ordinary K -theory is that it satisfies very useful *localization* theorems. Let X be a smooth variety, and let $\iota: X^T \rightarrow X$ be the inclusion of the T -fixed locus. As ι is a T -equivariant map, there is a pullback map $\iota^*: K_T(X) \rightarrow K_T(X^T)$. Because the T -action on X^T is trivial, we have $K_T(X^T) \cong K(X^T) \otimes_{\mathbb{Z}} K_T(\text{pt})$; see, e.g., [AP15, Lemma 2.3]. Surprisingly, ι^* is often injective, allowing us to describe $K_T(X)$ as a subring of a relatively simple ring. Results of this form date back to the work of Segal [Seg68]. For the rest of the section, we will make the following assumptions:

- (1) X is smooth and proper.
- (2) X^T is 0-dimensional.
- (3) There are finitely many T -fixed curves in X .

One sometimes calls a variety satisfying these assumptions a *GKM space*, after an influential paper of Goresky, Kottwitz, and MacPherson [GKM98].

It follows from [CGP15, Proposition A.8.10(2)] that X^T is a union of reduced points, and that each T -fixed curve is smooth. As the only smooth curve whose automorphism group contains a torus is \mathbb{P}^1 , each T -fixed curve is isomorphic to \mathbb{P}^1 . The action of T on this curve induces a map from T to $\text{Aut}(\mathbb{P}^1) = PGL_2$ whose image is a 1-dimensional torus. If we identify this 1-dimensional torus with $GL_1(k)$, then we obtain a

character of T . This character is well-defined up to inversion, as the identification of a 1-dimensional torus with $GL_1(k)$ is well-defined up to inversion. One has the following result; see, e.g., [VV03, Corollary 5.12].

Theorem 4.13. *With the assumptions above, the map $K_T(X) \rightarrow K_T(X^T)$ is injective, with image given by the subring of elements $(b_p) \in \prod_{p \in X^T} K_T(\text{pt})$ such that if p and q are two distinct points in X^T which are connected by a T -fixed curve C on which T acts by the character a , then $b_p - b_q$ is divisible by $1 - a$ in $K_T(\text{pt})$.*

The restriction in Theorem 4.13 can be interpreted as follows: for any distinct points $p, q \in X^T$ which are connected by a T -fixed \mathbb{P}^1 , the restriction map $K_T(X^T) \rightarrow K_T(p) \oplus K_T(q)$ factors through $K_T(C)$, so its image must be contained in the image of the map $K_T(C) \rightarrow K_T(p) \oplus K_T(q)$. The image of this map is exactly the classes satisfying the restriction in the theorem.

Theorem 4.13 will allow us to check that classes associated to realizable matroids are independent of the choice of realization. It will also let us construct classes in $K_T(X)$ by specifying a class in $K_T(\text{pt})$ for each T -fixed point of X and then checking the condition in Theorem 4.13. We will use this to construct classes associated to possibly non-realizable matroids.

Example 4.14. Let $X = \text{Gr}(r, n)$, and let $T = \mathbb{G}_m^n$. We described the T -fixed points and curves in $\text{Gr}(r, n)$ in Proposition 1.6. Theorem 4.13 then shows that $K_T(\text{Gr}(r, n))$ is the subring of $\prod_{S \in \binom{[n]}{r}} \mathbb{Z}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ given by classes (a_S) such that if $S_2 = (S_1 \setminus i) \cup j$, then $a_{S_1} - a_{S_2}$ is divisible by $1 - T_i/T_j$.

Furthermore, if $\pi: X \rightarrow Y$ is a T -equivariant map between GKM spaces, then it is possible to describe the pushforward map $\pi_*: K_T(X) \rightarrow K_T(Y)$ in terms of localization. This type of result is called an *Atiyah–Bott* formula, after [AB84], which described this type of formula in equivariant cohomology.

The action of T on X induces a canonical linearization of the tangent bundle of X . For a T -fixed point p of X , let $T_p X = \chi_1 \oplus \dots \oplus \chi_d$ be the decomposition of the T -representation on the tangent space to X at p into characters of T . Set $\epsilon_p := (1 - [\chi_1]) \cdots (1 - [\chi_d])$ in $K_T(p)$.

Theorem 4.15. *Let $\pi: X \rightarrow Y$ be a T -equivariant map of GKM spaces, and let ι_X and ι_Y be the inclusions of the T -fixed locus into X and Y , respectively. Let $[\mathcal{E}]$ be a class in $K_T(X)$ with $\iota_X^*[\mathcal{E}] = \prod_{p \in X^T} a_p \in \prod_{p \in X^T} K_T(\text{pt})$. Then*

$$\iota_Y^* \pi_*[\mathcal{E}] = \prod_{q \in Y^T} \epsilon_q \sum_{p \in X^T, \pi(p)=q} a_p \frac{1}{\epsilon_p}.$$

In particular, for each $q \in Y^T$, $\sum_{p \in X^T, \pi(p)=q} a_p \frac{\epsilon_q}{\epsilon_p}$ is a Laurent polynomial, and these Laurent polynomials satisfy the conditions in Theorem 4.13.

Proof. Let S denote the multiplicative subset of $K_T(\text{pt})$ generated by $\{1 - T_1^{a_1} \cdots T_n^{a_n} : (a_1, \dots, a_n) \in \mathbb{Z}^n\}$. Note that S does not contain 0. In [Nie74, Theorem 3.2], Nielsen shows that the map $\iota_X^*: K_T(X) \rightarrow K_T(X^T)$ becomes an isomorphism after inverting S . The inverse map is given by

$$(9) \quad y = \prod_{p \in X^T} y_p \mapsto \sum_{p \in X^T} y_p \frac{[\mathcal{O}_p]}{\epsilon_p}.$$

In particular, applying this to $[\mathcal{E}]$, we have

$$\pi_*[\mathcal{E}] = \pi_* \left(\sum_{p \in X^T} a_p \frac{[\mathcal{O}_p]}{\epsilon_p} \right) = \sum_{q \in Y^T} \sum_{p \in X^T, \pi(p)=q} a_p \frac{[\mathcal{O}_q]}{\epsilon_p}.$$

By considering the formula for the inverse to the map $K_T(Y) \rightarrow K_T(Y^T)$ given in (9), we see that $\pi_*[\mathcal{E}]$ is the image of the class in the statement of the theorem under the inverse to the map $K_T(Y) \rightarrow K_T(Y^T)$. \square

Using this result, we can describe the class of the structure sheaf of a torus-orbit closure in $K_T(\text{Gr}(r, n))$. We will realize the class $[\mathcal{O}_{\overline{T \cdot [L]}}] \in K_T(\text{Gr}(r, n))$ as a pushforward. It is the pushforward of the structure sheaf along the inclusion of $\overline{T \cdot [L]}$ into $\text{Gr}(r, n)$, but because torus-orbit closures in Grassmannians are rarely smooth, we cannot directly apply Theorem 4.15. Instead, we will use a particularly nice resolution of singularities for $\overline{T \cdot [L]}$.

The *permutohedral toric variety* X_n is the toric variety obtained by blowing up \mathbb{P}^{n-1} at the torus-fixed points, then the strict transforms of the torus-fixed lines, and so on. It is an $(n-1)$ -dimensional smooth projective toric variety whose fan is the Weyl chambers of the type A_{n-1} root system, and so its T -fixed points are labeled by permutations in the symmetric group S_n . It can be described in a number of other ways. For example, it is the torus-orbit closure of a general point in the variety of full flags in k^n , and it is the toric variety corresponding to the polytope $\sum_{r=1}^{n-1} P(U_{r,n})$, where the sum denotes Minkowski sum. The key property of the permutohedral toric variety is the following:

for any matroid M on $[n]$, the fan of X_n is a refinement of the normal fan of $P(M)$.

Indeed, by definition, each edge in $P(M)$ is parallel to a vector of the form $e_i - e_j$, so each codimension 1 cone in the normal fan of $P(M)$ is contained in a hyperplane which is normal to $e_i - e_j$. The fan of X_n is the fan obtained by intersecting all of these hyperplanes.

In particular, there is a T -equivariant map $X_n \rightarrow X_{P(M)}$, where $X_{P(M)}$ denotes the toric variety of the normal fan of $P(M)$. Using the description of torus-orbits on the Grassmannian from Section 1, for any r -dimensional linear subspace L of k^n , there is a T -equivariant map $p_L: X_n \rightarrow \text{Gr}(r, n)$ such that the identity of the torus in X_n maps to the point $[L]$ of $\text{Gr}(r, n)$, and so the image of p_L is $\overline{T \cdot [L]}$.

Lemma 4.16. *We have $p_{L*}\mathcal{O}_{X_n} = \mathcal{O}_{\overline{T \cdot [L]}}$ and $R^i p_{L*}\mathcal{O}_{X_n} = 0$ for $i > 0$.*

Proof. This follows from [CLS11, Theorem 9.2.5], using Proposition 1.5 and Corollary 1.19. \square

Lemma 4.16 implies that $p_{L*}[\mathcal{O}_{X_n}] = [\mathcal{O}_{\overline{T \cdot [L]}}]$ in $K_T(\text{Gr}(r, n))$. As X_n is a smooth projective toric variety, p_L is a map between GKM spaces, and so we can use Theorem 4.15 to calculate $p_{L*}[\mathcal{O}_{X_n}]$. To do this, we need to analyze the map $X_n^T \rightarrow \text{Gr}(r, n)^T$ induced by p_L . For a permutation $w \in S_n$ and a matroid M , let B_w^{\max} be the w -maximal basis of M , i.e., the basis of M which maximizes $\langle e_B, v_1 e_{w(1)} + \dots + v_n e_{w(n)} \rangle$ for any decreasing sequence $v_1 > v_2 > \dots > v_n$. That there is a unique such basis follows from the fact that the normal fan of $P(M)$ is a coarsening of the permutohedral fan.

Lemma 4.17. *The image of T -fixed point corresponding to $w \in S_n$ is the T -fixed point of $\text{Gr}(r, n)$ corresponding to B_w^{\max} .*

Proof. Choose an element $(a_1, \dots, a_n) \in \mathbb{Z}^n$ with $a_{w(1)} > \dots > a_{w(n)}$, and let $\lambda: \mathbb{G}_m \rightarrow T$ be the corresponding cocharacter. By [Ful93, p.38], inside of X_n , $\lim_{t \rightarrow \infty} \lambda(t)$ is the fixed point of X_n corresponding to w . On the other hand, acting by $\lambda(t)$ on $[L] \in \text{Gr}(r, n)$ scales the Plücker coordinate of $[L]$ corresponding to S by $\prod_{i \in S} t^{a_i}$. As $t \rightarrow \infty$, the largest coordinate of $\lambda(t) \cdot [L]$ is the one corresponding to B_w^{\max} , so $\lim_{t \rightarrow \infty} \lambda(t) \cdot [L]$ is the fixed point corresponding to B_w^{\max} . The result follows, because the map $X_n \rightarrow \text{Gr}(r, n)$ is continuous. \square

Proposition 4.18. *Let L be a linear subspace of k^n of dimension r which realizes a matroid M . Let ι_S be the inclusion of the T -fixed point of $\text{Gr}(r, n)$ indexed by S . Then*

$$\iota_S^*[\mathcal{O}_{\overline{T \cdot [L]}}] = \sum_{\substack{w \in S_n \\ B_w^{\max} = S}} \frac{\prod_{i \in S, j \notin S} (1 - T_i^{-1} T_j)}{\prod_{i=1}^{n-1} (1 - T_{w(i)}^{-1} T_{w(i+1)})} \text{ in } K_T(\text{pt}).$$

Proof. Let \mathcal{S} and \mathcal{Q} denote the universal subbundle and quotient bundle on $\text{Gr}(r, n)$. Then the tangent bundle of $\text{Gr}(r, n)$ is $\text{Hom}(\mathcal{S}, \mathcal{Q}) \cong \mathcal{S}^\vee \otimes \mathcal{Q}$; see [EH16, Theorem 3.5]. It follows from this that the class in $K_T(\text{pt})$ of the tangent space to $\text{Gr}(r, n)$ at the T -fixed point of $\text{Gr}(r, n)$ indexed by S is $\sum_{i \in S, j \notin S} T_i^{-1} T_j$. The class in $K_T(\text{pt})$ of the tangent space to X_n at the fixed point labeled by $w \in S_n$ is $\sum_{i=1}^{n-1} T_{w(i)}^{-1} T_{w(i+1)}$; see [BEST23, Section 2.3]. The result then follows from Theorem 4.15, Lemma 4.16, and Lemma 4.17. \square

Example 4.19. We compute the restriction of $[\mathcal{O}_{U_{2,4}}]$ to the T -fixed point of $\text{Gr}(2, 4)$ labeled by $\{1, 2\}$. We have $B_w^{\max} = \{1, 2\}$ if and only if $\{w(1), w(2)\} = \{1, 2\}$. There are 4 permutations with this property: 1234, 1243, 2134, and 2143. We therefore have

$$\begin{aligned} \iota_{12}^*[\mathcal{O}_{U_{2,4}}] &= (1 - T_1^{-1}T_3)(1 - T_1^{-1}T_4)(1 - T_2^{-1}T_3)(1 - T_2^{-1}T_4) \\ &\quad \left(\frac{1}{(1 - T_1^{-1}T_2)(1 - T_2^{-1}T_3)(1 - T_3^{-1}T_4)} + \frac{1}{(1 - T_1^{-1}T_2)(1 - T_2^{-1}T_4)(1 - T_4^{-1}T_3)} + \right. \\ &\quad \left. \frac{1}{(1 - T_2^{-1}T_1)(1 - T_1^{-1}T_3)(1 - T_3^{-1}T_4)} + \frac{1}{(1 - T_2^{-1}T_1)(1 - T_1^{-1}T_4)(1 - T_4^{-1}T_3)} \right) \\ &= 1 - \frac{T_3T_4}{T_1T_2}. \end{aligned}$$

Proposition 4.18 implies that the class $[\mathcal{O}_{T.[L]}]$ in $K_T(\text{Gr}(r, n))$ depends only on the matroid that L realizes. It follows from Theorem 4.15 that the formula in Proposition 4.18 is a Laurent polynomial when M is realizable. We will see that this holds for all matroids by using Theorem 4.10. We need the following lemma.

Lemma 4.20. *For a subset S in $\binom{[n]}{r}$, the function which assigns a matroid M of rank r on $[n]$ to*

$$\sum_{\substack{w \in S_n \\ B_w^{\max} = S}} \frac{\prod_{i \in S, j \notin S} (1 - T_i^{-1}T_j)}{\prod_{i=1}^{n-1} (1 - T_{w(i)}^{-1}T_{w(i+1)})} \text{ in } \mathbb{Q}(T_1, \dots, T_n)$$

is valutive.

Proof. It follows from [McM09, Theorem 4.6] that for any permutation $w \in S_n$ and any set $S \in \binom{[n]}{r}$, the function from the set of matroids of rank r on $[n]$ to \mathbb{Z} given by

$$M \mapsto \begin{cases} 1, & \text{if } B_w^{\max} = S \\ 0, & \text{if } B_w^{\max} \neq S \end{cases}$$

is valutive. The function in the statement is a sum of the functions resulting from composing this function with a certain map $\mathbb{Z} \rightarrow \mathbb{Q}(T_1, \dots, T_n)$. \square

Consider the function which assigns a matroid M of rank r on $[n]$ to the element $[\mathcal{O}_M]$ of $\prod_{S \in \binom{[n]}{r}} \mathbb{Q}(T_1, \dots, T_n)$ which has component labeled by S given by

$$[\mathcal{O}_M]_S = \sum_{\substack{w \in S_n \\ B_w^{\max} = S}} \frac{\prod_{i \in S, j \notin S} (1 - T_i^{-1}T_j)}{\prod_{i=1}^{n-1} (1 - T_{w(i)}^{-1}T_{w(i+1)})}.$$

It follows from Lemma 4.20 that this function is valutive as well. By Theorem 4.10, in $\text{Val}_{r,n}$ every matroid can be written as a linear combination of realizable matroids. If M is realizable, then Theorem 4.15 implies that $[\mathcal{O}_M]$ lies in the subring $K_T(\text{Gr}(r, n))$ of $\prod_{S \in \binom{[n]}{r}} \mathbb{Q}(T_1, \dots, T_n)$. It follows that this holds for all matroids.

To summarize, for each matroid M of rank r on $[n]$, we have constructed a class $[\mathcal{O}_M]$ in $K_T(\text{Gr}(r, n))$, and the function which assigns M to $[\mathcal{O}_M]$ is valutive.

See [FS12, Section 3] for a somewhat different approach to defining $[\mathcal{O}_M]$ and proving that it is valutive, without making use of the permutohedral toric variety.

5. SPEYER'S g -POLYNOMIAL OF A MATROID

We now introduce the invariant constructed by Speyer in [Spe09] in an attempt to bound the complexity of a matroid polytope subdivision. For each matroid M , he constructed a polynomial $g_M(t) \in \mathbb{Z}[t]$ which has the property that the function $M \mapsto (-1)^{c(M)+1} g_M(t)$ is valutive. When M is realizable over a field of characteristic 0, Speyer showed that its g -polynomial has non-negative coefficients, and that the bounds that it gives on the complexity of a matroid polytope subdivision are optimal, i.e., sufficient to prove Conjecture 2.11 in that case.

We will explain the definition of the g -polynomial. We begin with the case of realizable matroids, and then we will explain the definition for non-realizable matroids. We will describe the properties of the g -polynomial, and then prove the non-negativity of the coefficients when M is realizable over a field of characteristic 0.

5.1. Defining the g -polynomial for realizable matroids. An element $i \in [n]$ is a *loop* of a matroid M if i is not contained in any basis. A *coloop* is an element which is contained in every basis. We say that M is *loopless* if it does not have any loops, and we define *coloopless* similarly.

If M is realized by $L \subseteq k^n$, then M is loopless if and only if L is not contained in any coordinate hyperplane. The loops of M are the same as the coloops of the dual matroid M^\perp .

The following technical proposition is a consequence of Remark 2.14, because the matroid polytopes of matroids with loops or coloops are contained in the boundary of the matroid polytope of the uniform matroid, so in any matroid polytope subdivision of a loopless and coloopless matroid, all matroids appearing as interior faces are loopless and coloopless. It can also be proved directly; see [BEST23, Proof of Lemma 5.9].

Proposition 5.1. *Let Val_n^{cl} be the subgroup of Val_n generated by matroids with loops or coloops, and let Val_n^{ncl} be the subgroup generated by matroids without loops or coloops. Then $\text{Val}_n = \text{Val}_n^{cl} \oplus \text{Val}_n^{ncl}$.*

There are several different possible definitions of the g -polynomial. However, for matroids with loops or coloops, these definitions can fail to be equivalent, so we will define the g -polynomial to be 0 on matroids with loops or coloops.

Let $L \subseteq k^n$ be a realization of a loopless and coloopless matroid M of rank r , so $\dim L = r$ and L is not contained in any coordinate hyperplane. Let $L^\perp \subseteq k^n$ be the orthogonal complement to L , so L^\perp realizes M^\perp and is not contained in any coordinate hyperplane. There is a rational map

$$\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1},$$

given, in homogeneous coordinates, by $(x_1, \dots, x_n) \times (y_1, \dots, y_n) \mapsto (x_1 y_1, \dots, x_n y_n)$. Because M is loopless and coloopless, L and L^\perp are both non-zero vector spaces, so we can consider their projectivizations. Using the embeddings $\mathbb{P}L \hookrightarrow \mathbb{P}^{n-1}$ and $\mathbb{P}L^\perp \hookrightarrow \mathbb{P}^{n-1}$, we obtain a rational map

$$\mathbb{P}L \times \mathbb{P}L^\perp \xrightarrow{m} \mathbb{P}^{n-1}.$$

Note that the image of m is contained in the hyperplane $H = \{z_1 + \dots + z_n = 0\}$ of \mathbb{P}^{n-1} . As both H and $\mathbb{P}L \times \mathbb{P}L^\perp$ have dimension $n - 2$, m can be viewed as a rational map between two varieties of the same dimension. The g -polynomial of M will capture certain data about this map, such as its degree.

Let Z be a smooth projective variety resolving the indeterminacy of this rational map, i.e., Z is a smooth projective variety which fits into the following triangle:

$$\begin{array}{ccc}
Z & & \\
\downarrow p & \searrow m & \\
\mathbb{P}L \times \mathbb{P}L^\perp & \dashrightarrow & H,
\end{array}$$

where the map p is birational. For example, in characteristic 0, one can take the closure of the graph of m inside of $\mathbb{P}L \times \mathbb{P}L^\perp \times H$ and resolve its singularities. We will later see an explicit construction of a possible choice for Z , valid in any characteristic, in Section 5.3.

We will consider the line bundle $\mathcal{O}(1)$ on H , obtained by restricting the hyperplane class from \mathbb{P}^{n-1} . Let $[\mathcal{O}_D]$ denote the class of a generic section of $\mathcal{O}(1)$ in $K(H)$.

Definition 5.2. *The g -polynomial $g_M(t)$ of a realizable loopless and coloopless matroid M is the polynomial whose t^i coefficient is $(-1)^{i+c(M)}\chi(Z, m^*[\mathcal{O}(-1)] \cdot m^*[\mathcal{O}_D]^{n-i-1})$. If M is not loopless or not coloopless, then we set $g_M(t) = 0$.*

It is true, but not obvious, that the definition above does not depend on the choice of realization of M . This will be explained in Section 5.4, as well as how to extend this definition to non-realizable matroids.

The constant term of $g_M(t)$ is 0 by definition. In order to compute the coefficient of t^i , we slice H by $n - i - 1$ general hyperplanes, take the inverse image in Z , and compute the Euler characteristic of the pullback of $\mathcal{O}(-1)$. In particular, we see that the linear term of the g -polynomial is the degree of the map m .

In other words, by looking at the proof of Theorem 4.5 in the case of projective space, we can compute the coefficients of $g_M(t)$ by writing the Hilbert polynomial of $\mathcal{O}(1)$ on Z as

$$\chi(Z, \mathcal{O}(a)) = \sum_{q=0}^{n-2} a_q \binom{a+q}{q}.$$

Then, for $i > 0$, the coefficient of t^i in $g_M(t)$ is $(-1)^{i+c(M)} a_{n-1-i}$.

The coordinate-wise multiplication of two projective varieties is sometimes called the *Hadamard product*. See [BC24] for a comprehensive survey of the literature. It often arises in rigidity theory, and we will discuss it more in Section 10.

At least over a field of characteristic 0, it is not hard to see that Definition 5.2 is independent of the choice of Z . For if we have another smooth projective variety Z' resolving the indeterminacy, then both Z and Z' can be dominated by another smooth variety \tilde{Z} also resolving the indeterminacy, and the projection formula (3) implies that the Euler characteristic does not change when we pull back from Z or Z' to \tilde{Z} .

Remark 5.3. In [Spe09], Speyer defines the coefficients of the g -polynomial as certain linear combinations of the invariants $a_\lambda(M)$. This definition agrees with Definition 5.2, at least for loopless and coloopless matroids. From the definition in [Spe09], it is tricky to see the non-negativity of the coefficients. In particular, the non-negativity does not follow from Theorem 4.5.

Example 5.4. [Spe09, Proposition 10.3] Let M be a loopless and coloopless matroid of rank 3 on $[n]$. Assume that M is *simple*, i.e., every pair of elements is contained in a basis. Let d_1, \dots, d_s be the sizes of the flats of rank 2 of M , i.e., the maximal subsets of $[n]$ which do not contain a basis of M . Then

$$g_M(t) = \left(\binom{n-2}{2} - \sum_i \binom{d_i-1}{2} \right) t + \left((n-3)(n-4) - \sum_i (d_i-2)^2 \right) t^2 + \left(\binom{n-4}{2} - \sum_i \binom{d_i-2}{2} \right) t^3.$$

Remark 5.5. The map m has appeared frequently in the literature on hyperplane arrangements and matroids, see [Sil96, DGS12, Huh13, Huh15, ADH23].

5.2. Properties of g -polynomial. We now discuss some fundamental properties of the g -polynomial. We give arguments which work for realizable matroids, but all of the properties discussed below hold for all matroids, as will be discussed in Section 5.4. One property is clear: if M is loopless and coloopless, then the definition of $g_M(t)$ does not change when we replace L by L^\perp . This is also true when M has a loop or a coloop, giving the following.

Proposition 5.6. *Let M be a realizable matroid. Then $g_M(t) = g_{M^\perp}(t)$.*

We now analyze the image of the map m . For a matroid M on $[n]$ and an element $i \in [n]$ which is not a coloop, the *deletion* $M \setminus i$ is the matroid whose bases are the bases of M that do not contain i . If i is not a loop, then the *contraction* M/i is the matroid whose bases are $\{B \setminus i : B \text{ basis of } M, i \in B\}$. If i is a loop or coloop, then we set $M \setminus i = M/i$. If M is realized by $L \subset k^n$, then the contraction is realized by $L \cap k^{[n] \setminus i}$, and the deletion is realized by the image of L under the projection $k^n \rightarrow k^{[n] \setminus i}$. We have the following combinatorial result.

Proposition 5.7. [Oxl11, Theorem 4.3.1] *Let M be a connected matroid on $[n]$, with $n > 1$. For each $i \in [n]$, either M/i or $M \setminus i$ is connected.*

Proposition 5.8. *Let L be a realization of a loopless and coloopless matroid M with c connected components S_1, \dots, S_c . Then the closure of the image of m is $\{(z_1, \dots, z_n) : \sum_{j \in S_i} z_j = 0 \text{ for } i = 1, \dots, c\}$.*

Proof. It follows from Proposition 1.5 that the image of m is contained in the subset $\{(z_1, \dots, z_n) : \sum_{j \in S_i} z_j = 0 \text{ for } i = 1, \dots, c\}$ of \mathbb{P}^{n-1} . For the other direction, it suffices to show that the dimension of the image of m is $n - c - 1$. This reduces to the case of connected matroids.

We induct on n ; it is easy to check the case $n = 2$. For each $i \in [n]$, either M/i or $M \setminus i$ is connected by Proposition 5.7. By switching L and L^\perp if necessary, we can assume that M/i is connected. This matroid is realized by $L \cap k^{[n] \setminus i}$. The image of L^\perp in $k^{[n] \setminus i}$ is the orthogonal complement to $L \cap k^{[n] \setminus i}$. By induction, we deduce that the closure of the image of m contains the subset $\{z_i = 0, \sum_{j=1}^n z_j = 0\}$ of \mathbb{P}^{n-1} . This locus has dimension $n - 3$. Varying over all i , we see that the closure of the image of m contains at least n distinct subvarieties of dimension $n - 3$. But $L \times L^\perp$ is irreducible, so the image of m is irreducible. This implies that the dimension of the image of m is $n - 2$. \square

If M is connected and realizable, then Proposition 5.8 implies that the map m is dominant, and in particular it has positive degree. We see that, if M is connected and realizable, the linear term of $g_M(t)$ is positive.

The dimension of the image of m can also be proved to be $n - c - 1$ by using [Ber22, Theorem 3.1]. In Proposition 5.26 below, we will compute the degree of the map m and see that it is positive for connected matroids in Proposition 5.28, giving another proof.

5.3. Wonderful varieties and tautological bundles. We will now relate the g -polynomial to the wonderful varieties of [dCP95] and the tautological bundles of linear subspaces that were introduced in [BEST23]. This will allow us to connect the g -polynomial to other well-studied invariants of matroids.

Given a subspace L of k^n which is not contained in any coordinate hyperplane, i.e., it realizes a loopless matroid, the *wonderful variety* W_L is the strict transform of $\mathbb{P}L$ under the map $X_n \rightarrow \mathbb{P}^{n-1}$. I.e., it is the closure of $\mathbb{P}L \cap \mathbb{G}_m^n / \mathbb{G}_m$ in X_n , where $\mathbb{G}_m^n / \mathbb{G}_m$ is the torus embedded in \mathbb{P}^{n-1} . In particular, the permutohedral toric variety is the wonderful variety of the Boolean arrangement $k^n \subseteq k^n$.

The wonderful variety is a smooth and projective variety of dimension $\dim L - 1$; it can be described as an iterated blow-up of $\mathbb{P}L$ along strict transforms of intersections of $\mathbb{P}L$ with coordinate subspaces. It is a simple normal crossings compactification of $\mathbb{P}L \cap \mathbb{G}_m^n / \mathbb{G}_m$.

In [BEST23], Berget, Eur, Spink, and Tseng constructed some important vector bundles $\mathcal{S}_L, \mathcal{Q}_L$ on the permutohedral variety X_n using a linear subspace L of k^n . These bundles form a short exact sequence

$$(10) \quad 0 \rightarrow \mathcal{S}_L \rightarrow \mathcal{O}_{X_n}^{\oplus n} \rightarrow \mathcal{Q}_L \rightarrow 0.$$

Over a point $t \in \mathbb{G}_m^n/\mathbb{G}_m$, the torus embedded in X_n , the fiber of \mathcal{S}_L in k^n is $t^{-1}L$. This uniquely determines \mathcal{S}_L : the total space of \mathcal{S}_L is the closure in $X_n \times k^n$ of the total space of the restriction of \mathcal{S}_L to $\mathbb{G}_m^n/\mathbb{G}_m$. It is a theorem that this is a subbundle: it is the pullback of a certain vector bundle from the torus-orbit closure of $[L^\perp]$ in $\text{Gr}(n-r, n)$; as discussed in Section 4.4, there is a map from X_n to the toric variety of any matroid polytope. The bundle \mathcal{Q}_L is defined as the quotient $\mathcal{O}_{X_n}^{\oplus n}/\mathcal{S}_L$.

The image of the section $(1, \dots, 1) \in H^0(X_n, \mathcal{O}_{X_n}^{\oplus n})$ in \mathcal{Q}_L transversely cuts out W_L [BEST23, Theorem 7.10]. This is easy to see on the torus in X_n : the section $(1, \dots, 1) \in H^0(X_n, \mathcal{Q}_L)$ vanishes at a point $t \in \mathbb{G}_m^n/\mathbb{G}_m$ if and only if $(1, \dots, 1) \in t^{-1}L$, which happens if and only if $t \in L$. This identifies the restriction of \mathcal{Q}_L to W_L with the normal bundle of W_L in X_n . The restriction of \mathcal{S}_L to W_L is closely related to the log tangent bundle of W_L , viewed as a compactification of $\mathbb{P}L \cap \mathbb{G}_m^n/\mathbb{G}_m$ [BEST23, Theorem 8.8].

The geometry of Speyer's g -polynomial is closely related to the restriction of \mathcal{Q}_L to W_L . The sequence (10) dualizes to give an injection $\mathcal{Q}_L^\vee \hookrightarrow \mathcal{O}_{X_n}^{\oplus n}$. Restricting to W_L and projectivizing, we obtain a subvariety $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$ of $W_L \times \mathbb{P}^{n-1}$.

Identifying k^n/L with L^\perp , the intersection of $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$ with $\mathbb{G}_m^n/\mathbb{G}_m \times \mathbb{G}_m^n/\mathbb{G}_m$ is the locus

$$\{(v, tv) : v \in \mathbb{P}L \cap \mathbb{G}_m^n/\mathbb{G}_m, t \in \mathbb{P}L^\perp \cap \mathbb{G}_m^n/\mathbb{G}_m\}.$$

If the matroid M corresponding to L is loopless and coloopless, then this variety is isomorphic to $(\mathbb{P}L \cap \mathbb{G}_m^n/\mathbb{G}_m) \times (\mathbb{P}L^\perp \cap \mathbb{G}_m^n/\mathbb{G}_m)$, via the map which sends (v, w) to (v, vw) . In particular, if p is the projection of $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee) \subseteq W_L \times \mathbb{P}^{n-1}$ onto \mathbb{P}^{n-1} , then p is a birational model for the map m considered in the definition of the g -polynomial. Let $\mathcal{O}(1)$ denote the relative hyperplane class on $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$, which is the restriction of the hyperplane class on \mathbb{P}^{n-1} . As a consequence, we can compute the g -polynomial in terms of the polynomial $a \mapsto \chi(\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee), \mathcal{O}(a))$: if we write

$$\chi(\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee), \mathcal{O}(a)) = \sum_{q=0}^{n-2} a_q \binom{a+q}{q},$$

then for each $i > 0$, the coefficient of t^i in $g_M(t)$ is $(-1)^{i+c(M)} a_{n-1-i}$. This will enable us to reduce problems involving the g -polynomial to computations on X_n , where there are more tools because it is a toric variety.

Let $\pi: \mathbb{P}_{W_L}(\mathcal{Q}_L^\vee) \rightarrow W_L$ be the projection. We can compute the Euler characteristic on $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$ by pushing forward along π . It follows from the fact that $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee) = \text{Proj}_{W_L} \text{Sym } \mathcal{Q}_L$ that, for $a \geq 0$, we have $\pi_* \mathcal{O}(a) = \text{Sym}^a \mathcal{Q}_L$. Furthermore, we have $R^i \pi_* \mathcal{O}(a) = 0$ for $i > 0$, $a \geq 0$, for example by the theorem on cohomology and base change [Vak25, Theorem 25.1.6]. We deduce the following formula for the g -polynomial. The function $a \mapsto \chi(W_L, \text{Sym}^a \mathcal{Q}_L)$ is a polynomial for $a \geq 0$.

Proposition 5.9. *Let L be a realization of a loopless and coloopless matroid M . If we write*

$$\chi(W_L, \text{Sym}^a \mathcal{Q}_L) = \sum_{q=0}^{n-2} a_q \binom{a+q}{q},$$

then for each $i > 0$, the coefficient of t^i in $g_M(t)$ is $(-1)^{i+c(M)} a_{n-1-i}$.

In other words, if we write $g_M(t) = g_1 t + \dots + g_{n-1} t^{n-1}$, then

$$(11) \quad \chi(W_L, \text{Sym}^a \mathcal{Q}_L) = \sum_{i=0}^{n-2} (-1)^{n-1-i+c(M)} g_{n-1-i} \binom{a+i}{i}.$$

Using this description of the g -polynomial, we can analyze how it behaves under direct sums, which will allow us to reduce computations to the case of connected matroids. It also explains the choice of indexing in the definition of the g -polynomial.

Proposition 5.10. [Spe09, Proposition 3.2] *Suppose that M is a realizable matroid with $M = M_1 \oplus \cdots \oplus M_c$. Then*

$$g_M(t) = g_{M_1}(t) \cdots g_{M_c}(t).$$

To prove this, we will make use of some maps between wonderful varieties. For a proper subset S of $[n]$, let L^S denote the image of L under the coordinate projection $k^n \rightarrow k^S$. This induces a rational map $\mathbb{P}L \dashrightarrow \mathbb{P}L^S$, which we claim extends to a morphism $\pi_S: W_L \rightarrow W_{L^S}$. Indeed, using the inclusion of W_L into X_n , it suffices to check that there is a morphism $X_n \rightarrow X_{|S|}$ induced by the coordinate projection onto S . This is a toric computation, which is carried out, for example, in [BHM⁺22, Proposition 3.1].

Lemma 5.11. *Let $L \subseteq k^n$ be a realization of a loopless matroid M , and let $[n] = S_1 \sqcup \cdots \sqcup S_c$ be a partition of $[n]$ such that each S_i is a union of components of M . Let $\pi: W_L \rightarrow W_{L^{S_1}} \times \cdots \times W_{L^{S_c}}$ be the map induced by $\pi_{S_1}, \pi_{S_2}, \dots, \pi_{S_c}$. Then $R^i \pi_* \mathcal{O}_{W_L} = 0$ for $i > 0$, and the natural map $\mathcal{O}_{W_{L^{S_1}} \times \cdots \times W_{L^{S_c}}} \rightarrow \pi_* \mathcal{O}_{W_L}$ is an isomorphism.*

It is possible to give a direct proof of Lemma 5.11, but it will be easiest to prove it in Section 8 using the tools developed there.

Proof of Proposition 5.10. It suffices to do the case $c = 2$. Let $\pi: W_L \rightarrow W_{L^{S_1}} \times W_{L^{S_2}}$ be the map described in Lemma 5.11. We claim that the restriction of \mathcal{Q}_L to W_L is isomorphic to $\pi^*(\mathcal{Q}_{L^{S_1}} \boxplus \mathcal{Q}_{L^{S_2}})$. Indeed, both of these vector bundles are quotients of $\mathcal{O}^{\oplus n}$, and they are identical over $\mathbb{P}L \cap \mathbb{G}_m^n / \mathbb{G}_m$, which is dense in W_L . In particular, for any $a \geq 0$, we have

$$\mathrm{Sym}^a \mathcal{Q}_L \cong \mathrm{Sym}^a \pi^*(\mathcal{Q}_{L^{S_1}} \boxplus \mathcal{Q}_{L^{S_2}}) \cong \pi^* \left(\bigoplus_{i+j=a} \mathrm{Sym}^i \mathcal{Q}_{L^{S_1}} \boxtimes \mathrm{Sym}^j \mathcal{Q}_{L^{S_2}} \right).$$

By Lemma 5.11 and the projection formula (3), we have

$$\chi(W_L, \mathrm{Sym}^a \mathcal{Q}_L) = \chi(W_{L^{S_1}} \times W_{L^{S_2}}, \mathrm{Sym}^a(\mathcal{Q}_{L^{S_1}} \boxplus \mathcal{Q}_{L^{S_2}})) = \sum_{i+j=a} \chi(W_{L^{S_1}}, \mathrm{Sym}^i \mathcal{Q}_{L^{S_1}}) \cdot \chi(W_{L^{S_2}}, \mathrm{Sym}^j \mathcal{Q}_{L^{S_2}}).$$

Set $n_1 = |S_1|$ and $n_2 = |S_2|$. Let $g_{M_1}(t) = b_1 t + \cdots + b_{n_1-1} t^{n_1-1}$, and let $g_{M_2}(t) = c_1 t + \cdots + c_{n_2-1} t^{n_2-1}$. Then we have

$$\begin{aligned} & \sum_{i+j=a} \chi(W_{L^{S_1}}, \mathrm{Sym}^i \mathcal{Q}_{L^{S_1}}) \cdot \chi(W_{L^{S_2}}, \mathrm{Sym}^j \mathcal{Q}_{L^{S_2}}) \\ &= \sum_{i+j=a} \left(\sum_{\ell=0}^{n_1-2} (-1)^{n_1+c(M_1)+\ell-1} b_{n_1-\ell-1} \binom{i+\ell}{\ell} \right) \left(\sum_{s=0}^{n_2-2} (-1)^{n_2+c(M_2)+s-1} c_{n_2-s-1} \binom{j+s}{s} \right) \\ &= (-1)^{n+c(M)} \sum_{\ell=0}^{n_1-2} \sum_{s=0}^{n_2-2} (-1)^{\ell+s} b_{n_1-\ell-1} c_{n_2-s-1} \sum_{i+j=a} \binom{i+\ell}{\ell} \binom{j+s}{s} \\ &= (-1)^{n+c(M)} \sum_{\ell=0}^{n_1-2} \sum_{s=0}^{n_2-2} (-1)^{\ell+s} b_{n_1-\ell-1} c_{n_2-s-1} \binom{a+\ell+s+1}{\ell+s+1} \\ &= (-1)^{n+c(M)} \sum_{u=1}^{n-3} (-1)^{u-1} \sum_{s+\ell+1=u} b_{n_1-\ell-1} c_{n_2-s-1} \binom{a+u}{u}. \end{aligned}$$

Comparing with (11) implies the result. \square

As mentioned previously, Proposition 5.8 implies that if M is connected, then the coefficient of t in $g_M(t)$ is positive. Proposition 5.10 then gives the following result.

Corollary 5.12. *If M is a realizable loopless and coloopless matroid with c connected components, then the coefficient of t^c in $g_M(t)$ is positive, and the coefficient of t^i in $g_M(t)$ is 0 for $i < c$.*

The realizability assumption in Corollary 5.12 is not necessary. In Corollary 5.29, we explicitly compute the coefficient of t^c in terms of a well-known combinatorial invariant of matroids, and see that it is positive for any loopless matroid.

5.4. Defining the g -polynomial for non-realizable matroids. We can use the formula in Proposition 5.9 to see that the g -polynomial does not depend on the choice of realization, and to define the g -polynomial for non-realizable matroids. This was first done in a different way in [FS12]. We will first need to discuss a few constructions related to matroids.

For a matroid M on $[n]$, the authors of [BEST23] constructed classes $[\mathcal{S}_M]$ and $[\mathcal{Q}_M]$ in $K_T(X_n)$. These are defined in terms of the description of $K_T(X_n)$ using Theorem 4.13, which describes $K_T(X_n)$ as a subring of $\prod_{w \in S_n} \mathbb{Z}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. For a permutation $w \in S_n$, let B_w denote the w -minimal basis of M , i.e., the basis B which minimizes $\langle e_B, v_1 e_{w(1)} + \dots + v_n e_{w(n)} \rangle$ for any decreasing sequence $v_1 > \dots > v_n$. We have

$$[\mathcal{S}_M] = \left(\sum_{i \in B_w} T_i^{-1} \right)_{w \in S_n} \quad \text{and} \quad [\mathcal{Q}_M] = \left(\sum_{i \in [n] \setminus B_w} T_i^{-1} \right)_{w \in S_n},$$

When M is realized by L , then $[\mathcal{S}_M] = [\mathcal{S}_L]$ and $[\mathcal{Q}_M] = [\mathcal{Q}_L]$ in $K_T(X_n)$; see [BEST23, Proposition 3.7]. In particular, the class $[\mathcal{Q}_L]$ does not depend on the choice of realization. We will usually consider $[\mathcal{S}_M]$ and $[\mathcal{Q}_M]$ in $K(X_n)$; there we have $[\mathcal{S}_M] + [\mathcal{Q}_M] = [\mathcal{O}_{X_n}^{\oplus n}]$, mirroring (10). We have the following key property, whose formulation makes use of the λ -ring structure on $K(X_n)$.

Proposition 5.13. [BEST23, Proposition 5.6] *The function which assigns a matroid to any fixed polynomial function in the symmetric or exterior powers of $[\mathcal{S}_M]$, $[\mathcal{Q}_M]$, and their duals is valutive.*

In [LLPP24], the authors defined a ring $K(M)$ for every loopless matroid M ; see [Lar24] for a simplified exposition. There is a surjective map from $K(X_n)$ to $K(M)$. If M is realized by L , then there is an identification of $K(W_L)$ with $K(M)$. For instance, the inclusion of W_L into X_n induces a restriction map $K(X_n) \rightarrow K(W_L)$, which is surjective. The kernel of the map from $K(X_n)$ to $K(M)$ is the same as the kernel of the map from $K(X_n)$ to $K(W_L)$. The ring $K(M)$ is the K -ring of a certain (non-proper) toric variety, and, as such, it has all of the properties of the K -ring of a smooth variety. In particular, it is a λ -ring.

The authors of [LLPP24] constructed an ‘‘Euler characteristic’’ map $\chi: K(M) \rightarrow \mathbb{Z}$ which satisfies an analogue of the projection formula. When M is realized by L , this Euler characteristic map coincides with the usual one. Suppose that L is a realization of M . Recall that there is a section of \mathcal{Q}_L which transversely cuts out W_L inside of X_n . This induces a Koszul resolution of \mathcal{O}_{W_L} : there is an exact sequence

$$0 \rightarrow \wedge^{n-r} \mathcal{Q}_L^\vee \rightarrow \wedge^{n-r-1} \mathcal{Q}_L^\vee \rightarrow \dots \rightarrow \mathcal{Q}_L^\vee \rightarrow \mathcal{O}_{X_n} \rightarrow \mathcal{O}_{W_L} \rightarrow 0.$$

This implies that $[\mathcal{O}_{W_L}] = \sum_{i \geq 0} (-1)^i \wedge^i [\mathcal{Q}_L^\vee]$ in $K(X_n)$. The projection formula (3) then shows that if M is realized by L and a is a class in $K(X_n)$, then

$$(12) \quad \chi(W_L, a) = \chi(X_n, a \cdot \sum_{i \geq 0} (-1)^i \wedge^i [\mathcal{Q}_L^\vee]).$$

By [LLPP24, Proposition 5.6], for any class $a \in K(X_n)$ and any loopless matroid M , we have

$$(13) \quad \chi(M, a) = \chi(X_n, a \cdot \sum_{i \geq 0} (-1)^i \wedge^i [\mathcal{Q}_M^\vee]).$$

We will need another description of the Euler characteristic map on $K(M)$. In [FY04], Feichtner and Yuzvinsky defined a ring $A^\bullet(M)$ for each loopless matroid M . If M is realized by L , there is an identification of the Chow ring of W_L with $A^\bullet(M)$; if $k = \mathbb{C}$, then the Chow ring of W_L is isomorphic to its singular cohomology ring. The ring $A^\bullet(M)$ is graded: if M has rank r , then there is a decomposition $A^\bullet(M) = A^0(M) \oplus \cdots \oplus A^{r-1}(M)$. In [AHK18, Definition 5.9], Adiprasito, Huh, and Katz show that $A^{r-1}(M) \cong \mathbb{Z}$, and they construct a distinguished isomorphism $\deg_M: A^{r-1}(M) \rightarrow \mathbb{Z}$. We extend \deg_M to a map from all of $A^\bullet(M)$ to \mathbb{Z} which is 0 on the other graded pieces. When M is realized by L , \deg_M coincides with the degree map in the Chow ring of W_L . See [Lar25] for a simple exposition of the basic algebraic properties of $A^\bullet(M)$.

The relationship between $K(M)$ and $A^\bullet(M)$ is the same as the relationship between the K -ring and Chow ring of a smooth variety; see Section 3.2. For example, there is a Chern character ring homomorphism from $K(M)$ to $A^\bullet(M)_\mathbb{Q}$ which satisfies a Hirzebruch–Riemann–Roch type formula; see [EL26, Proposition 4.11] and [Che25, Section 3]. Importantly, any class $a \in K(M)$ has a total Chern class $c(a) \in A^\bullet(M)^\times$.

There is an important class $\alpha \in A^1(M)$; see [AHK18, Definition 5.7]. When M is realized by L , α is the pullback from $\mathbb{P}L$ to W_L of the hyperplane class. The following result will be useful to compute Euler characteristics in $K(M)$.

Proposition 5.14. [LLPP24] *Let M be a loopless matroid of rank r . There is a ring isomorphism $\zeta: K(M) \rightarrow A^\bullet(M)$ such that, for any $a \in K(M)$, we have*

$$\chi(M, a) = \deg_M(\zeta(a) \cdot (1 + \alpha + \cdots + \alpha^{r-1})).$$

In the case of the permutohedral toric variety, i.e., the wonderful variety of the Boolean matroid, Proposition 5.14 was proved in [BEST23, Theorem D]. Proposition 5.14 resembles the Hirzebruch–Riemann–Roch theorem, but it is more useful because the map ζ , which is described in [LLPP24, Section 5], behaves nicely with respect to natural classes in $K(M)$. A computer program for calculating Euler characteristics in K -rings of matroids can be found here.

We say that a class a in $K(M)$ is the class of a line bundle if it is the image of the class of a line bundle in $K(X_n)$ under the map $K(X_n) \rightarrow K(M)$; this implies that a is a unit in $K(M)$. The class a is determined by its first Chern class $c_1(a) \in A^1(M)$, and for every class $y \in A^1(M)$, there is a unique line bundle in $K(M)$ with total Chern class $1 + y$. The isomorphism ζ constructed in Proposition 5.14 satisfies

$$\zeta(a) = 1 + a + \text{terms of degree at least 2}$$

for any a which is the class of a line bundle [LLPP24, Lemma 5.5]. The following technical result will be useful.

Lemma 5.15. *Let $\mathcal{L}_1, \dots, \mathcal{L}_m$ be classes of line bundles in $K(M)$. Then the function $(a_1, \dots, a_m) \mapsto \chi(M, \mathcal{L}_1^{a_1} \cdots \mathcal{L}_m^{a_m})$ is a polynomial in (a_1, \dots, a_m) of degree at most $r - 1$. For any non-negative integers s_1, \dots, s_m with $s_1 + \cdots + s_m = r - 1$, the coefficient of $a_1^{s_1} \cdots a_m^{s_m}$ is $\deg_M(c_1(\mathcal{L}_1)^{s_1} \cdots c_1(\mathcal{L}_m)^{s_m}) / s_1! \cdots s_m!$.*

Proof. Let $\sigma_i = 1 - \mathcal{L}_i^{-1}$. Note that $\zeta(\sigma_i) = c_1(\mathcal{L}_i) + \text{terms of degree at least 2}$, so $\zeta(\sigma_i)^r = 0$. This implies that $\sigma_i^r = 0$ because ζ is an isomorphism. Therefore, for any integer a_i , we have

$$\mathcal{L}_i^{a_i} = \frac{1}{(1 - \sigma_i)^{a_i}} = \sum_{d=0}^{r-1} \frac{a_i(a_i + 1) \cdots (a_i + d - 1)}{d!} \sigma_i^d.$$

We see that

$$\chi(\mathbb{M}, \mathcal{L}_1^{\alpha_1} \cdots \mathcal{L}_m^{\alpha_m}) = \sum_{d_1 + \cdots + d_m = r-1} \chi(\mathbb{M}, \sigma_1^{d_1} \cdots \sigma_m^{d_m}) \frac{a_1(a_1+1) \cdots (a_1+d_1-1)}{d_1!} \cdots \frac{a_m(a_m+1) \cdots (a_m+d_m-1)}{d_m!},$$

which is clearly a polynomial. For any non-negative integers s_1, \dots, s_m with $s_1 + \cdots + s_m = r-1$, the coefficient of $a_1^{s_1} \cdots a_m^{s_m}$ is $\chi(\mathbb{M}, \sigma_1^{s_1} \cdots \sigma_m^{s_m}) / s_1! \cdots s_m!$. Using Proposition 5.14, we see that this is equal to $\deg_{\mathbb{M}}(c_1(\mathcal{L}_1)^{s_1} \cdots c_1(\mathcal{L}_m)^{s_m}) / s_1! \cdots s_m!$. \square

The polynomial in Lemma 5.15 is called the *Snapper polynomial* of $\mathcal{L}_1, \dots, \mathcal{L}_m$. Using these results, we can show that the definition of the g -polynomial of a realizable matroid given previously indeed depends only on the matroid, and not on the choice of realization.

Corollary 5.16. *Let L be a realization of a loopless and coloopless matroid \mathbb{M} . Then, for any $a \geq 0$,*

$$\chi(W_L, \text{Sym}^a \mathcal{Q}_L) = \chi(\mathbb{M}, \text{Sym}^a[\mathcal{Q}_{\mathbb{M}}]).$$

In particular, the function $a \mapsto \chi(\mathbb{M}, \text{Sym}^a[\mathcal{Q}_{\mathbb{M}}])$ is a polynomial in a .

Note, however, that the cohomology of \mathcal{Q}_L on W_L *does* depend on the realization; see Section 7.3.

We will use the formula in Proposition 5.9 to define the g -polynomial for possibly non-realizable matroids. To do this, we will need some properties of the function $a \mapsto \chi(\mathbb{M}, \text{Sym}^a[\mathcal{Q}_{\mathbb{M}}])$. These properties will follow from the following combinatorial result. Let $\mathbb{N} = \{0, 1, 2, \dots\}$.

Proposition 5.17. *Let $q(b_1, \dots, b_\ell) = \sum_{\alpha=(\alpha_1, \dots, \alpha_\ell)} c_\alpha b_1^{\alpha_1} \cdots b_\ell^{\alpha_\ell}$ be a polynomial of degree s . Then the function*

$$a \mapsto \sum_{\substack{(p_1, \dots, p_\ell) \in \mathbb{N}^\ell \\ p_1 + \cdots + p_\ell = a}} q(p_1, \dots, p_\ell)$$

is a polynomial \tilde{q} of degree at most $s + \ell - 1$. We have $\tilde{q}(-a) = 0$ for $a \in \{1, \dots, \ell - 1\}$, $\tilde{q}(-\ell) = (-1)^{\ell-1} q(-1, \dots, -1)$, and

$$\tilde{q}(a) = \sum_{\substack{\alpha=(\alpha_1, \dots, \alpha_\ell) \\ \alpha_1 + \cdots + \alpha_\ell = s}} c_\alpha \frac{\alpha_1! \cdots \alpha_\ell!}{(s + \ell - 1)!} a^{s+\ell-1} + \text{lower order terms.}$$

Proof. It suffices to verify the properties when $q(b_1, \dots, b_\ell) = b_1^{\alpha_1} \cdots b_\ell^{\alpha_\ell}$ is a monomial. Let $S(n, j)$ denote the Stirling number of the second kind, so we have

$$q(b_1, \dots, b_\ell) = \prod_{i=1}^{\ell} \left(\sum_{j_i=0}^{\alpha_i} S(\alpha_i, j_i) j_i! \binom{b_i}{j_i} \right).$$

Using the definition of \tilde{q} and rearranging the summation, we have

$$\begin{aligned} \tilde{q}(a) &= \sum_{j_1=0}^{\alpha_1} \cdots \sum_{j_\ell=0}^{\alpha_\ell} \left(\prod_{i=1}^{\ell} S(\alpha_i, j_i) j_i! \right) \sum_{\substack{(p_1, \dots, p_\ell) \in \mathbb{N}^\ell \\ p_1 + \cdots + p_\ell = a}} \binom{p_1}{j_1} \cdots \binom{p_\ell}{j_\ell} \\ &= \sum_{j_1=0}^{\alpha_1} \cdots \sum_{j_\ell=0}^{\alpha_\ell} \left(\prod_{i=1}^{\ell} S(\alpha_i, j_i) j_i! \right) \binom{a + \ell - 1}{\ell - 1 + \sum_{i=1}^{\ell} j_i}. \end{aligned}$$

Using this formula, all of the claims can be easily verified. \square

For a class $a \in K(\mathbb{M})$, we set $s(a)$ to be its *total Segre class*, defined by the property that $s(a) = c(a)^{-1}$. We write $s_i(a)$ for the degree i part of $s(a)$, so $s(a) = s_0(a) + s_1(a) + \cdots + s_{r-1}(a)$.

Lemma 5.18. *The function $a \mapsto \chi(\mathbb{M}, \text{Sym}^a[\mathcal{Q}_{\mathbb{M}}])$ for $a \geq 0$ is a polynomial $q(a)$ of degree at most $n-2$. We have $q(-s) = 0$ for $s \in \{1, \dots, n-r-1\}$, and $q(-n+r) = (-1)^{n-r-1} \chi(\mathbb{M}, (\wedge^{n-r}[\mathcal{Q}_{\mathbb{M}}])^{-1})$. The coefficient of a^{n-2} in q is $\deg_{\mathbb{M}}(s_{r-1}(\mathcal{Q}_{\mathbb{M}}^{\vee})) / (n-2)!$.*

Proof. Although it isn't really necessary, we will make use of the following fact from [BEST23, Appendix III]: there are classes of line bundles $\mathcal{L}_1, \dots, \mathcal{L}_{n-r}$ on X_n such that $[\mathcal{Q}_{\mathbb{M}}] = [\mathcal{L}_1] + \cdots + [\mathcal{L}_{n-r}]$. This will make it easy to compute $\text{Sym}^a[\mathcal{Q}_{\mathbb{M}}]$: in either $K(X_n)$ or $K(\mathbb{M})$, we have

$$\text{Sym}^a[\mathcal{Q}_{\mathbb{M}}] = \sum_{\substack{(p_1, \dots, p_{n-r}) \in \mathbb{N}^{n-r} \\ p_1 + \cdots + p_{n-r} = a}} [\mathcal{L}_1]^{p_1} \cdots [\mathcal{L}_{n-r}]^{p_{n-r}}.$$

Let q be the Snapper polynomial of $\mathcal{L}_1, \dots, \mathcal{L}_{n-r}$. Then we have

$$\begin{aligned} \chi(\mathbb{M}, \text{Sym}^a[\mathcal{Q}_{\mathbb{M}}]) &= \sum_{\substack{(p_1, \dots, p_{n-r}) \in \mathbb{N}^{n-r} \\ p_1 + \cdots + p_{n-r} = a}} \chi(\mathbb{M}, [\mathcal{L}_1]^{p_1} \cdots [\mathcal{L}_{n-r}]^{p_{n-r}}) \\ &= \sum_{\substack{(p_1, \dots, p_{n-r}) \in \mathbb{N}^{n-r} \\ p_1 + \cdots + p_{n-r} = a}} q(p_1, \dots, p_{n-r}). \end{aligned}$$

Because q is a polynomial of degree at most $r-1$, it follows from Proposition 5.17 that this is a polynomial of degree at most $n-2$. The statements about the evaluations of this polynomial at the negative integers follow from Proposition 5.17; the class $(\wedge^{n-r}[\mathcal{Q}_{\mathbb{M}}])^{-1}$ is equal to $\mathcal{L}_1^{-1} \cdots \mathcal{L}_{n-r}^{-1}$.

The i th Segre class of $[\mathcal{Q}_{\mathbb{M}}^{\vee}]$ is equal to the i th complete symmetric function evaluated at $c_1(\mathcal{L}_1), \dots, c_1(\mathcal{L}_{n-r})$. In particular,

$$s_{r-1}(\mathcal{Q}_{\mathbb{M}}^{\vee}) = \sum_{\substack{\alpha = (\alpha_1, \dots, \alpha_{n-r}) \\ \alpha_1 + \cdots + \alpha_{n-r} = r-1}} c_1(\mathcal{L}_1)^{\alpha_1} \cdots c_1(\mathcal{L}_{n-r})^{\alpha_{n-r}}.$$

The claim about the coefficient of a^{n-2} follows from comparing Proposition 5.17 with Lemma 5.15. \square

Using Lemma 5.18, we can extend the definition of the g -polynomial to all matroids.

Definition 5.19. *Let \mathbb{M} be a matroid. If \mathbb{M} has a loop or coloop, then define $g_{\mathbb{M}}(t)$ to be 0. Otherwise, define the g -polynomial as follows: if we write*

$$\chi(\mathbb{M}, \text{Sym}^a[\mathcal{Q}_{\mathbb{M}}]) = \sum_{q=0}^{n-2} a_q \binom{a+q}{q},$$

then for each $i > 0$, set the coefficient of t^i in $g_{\mathbb{M}}(t)$ to be $(-1)^{i+c(\mathbb{M})} a_{n-1-i}$. Set the other coefficients to be 0.

Proposition 5.20. *The assignment $\mathbb{M} \mapsto (-1)^{c(\mathbb{M})} g_{\mathbb{M}}(t)$ is a valuative invariant of matroids.*

Proof. By Proposition 5.1, we may restrict our attention to loopless and coloopless matroids. The polynomial $a \mapsto \chi(\mathbb{M}, \text{Sym}^a[\mathcal{Q}_{\mathbb{M}}])$ has degree at most $n-2$, and so the coefficients which express it in the basis $\binom{a}{0}, \binom{a+1}{1}, \dots, \binom{a+n-2}{n-2}$ for the space of polynomials of degree at most $n-2$ are certain fixed linear combinations of $\chi(\mathbb{M}, \text{Sym}^0[\mathcal{Q}_{\mathbb{M}}]), \dots, \chi(\mathbb{M}, \text{Sym}^{n-2}[\mathcal{Q}_{\mathbb{M}}])$. It follows from Proposition 5.13 and (13) that, for any $a \geq 0$, the function which assigns a loopless and coloopless matroid \mathbb{M} to $\chi(\mathbb{M}, \text{Sym}^a[\mathcal{Q}_{\mathbb{M}}])$ is valuative. The result follows from Definition 5.19. \square

This implies that the above definition of the g -polynomial agrees with other definitions of the g -polynomial in the literature, because they agree for realizable matroids; see Corollary 4.11.

Proposition 5.21. *Let M be a matroid. Then $g_M(t) = g_{M^\perp}(t)$.*

Proof. If M has a loop or a coloop, then this is obvious. The case when M is realizable is Proposition 5.6. The map which sends a matroid to the class of its dual defines an isomorphism from $\text{Val}_{r,n}$ to $\text{Val}_{n-r,n}$, and so the function which sends M to $(-1)^{c(M^\perp)}g_{M^\perp}(t)$ is valuative. As $c(M) = c(M^\perp)$, the result follows from Corollary 4.11. \square

Corollary 5.22. *Let M be a matroid of rank r on $[n]$. Then $g_M(t)$ has degree at most $\min\{r, n-r\}$.*

Proof. By Proposition 5.21, it suffices to show that the degree of $g_M(t)$ is at most r . Let $p(a)$ be the polynomial which agrees with $a \mapsto \chi(M, \text{Sym}^a[\mathcal{Q}_M])$ for $a \geq 0$. By Lemma 5.18, $p(-1) = \cdots = p(-n+r) = 0$. This implies that, if we write

$$p(a) = \sum_{q=0}^{n-2} a_q \binom{a+q}{q},$$

then $a_q = 0$ for $q < n-r-1$, so this follows from Definition 5.19. \square

We define \mathcal{L}_{M^\perp} to be the class in $K(M)$ given by $\wedge^{n-r}[\mathcal{Q}_M]$. This is the class of a line bundle: as we will discuss in Section 9.1, it is the restriction of the line bundle on X_n corresponding to the matroid polytope of M^\perp .

Let M be a matroid of rank r . The coefficient of t^r in $g_M(t)$ has been closely studied in [FSS24], where it is called the *omega invariant* $\omega(M)$ of a matroid. Corollary 5.22 implies that $\omega(M) = 0$ if $r > n/2$. We note the following formula for $\omega(M)$.

Proposition 5.23. *Let M be a loopless and coloopless matroid of rank r . Then $\omega(M) = (-1)^{r+c(M)}\chi(M, \mathcal{L}_{M^\perp}^{-1})$.*

Proof. Let $p(a)$ be the polynomial which agrees with $a \mapsto \chi(M, \text{Sym}^a[\mathcal{Q}_M])$ for $a \geq 0$. By Corollary 5.22, we can write

$$p(a) = \sum_{q=n-r-1}^{n-2} a_q \binom{a+q}{q}.$$

Evaluating this at $a = -n+r$ and using Lemma 5.18, we see that $(-1)^{n-r-1}\chi(M, \mathcal{L}_{M^\perp}^{-1}) = (-1)^{n-r-1}a_{n-r-1}$, so the t^r coefficient of $g_M(t)$ is equal to $(-1)^{r+c(M)}\chi(M, \mathcal{L}_{M^\perp}^{-1})$. \square

Finally, we mention a constraint on the coefficients of the g -polynomial.

Proposition 5.24. [Spe09, Proposition 6.4] *Let M be a loopless and coloopless matroid. Then $g_M(-1) = (-1)^{c(M)}$.*

Proof. This follows from the fact that $\chi(M, \text{Sym}^0[\mathcal{Q}_M]) = 1$ and Definition 5.19. \square

Proposition 5.25. *Suppose that M is a matroid with $M = M_1 \oplus \cdots \oplus M_c$. Then*

$$g_M(t) = g_{M_1}(t) \cdots g_{M_c}(t).$$

Proof. It suffices to do the case $c = 2$. Let n_1 be the size of the ground set of M_1 , and let n_2 be the size of the ground set of M_2 . By Proposition 5.20, there is a map $\text{Val}_{n_1} \otimes \text{Val}_{n_2} \rightarrow \mathbb{Z}[t]$ given by $(M_1, M_2) \mapsto (-1)^{c(M_1)+c(M_2)}g_{M_1}(t) \cdot g_{M_2}(t)$. On the other hand, there is a map $\text{Val}_{n_1} \otimes \text{Val}_{n_2} \rightarrow \text{Val}_{n_1+n_2}$ given by $(M_1, M_2) \mapsto M_1 \oplus M_2$. This gives a map $\text{Val}_{n_1} \otimes \text{Val}_{n_2} \rightarrow \mathbb{Z}[t]$ given by $(M_1, M_2) \mapsto (-1)^{c(M_1 \oplus M_2)}g_{M_1 \oplus M_2}(t) = (-1)^{c(M_1)+c(M_2)}g_{M_1 \oplus M_2}(t)$. When M_1 and M_2 are realizable over the same field, the images of (M_1, M_2) under these two maps agree by Proposition 5.10. As $\text{Val}_{n_1} \otimes \text{Val}_{n_2}$ is generated by classes of pairs of matroids which are realizable over any infinite field by Theorem 4.10, this implies the result. \square

5.5. The beta invariant of a matroid. Now that we have defined the g -polynomial for all matroids, we will compute its linear term. This computation will show that the linear term of the g -polynomial is strictly positive for connected matroids, which will be crucial for applications to Conjecture 2.11.

The *beta invariant* $\beta(M)$ is an important invariant of matroids. It can be defined using a deletion-contraction recursion. We have $\beta(U_{0,1}) = 0$, $\beta(U_{1,1}) = 1$, and if $i \in [n]$ is an element which is not a loop or coloop, then $\beta(M) = \beta(M/i) + \beta(M \setminus i)$. It can also be defined as the coefficient of x in the Tutte polynomial of M .

Proposition 5.26. *Let M be a loopless and coloopless matroid. The coefficient of t in $g_M(t)$ is $\beta(M)$.*

This can be proved in several ways. In [Spe09], Speyer directly verified that the linear term of the g -polynomial satisfied the recursion defining the beta invariant when the matroid is realizable. This then extends to all matroids by Proposition 5.20 and Corollary 4.11, as the β invariant is known to be valutive; see [Spe08, Lemma 6.4].

Proof of Proposition 5.26. It follows from Lemma 5.18 that the linear term of the g -polynomial is equal to $\deg_M(s_{r-1}(\mathcal{Q}_M^\vee))$. In the realizable case, this is because the coefficient of t in $g_M(t)$ is the top self-intersection of $\mathcal{O}(1)$ on $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$. Because $[\mathcal{S}_M] + [\mathcal{Q}_M] = [\mathcal{O}^{\oplus n}]$, we have that $[\mathcal{S}_M^\vee] + [\mathcal{Q}_M^\vee] = [\mathcal{O}^{\oplus n}]$. This implies that $c(\mathcal{S}_M^\vee)c(\mathcal{Q}_M^\vee) = c(\mathcal{O}^{\oplus n}) = 1$, so the Segre classes of $[\mathcal{Q}_M^\vee]$ are the Chern classes of $[\mathcal{S}_M^\vee]$. The result then follows from [BEST23, Theorem 6.2]. \square

Remark 5.27. When M is realized by L , one can deduce that $\deg_M(c_{r-1}(\mathcal{S}_M^\vee)) = \beta(M)$ by using the logarithmic Poincaré-Hopf theorem, as follows. In [BEST23, Theorem 8.8], it is shown that \mathcal{S}_L is an extension of the log tangent bundle of W_L by a trivial bundle, so $c_{r-1}(\mathcal{S}_L)$ is the top Chern class of the log tangent bundle. The logarithmic Poincaré-Hopf theorem states that the degree of the top Chern class of the log cotangent bundle of a simple normal crossings compactification of a variety U is $(-1)^{\dim U}$ times the topological Euler characteristic of U ; see [Sil96, Theorem 4.1]. It is known that the Euler characteristic of $\mathbb{P}L \cap \mathbb{G}_m^n / \mathbb{G}_m$ is $(-1)^{r-1}\beta(M)$; see [OT92, Theorem 3.68, Theorem 5.90].

In the applications to Conjecture 2.11, we will use the following combinatorial property of the beta invariant. The difficult part is a consequence of the deletion-contraction recursion for $\beta(M)$ and Proposition 5.7.

Proposition 5.28. [Cra67] *A matroid M on at least 2 elements is connected if and only if $\beta(M)$ is positive.*

The next result follows from Proposition 5.26 and Proposition 5.25.

Corollary 5.29. *If M is a loopless and coloopless matroid with connected components M_1, \dots, M_c , then the coefficient of t^c in $g_M(t)$ is $\beta(M_1) \cdots \beta(M_c)$. In particular, the coefficient of t^c is strictly positive.*

Proposition 5.25 implies that the coefficient of t^i in $g_M(t)$ vanishes for $i < c(M)$. We see that the polynomial $a \mapsto \chi(M, \text{Sym}^a[\mathcal{Q}_M])$ has degree $n - c(M) - 1$.

Finally, we note the following interpretation of the beta invariant.

Proposition 5.30. [Spe09, Theorem 5.1] *Let $\lambda = (n - r, 1^{r-1})$ be the hook shape. Then $\beta(M) = a_\lambda(M)$.*

Proof. Using Corollary 4.11, that the β invariant is valutive [Spe08, Lemma 6.4], and that $a_\lambda(M)$ is valutive, it suffices to prove this when M is realized by a linear subspace $L \subseteq k^n$. If M is disconnected, then $\dim \overline{T \cdot [L]} < n - 1$, so $a_\lambda(M) = 0$. It therefore suffices to consider the case when M is connected.

Let $\text{Gr}_e = \{[L] \in \text{Gr}(r, n) : (1, \dots, 1) \in L\}$. This subvariety is isomorphic to $\text{Gr}(r - 1, n - 1)$, and its class in $A^\bullet(\text{Gr}(r, n))$ is the class of the Schubert variety Ω_μ , where $\mu = ((n - r)^{r-1})$. Indeed, $\Omega_\mu = \{[L] \in \text{Gr}(r, n) : (1, 0, \dots, 0) \in L\}$, which is a degeneration of Gr_e .

The variety $\mathcal{V}_L := \overline{T \cdot [L]} \cap \text{Gr}_e$ is called the *visible contours* compactification of the hyperplane arrangement complement $\mathbb{P}L \cap \mathbb{G}_m^n / \mathbb{G}_m$. This variety was introduced by Kapranov [Kap93], and it is generally singular.

The intersection of \mathcal{V}_L with the embedded torus $\mathbb{G}_m^n/\mathbb{G}_m$ in $\overline{T \cdot [L]}$ is isomorphic to $\mathbb{P}L \cap \mathbb{G}_m^n/\mathbb{G}_m$: we have $t \cdot [L] \in \text{Gr}_e$ if and only if $t^{-1} \in L$, so applying the inversion map on the torus identifies it with $\mathbb{P}L \cap \mathbb{G}_m^n/\mathbb{G}_m$. In fact, the intersection of $\overline{T \cdot [L]}$ with Gr_e is transverse in an appropriate sense; see [Tev07, Theorem 1.7]. In particular, this implies that $T \cdot [L] \cap \text{Gr}_e$ is dense in \mathcal{V}_L , and that, in $A^\bullet(\text{Gr}(r, n))$, we have $[\mathcal{V}_L] = \overline{[T \cdot [L]]} \cdot [\text{Gr}_e]$.

Let \mathcal{S} denote the rank r tautological subbundle on $\text{Gr}(r, n)$. It follows from Pieri's rule that $[\text{Gr}_e] \cdot c_{r-1}(\mathcal{S}^\vee)$ is the Poincaré dual class to $[\Omega_\lambda]$, so $a_\lambda(\text{M})$ is the intersection number $\int_{\mathcal{V}_L} c_{r-1}(\mathcal{S}^\vee)$. There is a birational map $\pi: W_L \rightarrow \mathcal{V}_L$, and $\pi^* \mathcal{S}^\vee = \mathcal{S}_L^\vee$. The result follows, because $\deg_{\text{M}}(c_{r-1}(\mathcal{S}_M^\vee)) = \beta(\text{M})$, as discussed above. \square

Remark 5.31. The map $W_L \rightarrow \mathcal{V}_L$ in the proof of the previous result is induced by the complete linear series of the line bundle $\mathcal{L}_{\text{M}^\perp}$. It follows from Theorem 9.3 that $\mathcal{V}_L \cong \text{Proj} \oplus_{a \geq 0} H^0(W_L, \mathcal{L}_{\text{M}^\perp}^{\otimes a})$. The line bundle $\mathcal{L}_{\text{M}^\perp}$ is the log-canonical bundle of W_L : it is the line bundle corresponding to the sum of the canonical divisor with the divisor $W_L \setminus (\mathbb{P}L \cap \mathbb{G}_m^n/\mathbb{G}_m)$. At least if k has characteristic 0, then this implies that \mathcal{V}_L is the log-canonical model of $\mathbb{P}L \cap \mathbb{G}_m^n/\mathbb{G}_m$: if X is any simple normal crossings compactification of $\mathbb{P}L \cap \mathbb{G}_m^n/\mathbb{G}_m$ with log-canonical bundle \mathcal{L} , then $\mathcal{V}_L \cong \text{Proj} \oplus_{a \geq 0} H^0(X, \mathcal{L}^{\otimes a})$.

5.6. Bounding the complexity of a matroid polytope subdivision. We now explain how the g -polynomial can be applied to Conjecture 2.11. This rests upon the following conjecture.

Conjecture 5.32. *Let M be a matroid. Then the coefficients of $g_{\text{M}}(t)$ are non-negative.*

We will discuss the known cases of Conjecture 5.32 in Section 5.7. The most important evidence for Conjecture 5.32 is the following result.

Theorem 5.33. [Spe09, Proposition 3.3] *Let M be a matroid which is realizable over a field of characteristic 0. Then the coefficients of $g_{\text{M}}(t)$ are non-negative.*

Proof of Theorem 5.33. We use the notation in Definition 5.2. By Proposition 5.25, we may assume that M is connected, i.e., $c(\text{M}) = 1$. Let Z_i be the inverse image under m of the result of slicing H by $n - i - 1$ generic hyperplanes. By Proposition 3.4, $m^*[\mathcal{O}_D]^{n-i-1} = [\mathcal{O}_{Z_i}]$. By Bertini's theorem, as in the proof of Theorem 4.5, Z_i is a smooth variety of dimension $i - 1$. By Proposition 5.8, because M is connected, m is dominant, and so the restriction of $m^*\mathcal{O}(1)$ to Z_i is a nef and big line bundle. Corollary 4.7 then implies that $(-1)^{i-1} \chi(Z_i, m^*\mathcal{O}(-1)) \geq 0$, implying the result. \square

Proposition 5.34. [Spe09, Proposition 3.1] *For each $0 < r < n$, we have*

$$g_{U_{r,n}}(t) = \sum_{i=1}^{\min\{r, n-r\}} \frac{(n-i-1)!}{(r-i)!(n-r-i)!(i-1)!} t^i.$$

Proof. The statement is clearly true for $r = 0$ or $r = n$, so we may assume that $0 < r < n$. Let $L \subseteq k^n$ be a linear subspace realizing $U_{r,n}$. We claim that the rational map $m: \mathbb{P}L \times \mathbb{P}L^\perp \dashrightarrow H$ is a morphism, i.e., is defined everywhere. To check this, suppose that (x_1, \dots, x_n) and (y_1, \dots, y_n) are non-zero vectors in L and L^\perp , respectively, such that $x_i y_i = 0$ for all i . Let $F = \{i \in [n] : x_i = 0\}$, and let $G = \{j \in [n] : y_j = 0\}$. Because L and L^\perp realize the uniform matroid, we have that $|F| \leq r - 1$ and $|G| \leq n - r - 1$. So $|F| + |G| \leq n - 2$, a contradiction.

We can therefore take $Z = \mathbb{P}L \times \mathbb{P}L^\perp$. Note that $m^*\mathcal{O}(1)$ is the line bundle $\mathcal{O}(1, 1)$ on $\mathbb{P}L \times \mathbb{P}L^\perp$. The Hilbert polynomial of $\mathcal{O}(1, 1)$ is $a \mapsto \binom{a+r-1}{r-1} \cdot \binom{a+n-r-1}{n-r-1}$. The claim then follows from the computation that

$$\begin{aligned} \binom{a+r-1}{r-1} \binom{a+n-r-1}{n-r-1} &= (-1)^n \binom{-a-1}{r-1} \binom{-a-1}{n-r-1} \\ &= (-1)^n \sum_{q=\max\{r, n-r\}-1}^{n-2} \binom{q}{n-r-1} \binom{n-r-1}{q-r+1} \binom{-a-1}{q} \\ &= \sum_{q=\max\{r, n-r\}-1}^{n-2} \frac{(-1)^{n-q} q!}{(q+r-n+1)!(q-r+1)!(n-2-q)!} \binom{a+q}{q}. \quad \square \end{aligned}$$

Theorem 5.35. *A matroid polytope subdivision of $P(U_{r,n})$ where all matroids M appearing have all coefficients of $g_M(t)$ non-negative has at most $\frac{(n-c-1)!}{(r-c)!(n-r-c)!(c-1)!}$ interior faces of dimension $n-c$ for $c \leq \min\{r, n-r\}$, and it has no interior faces of dimension less than $\max\{r, n-r\}$.*

In particular, Conjecture 5.32 implies Conjecture 2.11.

Proof of Theorem 5.35. The matroid polytope of a matroid with a loop or coloop is contained in the boundary of $P(U_{r,n})$, so all interior faces of a matroid polytope subdivision of $P(U_{r,n})$ correspond to loopless and coloopless matroids. Using (2) and Proposition 5.20, we deduce that

$$-g_{U_{r,n}}(t) = \sum_{M, P(M) \text{ interior face}} (-1)^{c(M)+1} (-1)^{c(M)} g_M(t).$$

Looking at the coefficient of t^c for $c \leq \min\{r, n-r\}$, we see that $\frac{(n-c-1)!}{(r-c)!(n-r-c)!(c-1)!}$ is a sum of the coefficients of t^c in $g_M(t)$ over all matroids M where $P(M)$ is an interior face. By Corollary 5.29 (or Corollary 5.12 for realizable matroids), each interior face of dimension $n-c$ contributes at least 1 to this sum, and each other face contributes non-negatively, and so there are at most $\frac{(n-c-1)!}{(r-c)!(n-r-c)!(c-1)!}$ faces of dimension $n-c$. If $c > \min\{r, n-r\}$, the coefficient of t^c on the left-hand side is 0, so there can be no faces of dimension $n-c$. \square

Remark 5.36. In [Spe09], Speyer shows that if M is a series-parallel matroid, then $g_M(t) = t$. It is known that if M is a matroid on at least 2 elements, then $\beta(M) = 1$ if and only if M is a series-parallel matroid [Bry71, Theorem 7.6]. This implies that a loopless and coloopless matroid M has $g_M(t) = t^{c(M)}$ if and only if it is a direct sum of series-parallel matroids, i.e., if and only if it is quasi series-parallel. A face of the matroid polytope of a quasi series-parallel matroid is a quasi series-parallel matroid, and so if all $(n-1)$ -dimensional faces in a matroid polytope subdivision of $P(U_{r,n})$ are series-parallel, then all interior faces are quasi series-parallel, and the bounds in Conjecture 2.11 are sharp. Furthermore, if any $(n-1)$ -dimensional face is not a series-parallel matroid, then there are fewer than $\binom{n-2}{r-1}$ $(n-1)$ -dimensional faces.

Remark 5.37. Assume that Conjecture 5.32 holds. The proof of Theorem 5.35 shows that if M is connected, then the number of interior faces of dimension $n-c$ in a matroid polytope subdivision of $P(M)$ is at most the coefficient of t^c in $g_M(t)$. It follows from Example 2.8 that $g_M(t) \leq g_{U_{r,n}}(t)$ coefficient-wise and that $\beta(M) < \beta(U_{r,n})$ if M is not uniform, so this is a strictly stronger bound if M is not uniform. This bound on the number of interior faces of dimension $n-c$ is sharp for every c if and only if $P(M)$ admits a matroid polytope subdivision into matroid polytopes of quasi series-parallel matroids. This holds for Schubert matroids [CRA11], but no such subdivision exists for many families of matroids. For example, if M is a connected matroid which is realizable over \mathbb{F}_2 , then $P(M)$ is the only $(n-1)$ -dimensional matroid

polytope which is contained in $P(M)$ [Luc75, Theorem 6.18]. In particular, if M is not series-parallel, then there is no matroid polytope subdivision into matroid polytopes of quasi series-parallel matroids.

5.7. Known cases of Conjecture 5.32. Besides Theorem 5.33 and Corollary 5.29, the most important piece of evidence for Conjecture 5.32 is the following result. Recall that if M is a matroid, the coefficient of t^r in $g_M(t)$ is denoted $\omega(M)$.

Theorem 5.38. [BF24] *Let M be a matroid. Then $\omega(M) \geq 0$.*

A different proof of this result was given in [EFL25]. We will discuss the proofs of Theorem 5.38 in Section 9. The known results imply that Conjecture 5.32 holds for matroids of rank at most 3.

Corollary 5.39. *Let M be a matroid of rank at most 3. Then the coefficients of $g_M(t)$ are non-negative.*

Proof. If M has rank at most 2, then M is realizable over every field, and so Theorem 5.33 applies; these cases can also be checked directly. If M is disconnected, then it is a direct sum of matroids of rank at most 2, and so the result follows from Proposition 5.25.

The remaining case is when M is connected of rank 3. Write $g_M(t) = \beta(M)t + g_2t^2 + \omega(M)t^3$. By Proposition 5.24, we have $g_2 = \omega(M) + \beta(M) - 1$. As M is connected, $\beta(M) \geq 1$ by Proposition 5.28, so $g_2 \geq 0$. \square

See [FSS24, Corollary 11.2] for a direct combinatorial proof that the omega invariant of a rank 3 matroid is non-negative using Example 5.4. Conjecture 5.32 is open for matroids of rank at least 4. There are some families of matroids where Conjecture 5.32 can be checked, such as for “sparse paving matroids” [FS24, Theorem 9.21].

6. POSITIVITY VIA COHEN–MACAULAYNESS

The driving force behind all of the K -theoretic positivity phenomena that we have seen has been Corollary 4.7. In situations where the hypotheses of Corollary 4.7 are not satisfied, it has been difficult to prove directly that the conclusion holds. Speyer’s proof of Theorem 5.33 involves applying Corollary 4.7 to general slices of a variety, and these seem even harder to control. One could try to prove Conjecture 5.32 by finding a combinatorial interpretation of the coefficients of the g -polynomial, or by finding a non-negative recursive formula. It is possible to prove that the β invariant of a matroid, i.e., the linear coefficient of the g -polynomial, is non-negative in either of these ways. But no one has been able to do this for the other coefficients.

Corollary 4.7 concerns the value of the Hilbert polynomial of \mathcal{L} at -1 . This is difficult to get at directly because it involves understanding the higher cohomology of \mathcal{L}^{-1} . It is usually much easier to analyze the sections and cohomology of positive twists of \mathcal{L} . It is sometimes possible to prove the conclusion of Corollary 4.7 by proving a stronger result about the sections of positive twists, known as *arithmetic Cohen–Macaulayness*. The significance of Cohen–Macaulayness in combinatorial problems was first realized by Stanley [Sta75, Sta80].

Let $A^\bullet = A^0 \oplus A^1 \oplus \cdots$ be a finitely generated graded algebra of Krull dimension d , with A^0 a finite dimensional vector space over k . Then A^\bullet is *Cohen–Macaulay* if there are homogeneous elements $\theta_1, \dots, \theta_d$ such that, for $i = 1, \dots, d$,

$$\theta_i \text{ is not a zero-divisor in } A^\bullet / (\theta_1, \dots, \theta_{i-1}).$$

The elements $\theta_1, \dots, \theta_d$ are called a *homogeneous system of parameters*. Any homogeneous elements $\theta_1, \dots, \theta_d$ such that $A^\bullet / (\theta_1, \dots, \theta_d)$ has Krull dimension 0 are a homogeneous system of parameters [BH93, Theorem 2.1.2]. In most cases that are relevant to combinatorics, the ideal generated by A^1 is primary to the homogeneous maximal ideal. For example, this is true if A^\bullet is a finitely generated module over the subalgebra

generated by A^1 . As k is infinite, the ideal generated by A^1 being primary to the homogeneous maximal ideal is equivalent to the existence of a homogeneous system of parameters consisting of elements of A^1 [BH93, Proposition 1.5.12], which we call a *linear system of parameters* (l.s.o.p.). Any d general elements of A^1 will do.

For example, A^\bullet is automatically Cohen–Macaulay if $d = 0$. If $d = 1$, then A^\bullet is Cohen–Macaulay if and only if there is a homogeneous element which is not a zero-divisor. This holds if and only if $\text{Spec } A^\bullet$ is generically reduced.

More generally, let M be a graded module over a finitely generated graded ring A^\bullet where A^0 is a finitely generated vector space over k . Let d be the dimension of the support of M . Then M is said to be *Cohen–Macaulay* if there are homogeneous elements $\theta_1, \dots, \theta_d$ such that, for each i , θ_i is not a zero-divisor on $M/(\theta_1, \dots, \theta_{i-1})M$. The ring A^\bullet is Cohen–Macaulay as an A^\bullet -module if and only if it is a Cohen–Macaulay ring.

A graded ring being Cohen–Macaulay gives very strong restrictions on the Hilbert function of A^\bullet . For a \mathbb{Z} -graded module M^\bullet , let $\text{Hilb}_M(z) = \sum_{i=-\infty}^{\infty} \dim M^i \cdot z^i$ denote the Hilbert series of M^\bullet . Here M^i is the i th graded piece of M^\bullet .

Proposition 6.1. *Assume that A^\bullet admits an l.s.o.p. If A^\bullet is Cohen–Macaulay of Krull dimension d , then we can write*

$$\text{Hilb}_A(z) = \frac{h_0 + h_1 z + \dots + h_e z^e}{(1-z)^d},$$

where $h_i \geq 0$ for each i . If A^\bullet is generated in degree 1, then (h_0, \dots, h_e) is the Hilbert function of an algebra which is generated in degree 1.

Proof. Choose an l.s.o.p. $\theta_1, \dots, \theta_d$. There is an exact sequence

$$0 \rightarrow (\theta_1) \rightarrow A^\bullet \rightarrow A^\bullet/(\theta_1) \rightarrow 0.$$

Because θ_1 is not a zero-divisor, we see that the Hilbert series of (θ_1) is $z \text{Hilb}_A(z)$. Using the additivity of Hilbert series in short exact sequences, we see that

$$(1-z) \text{Hilb}_A(z) = \text{Hilb}_{A/(\theta_1)}(z).$$

Repeating this argument, and using that θ_i is not a zero-divisor on $A^\bullet/(\theta_1, \dots, \theta_{i-1})$, we see that

$$(1-z)^d \text{Hilb}_A(z) = \text{Hilb}_{A/(\theta_1, \dots, \theta_d)}(z).$$

Note that $A^\bullet/(\theta_1, \dots, \theta_d)$ is a graded algebra of Krull dimension 0, and so its Hilbert series is a polynomial with non-negative coefficients, proving the claim about the form on the Hilbert series and the non-negativity of the h_i . If A^\bullet is generated in degree 1, then so is $A^\bullet/(\theta_1, \dots, \theta_d)$, proving the second part. \square

Let A^\bullet be a graded ring which is finitely generated over A^0 , where A^0 is a finite dimensional vector space over k . If A^\bullet is finitely generated over the subring generated by A^0 and A^1 , then for a sufficiently large, the Hilbert function of A^\bullet agrees with a polynomial, called the *Hilbert polynomial* [BH93, Theorem 4.1.3]. Often, the Hilbert function will be a polynomial for all $a \geq 0$. This holds in many cases which are relevant to combinatorics. If p is a polynomial of degree d , then we can write

$$(14) \quad \sum_{a \geq 0} p(a) t^a = \frac{h_0^* + h_1^* t + \dots + h_d^* t^d}{(1-t)^{d+1}},$$

where the coefficients (h_0^*, \dots, h_d^*) are related to p by the equation

$$(15) \quad p(a) = \sum_{q=0}^d h_q^* \binom{a+d-q}{d}.$$

See [Sta12, Section 4.3]. Note that (15) implies that $(-1)^d p(-1) = h_d^*$, and that the leading coefficient of p is $(h_0^* + \cdots + h_d^*)/d!$.

Note that the polynomial $a \mapsto \binom{a+d-q}{d}$ is the Hilbert polynomial of $\mathcal{O}(-q)$ on \mathbb{P}^d . The polynomials $\{\binom{a+d-q}{d}\}_{q=0,\dots,d}$ form a basis for the space of numerical polynomials of degree at most d , and computing the numerator of the series (14) for a polynomial of degree d is the same thing as expanding it in this basis.

If X is a d -dimensional subscheme of \mathbb{P}^N , then $[\mathcal{O}_X]$ lies in the piece F_{N-d} of the coniveau filtration on $K(\mathbb{P}^N)$. The classes $\{[\mathcal{O}_H]^{N-d}[\mathcal{O}(-q)]\}_{q=0,\dots,d}$ form a basis for F_{N-d} , and computing the numerator of the series $\sum_{a \geq 0} \chi(X, \mathcal{O}(a))t^a$ is the same as computing the expansion of $[\mathcal{O}_X]$ in this basis.

Remark 6.2. The K -polynomial of a subscheme X of \mathbb{P}^N is $(1-t)^{N+1}$ times the Hilbert series of the homogeneous coordinate ring of X [MS05, Definition 1.12]. This polynomial is not determined by the class $[\mathcal{O}_X]$ in $K(\mathbb{P}^N)$. Rather, it encodes the class of the affine cone over X in $K_{\mathbb{G}_m^{N+1}}(\mathbb{A}^{N+1})$.

Using that $[\mathcal{O}(-1)] = [\mathcal{O}] - [\mathcal{O}_H]$, we see that $[\mathcal{O}(-q)] = ([\mathcal{O}] - [\mathcal{O}_H])^q$. Equating the Hilbert polynomials of these classes, we see that

$$(16) \quad \binom{a+d-q}{d} = \sum_{i=0}^q (-1)^i \binom{q}{i} \binom{a+d-i}{d-i}.$$

The sign-definiteness of this transformation implies that if the numerator in (14) is non-negative, then the coefficients in the expansion of p into the polynomials $\{\binom{a+d-q}{d}\}$ have alternating signs. For example, we have the following corollary.

Corollary 6.3. *Let X be a subscheme of \mathbb{P}^N of dimension d such that the ring $\bigoplus_{a \geq 0} H^0(X, \mathcal{O}(a))$ is a Cohen–Macaulay ring. Then in the expansion $[\mathcal{O}_X] = \sum a_i [\mathcal{O}_H]^{N-d+i}$, we have $(-1)^i a_i \geq 0$.*

Proof. To simplify the proof, we assume that $\chi(X, \mathcal{O}(a)) = \dim H^0(X, \mathcal{O}(a))$ for $a \geq 0$; for the general case, see [BH93, Corollary 4.1.10]. Because the restriction of $\mathcal{O}(1)$ to X is globally generated, the ring $\bigoplus_{a \geq 0} H^0(X, \mathcal{O}(a))$ is finitely generated as a module over the subalgebra which is generated in degree 1; see [Laz04, Example 2.1.30]. This implies that $\bigoplus_{a \geq 0} H^0(X, \mathcal{O}(a))$ admits an l.s.o.p., and so Proposition 6.1 and the assumption that $\chi(X, \mathcal{O}(a)) = \dim H^0(X, \mathcal{O}(a))$ for $a \geq 0$ implies that, if we write

$$\sum_{a \geq 0} \chi(X, \mathcal{O}(a))t^a = \frac{h_0^* + h_1^*t + \cdots + h_d^*t^d}{(1-t)^{d+1}},$$

then $h_i^* \geq 0$ for all i . From (16), we deduce that

$$a_i = (-1)^i \sum_{q=i}^d \binom{q}{i} h_q^*. \quad \square$$

Note that Corollary 6.3 does not require the characteristic to be 0 or for X to have rational singularities (or even be integral). This opens the door to verifying the hypotheses of Corollary 6.3 by degenerating X to a reducible scheme. Additionally, while it is difficult to define varieties whose geometry is related to the combinatorics of a non-realizable matroid, it is not hard to construct reducible schemes related to non-realizable matroids. One can hope to prove that these schemes satisfy the hypothesis of Corollary 6.3. We will later discuss cohomological criteria for the section ring of a line bundle to be Cohen–Macaulay.

Example 6.4. Let $A^\bullet = k[x, y, z, w]/(x^4 - z^2, y^4 - w^2)$, with x and y in degree 1 and z and w in degree 2. Then A^\bullet is Cohen–Macaulay, with

$$\text{Hilb}_{A^\bullet}(t) = \frac{1 + 2t^2}{(1-t)^2} = \frac{-1 + 4t}{(1-t)^2} + 2.$$

I.e., the numerator of the Hilbert series has non-negative coefficients, but this is no longer true if we replace the Hilbert series by the Hilbert polynomial. Note that the Hilbert polynomial is equal to $3a - 1 = 3\binom{1+a}{1} - 4\binom{a}{0}$, so the inequalities in Theorem 4.5 hold (viewing $\text{Proj } A^\bullet$ as a subscheme of \mathbb{P}^3).

Example 6.5. Let Δ be a simplicial complex of dimension $d - 1$ on $[n]$, and let $k[\Delta]$ be the *Stanley–Reisner ring* of Δ , i.e., the quotient of $k[x_1, \dots, x_n]$ by the ideal generated by monomials corresponding to non-faces of Δ . For each i , let f_i denote the number of faces of Δ of dimension i , and set $f_{-1} = 1$. The non-zero monomials in $k[\Delta]$ are in bijection with faces F of Δ together with positive integers for each vertex. Grouping monomials by the vertices which have non-zero coefficients, we see that

$$(17) \quad \text{Hilb}_{k[\Delta]}(t) = \sum_{i=0}^d f_{i-1} \frac{t^i}{(1-t)^i} = \frac{(1-t)^d f_{-1} + t(1-t)^{d-1} f_0 + \dots + t^d f_{d-1}}{(1-t)^d}.$$

The expression in the numerator of (17) is called the *h-polynomial* of Δ , and its coefficients are called the *h-vector* of Δ . The Hilbert polynomial and Hilbert function of $k[\Delta]$ need not agree, so the *h-polynomial* contains slightly more information than the Hilbert polynomial of $k[\Delta]$. Proposition 6.1 shows that if $k[\Delta]$ is Cohen–Macaulay, then the *h-polynomial* of Δ has non-negative coefficients. If Δ is the simplicial complex in Figure 3, then

$$k[\Delta] = \frac{k[x_1, x_2, x_3, x_4, x_5]}{(x_1 x_4, x_1 x_5, x_2 x_4, x_2 x_5)}.$$

The *h-polynomial* of this complex is $1 + 2t - t^2$. The Hilbert polynomial of $k[\Delta]$ is $1 + 3a + a^2 = 2\binom{a+2}{2} - \binom{a}{0}$, so the inequalities in Theorem 4.5 are not satisfied. See [Sta96] for an extensive discussion of algebraic properties of Stanley–Reisner rings.

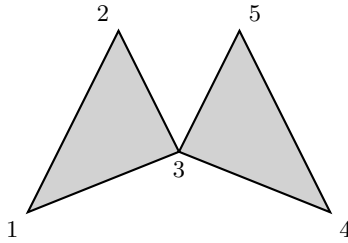


FIGURE 3. A non-Cohen–Macaulay simplicial complex

Even though the inequalities given by the Cohen–Macaulayness of the section ring of a variety are much stronger than those given by Theorem 4.5, they can be easier to prove. Unfortunately, these inequalities are so strong that they fail in the case of $\mathcal{O}(1)$ on $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$, the case most relevant for Theorem 5.35. Recall that the polynomial $a \mapsto \chi(M, \text{Sym}^a \mathcal{Q}_M)$ has degree $n - c(M) - 1$. If we write

$$\sum_{a \geq 0} \chi(M, \text{Sym}^a[\mathcal{Q}_M]) t^a = \frac{h_0^* + h_1^* t + \dots + h_{n-c(M)-1}^* t^{n-c(M)-1}}{(1-t)^{n-c(M)}},$$

then it follows from Definition 5.19 and (16) that

$$(18) \quad g_M(t)/t^c = h_0^* + h_1^*(t+1) + \dots + h_{n-c(M)-1}^*(t+1)^{n-c(M)-1}.$$

In particular, by Corollary 5.22, we have $h_i^* = 0$ for $i > r - c$. So if M is a realizable loopless and coloopless matroid where the section ring of $\mathcal{O}(1)$ on $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$ is Cohen–Macaulay and has a Hilbert function which

is a polynomial, then the evaluation of the polynomial $g_M(t)/t^c$ at $t - 1$ has non-negative coefficients. Unfortunately, this can fail.

Example 6.6. Let M be the Fano matroid F_7 . Then Example 5.4 implies that $g_M(t) = 3t + 5t^2 + 3t^3$, so

$$\begin{aligned} \chi(\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee), \mathcal{O}(a)) &= 3 \binom{a+5}{5} - 5 \binom{a+4}{4} + 3 \binom{a+3}{3} \\ &= \frac{a^5}{40} + \frac{a^4}{6} + \frac{13a^3}{24} + \frac{4a^2}{3} + \frac{29a}{15} + 1 \\ &= \binom{a+5}{5} - \binom{a+4}{5} + 3 \binom{a+3}{5}, \end{aligned}$$

i.e., the numerator of the series (14) is $3 + 5(t-1) + 3(t-1)^2 = 1 - t + 3t^2$. There are also counterexamples to the inequalities in Proposition 6.1 which are realizable over a field of characteristic 0; see Example 7.15 and [Fer23, Table 1].

In the case when A^\bullet is generated in degree 1, one obtains bounds on how fast the numerator of the Hilbert series can grow. These bounds are given explicitly by the following result.

Definition 6.7. A sequence (h_0, h_1, \dots, h_d) of integers is a Macaulay vector if $(h_0, h_1, \dots, h_d, 0, 0, \dots)$ is the Hilbert function of a graded artinian k -algebra A^\bullet which is generated in degree 1 and has $A^0 = k$.

Macaulay vectors are also called M-vectors and O-sequences. Macaulay gave an explicit description of these vectors, as follows [BH93, Theorem 4.2.10]. Given positive integers n and d , there is a unique expression

$$n = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \dots + \binom{k_\delta}{\delta}, \quad k_d > k_{d-1} > \dots > k_\delta \geq \delta \geq 1.$$

Set $n^{(d)} = \binom{k_d+1}{d+1} + \dots + \binom{k_\delta+1}{\delta+1}$, and define $0^{(i)} = 0$ for all i . Then $(1, a_1, \dots, a_d)$ is a Macaulay vector if and only if $0 \leq a_{t+1} \leq a_t^{(t)}$ for all $t \geq 1$.

If A^\bullet is a graded Cohen–Macaulay algebra which is generated in degree 1 and has $A^0 = k$, it follows from Proposition 6.1 that the coefficients of the numerator of the series (14) form a Macaulay vector.

Example 6.8. Let M be the non-Fano matroid F_7^+ . Then Example 5.4 implies that $g_M(t) = 4t + 6t^2 + 3t^3$, so the numerator of the series (14) is $1 + 3t^2$. Note that $(1, 0, 3)$ is not a Macaulay vector, so the function $a \mapsto \chi(M, \text{Sym}^a[\mathcal{Q}_M])$ is not the Hilbert function of a graded Cohen–Macaulay algebra A^\bullet which is generated in degree 1 and has $A^0 = k$.

For example, let P be a lattice polytope in \mathbb{R}^n , i.e., the convex hull of a finite number of points in \mathbb{Z}^n . For a natural number a , let $\text{ehr}_P(a)$ be the number of lattice points in the a th dilate of P . It is known that ehr_P is a polynomial in a , called the *Ehrhart polynomial*. Let R_P denote the semigroup algebra of P ; see Definition 1.17. The following result is due to Hochster.

Proposition 6.9. [BH93, Theorem 6.3.5] *Let P be a lattice polytope. Then the semigroup algebra R_P is Cohen–Macaulay.*

Furthermore, R_P is a finitely generated module over the subalgebra of R_P which is generated in degree 1. This implies the existence of an l.s.o.p. for R_P . Note that the Krull dimension of R_P is $\dim P + 1$.

Corollary 6.10. *Let P be a lattice polytope of dimension d with Ehrhart polynomial $\text{ehr}_P(a)$. Then we can write*

$$\sum_{a \geq 0} \text{ehr}_P(a) t^a = \frac{h_0^* + h_1^* t + \dots + h_d^* t^d}{(1-t)^{d+1}},$$

with $h_i^* \geq 0$ for each i . If, for each a , every lattice point in aP is a sum of a lattice points in P , then (h_0^*, \dots, h_d^*) is a Macaulay vector.

Proof. The Hilbert function of R_P is given by the Ehrhart polynomial, so the result follows from Proposition 6.1 and Proposition 6.9. \square

Remark 6.11. For a lattice polytope P of dimension d , h_d^* is the number of interior lattice points in P , which gives a combinatorial proof that $h_d^* \geq 0$. There are a number of combinatorial proofs of the non-negativity of the h^* -vector (h_0^*, \dots, h_d^*) ; see, e.g., [BS07], but in general there is no simple combinatorial interpretation of the entries of the h^* -vector.

Suppose that P is a lattice polytope in \mathbb{R}^n with a regular unimodular triangulation, given by a height function h . Let Δ be the simplicial complex corresponding to this triangulation. Then R_P is generated in degree 1, with variables x_1, \dots, x_m corresponding to the vertices of the triangulation. There is a degeneration of R_P to the Stanley–Reisner ring $k[\Delta]$. This can be thought of as a Gröbner degeneration (see [Stu91]), or viewed geometrically, as follows.

Because R_P is generated in degree 1, the toric variety X_P can be viewed as the closure of the projectivization of the image of the map from $\mathbb{G}_m^n \rightarrow \mathbb{G}_m^m$ given by the monomials corresponding to the lattice points in P . This realizes X_P as a closed subvariety of \mathbb{P}^{m-1} . By perturbing and scaling h , we can assume that it is integer valued. Then h induces an action of \mathbb{G}_m on \mathbb{P}^{m-1} , where an element t scales the i th coordinate of \mathbb{P}^{m-1} by the height of the i th lattice point. Then, as described in Section 2, the theory of projective toric varieties over a discrete valuation ring shows that the flat limit $\lim_{t \rightarrow 0} t \cdot X_P$ is a reduced union of coordinate subspaces, glued according to the triangulation. The homogeneous coordinate ring of this scheme is $k[\Delta]$, and this projective flat degeneration induces a flat degeneration of the homogeneous coordinate rings.

The topological space of the triangulation of P is a ball. It is known that the Stanley–Reisner ring of any triangulation of a ball is Cohen–Macaulay; see [BH93, Corollary 5.3.9]. In particular, as the Hilbert function of R_P is equal to the Hilbert function of $k[\Delta]$ (i.e., the h^* -vector of P is equal to the h -vector of the triangulation), this gives another way to see the non-negativity of the coefficients of the h^* -vector in the case where P has a regular unimodular triangulation.

When P has a unimodular triangulation Δ which is not regular, then there is no obvious geometric or algebraic connection between R_P and $k[\Delta]$. However, R_P and $k[\Delta]$ still have the same Hilbert function and $k[\Delta]$ is still Cohen–Macaulay, so this can still be used to prove the non-negativity of the coefficients of the h^* -vector.

This is a common technique: one proves the Cohen–Macaulayness of a ring by degenerating to the Stanley–Reisner ring of a simplicial complex, or some other ring which can be concretely analyzed. When one degenerates a variety, it often becomes reducible and so does not have rational singularities. In particular, the argument used to prove Theorem 4.5 cannot be applied. For this reason, we will develop techniques which are capable of proving Cohen–Macaulayness of reducible schemes.

6.1. Section rings. We now discuss one of the main sources of Cohen–Macaulay graded rings, namely section rings of line bundles on projective schemes. Let X be a projective scheme over k , and let \mathcal{L} be a globally generated line bundle on X . Then the complete linear series of \mathcal{L} induces a surjective map f from X to a projective scheme Y . There is a very ample line bundle $\mathcal{O}(1)$ on Y such that $\mathcal{L} \cong f^*\mathcal{O}(1)$. Let $d = \dim Y$, let $R_{\mathcal{L}} = \bigoplus_{a \in \mathbb{Z}} H^0(X, \mathcal{L}^{\otimes a})$ be the section ring of \mathcal{L} , and let $R_Y = \bigoplus_{a \in \mathbb{Z}} H^0(Y, \mathcal{O}(a))$. Note that $R_{\mathcal{L}}$ is a finitely generated module over R_Y , and so $R_{\mathcal{L}}$ has Krull dimension $d+1$. The following criterion for $R_{\mathcal{L}}$ to be Cohen–Macaulay will be useful.

Theorem 6.12. *Suppose that $H^i(X, \mathcal{L}^{\otimes a}) = 0$ for all $a \in \mathbb{Z}$ and all $0 < i < \dim Y$. Then $R_{\mathcal{L}}$ is Cohen–Macaulay.*

Before proving Theorem 6.12, we will need the following standard consequence of the Leray spectral sequence; see, e.g., [Hyr99, Lemma 2.1].

Proposition 6.13. *Let $\pi: Z \rightarrow W$ be a proper morphism, and let \mathcal{M} be an ample line bundle on W . Then, for each $i > 0$, $R^i\pi_*\mathcal{O}_Z = 0$ if and only if $H^i(Z, \pi^*\mathcal{M}^{\otimes a}) = 0$ for all a sufficiently large.*

Proof. Given a coherent sheaf \mathcal{F} on Z , recall that the Leray spectral sequence is a spectral sequence with $E_2^{p,q} = H^p(W, R^q\pi_*\mathcal{F})$ which converges to $H^{p+q}(Z, \mathcal{F})$. By the projection formula, $R^i\pi_*\pi^*\mathcal{M}^{\otimes a} \cong \mathcal{M}^{\otimes a} \otimes R^i\pi_*\mathcal{O}_Z$. By Serre vanishing,

$$H^j(W, \mathcal{M}^{\otimes a} \otimes R^i\pi_*\mathcal{O}_Z) = 0 \text{ for } j > 0 \text{ and } a \text{ sufficiently large.}$$

The Leray spectral sequence then implies that, for a sufficiently large, $H^i(Z, \pi^*\mathcal{M}^{\otimes a}) = H^0(W, \mathcal{M}^{\otimes a} \otimes R^i\pi_*\mathcal{O}_Z)$. As $\mathcal{M}^{\otimes a} \otimes R^i\pi_*\mathcal{O}_Z$ is globally generated for a sufficiently large, $R^i\pi_*\mathcal{O}_Z = 0$ if and only if $H^0(W, \mathcal{M}^{\otimes a} \otimes R^i\pi_*\mathcal{O}_Z) = 0$ for all a sufficiently large, which implies the result. \square

We will also need a cohomological criterion for the Cohen–Macaulayness of the section module of a sheaf.

Proposition 6.14. *Let X be a projective variety with a very ample line bundle $\mathcal{O}(1)$, and let \mathcal{F} be a coherent sheaf on X . Let $A^\bullet = \bigoplus_{a \in \mathbb{Z}} H^0(X, \mathcal{O}(a))$. Then the A^\bullet -module $\bigoplus_{a \in \mathbb{Z}} H^0(X, \mathcal{F} \otimes \mathcal{O}(a))$ is Cohen–Macaulay if and only if $H^i(X, \mathcal{F} \otimes \mathcal{O}(a)) = 0$ for all $0 < i < \dim \text{supp } \mathcal{F}$ and all $a \in \mathbb{Z}$.*

Proof. The Cohen–Macaulayness of a module can be described in terms of local cohomology: if \mathfrak{m} is the irrelevant ideal of A^\bullet , then $M := \bigoplus_{a \in \mathbb{Z}} H^0(X, \mathcal{F} \otimes \mathcal{O}(a))$ is a Cohen–Macaulay module if and only if $H_{\mathfrak{m}}^i(A^\bullet; M) = 0$ for all $i < \dim \text{supp } \mathcal{F} + 1$. We now interpret this in terms of sheaf cohomology; see [Smi97, Section 1] or [BS98, Theorem 20.4.4].

We have $H_{\mathfrak{m}}^0(A^\bullet; M) = H_{\mathfrak{m}}^1(A^\bullet; M) = 0$ because M , by construction, is a saturated A^\bullet -module. For $i > 1$, we have

$$H_{\mathfrak{m}}^i(A^\bullet; M) = \bigoplus_{a \in \mathbb{Z}} H^{i-1}(X, \mathcal{F} \otimes \mathcal{O}(a)).$$

That this vanishes for $1 < i < \dim \text{supp } \mathcal{F} + 1$ is exactly the hypothesis on \mathcal{F} . \square

Proof of Theorem 6.12. Let $\mathcal{O}(1)$ be the very ample line bundle on Y which pulls back to \mathcal{L} , and let $R_Y := \bigoplus_{a \in \mathbb{Z}} H^0(Y, \mathcal{O}(a))$ be the section ring of $\mathcal{O}(1)$. It suffices to show that $R_{\mathcal{L}}$ is a Cohen–Macaulay module over R_Y , as $\dim R_{\mathcal{L}} = \dim R_Y$. Note that

$$(19) \quad R_{\mathcal{L}} = \bigoplus_{a \in \mathbb{Z}} H^0(Y, f_*\mathcal{O}_X \otimes \mathcal{O}(a))$$

as $f_*\mathcal{L}^{\otimes a} = f_*\mathcal{O}_X \otimes \mathcal{O}(a)$ by the projection formula. We use the Leray spectral sequence to relate the cohomology of $f_*\mathcal{O}_X \otimes \mathcal{O}(a)$ on Y to the cohomology of $\mathcal{L}^{\otimes a}$ on X . By Proposition 6.13, we have $R^i f_*\mathcal{O}_X = 0$ for $0 < i < \dim Y$. The E_2 page of the Leray spectral sequence computing the cohomology of $\mathcal{L}^{\otimes a}$ looks like the following.

$$\begin{array}{c|cccc}
\vdots & \vdots & \vdots & \dots & \vdots \\
d & H^0(Y, R^d f_* \mathcal{O}_X \otimes \mathcal{O}(a)) & H^1(Y, R^d f_* \mathcal{O}_X \otimes \mathcal{O}(a)) & \dots & H^d(Y, R^d f_* \mathcal{O}_X \otimes \mathcal{O}(a)) \\
d-1 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \dots & \vdots \\
1 & 0 & 0 & \dots & 0 \\
0 & H^0(Y, f_* \mathcal{O}_X \otimes \mathcal{O}(a)) & H^1(Y, f_* \mathcal{O}_X \otimes \mathcal{O}(a)) & \dots & H^d(Y, f_* \mathcal{O}_X \otimes \mathcal{O}(a)) \\
\hline
& 0 & 1 & \dots & d
\end{array}$$

The differential on the E_i page goes i to the right and $i - 1$ down. In particular, none of the differentials involving the 0th row can be non-zero, so the 0th row survives to the E_∞ page. As only the 0th row contributes to $H^i(X, \mathcal{L}^{\otimes a})$ for $i < d$, we deduce that $H^i(Y, f_* \mathcal{O}_X \otimes \mathcal{O}(a)) = H^i(X, \mathcal{L}^{\otimes a})$ for $i < d$. This vanishes for $0 < i < d$ by assumption. This proves the Cohen–Macaulayness by Proposition 6.14. \square

Example 6.15. Let \mathcal{L} be a nef line bundle on a normal toric variety X . Then \mathcal{L} corresponds to a lattice polytope P whose normal fan is a coarsening of the normal fan of X . The line bundle \mathcal{L} is globally generated, and it induces a map from X to the toric variety of the normal fan of P , which has dimension equal to $\dim P$. It is known that, for $a \geq 0$, $H^0(X, \mathcal{L}^{\otimes a})$ has a basis labeled by lattice points in the dilate aP , and $H^i(X, \mathcal{L}^{\otimes a}) = 0$ for $i > 0$. Furthermore, assuming that $\dim P > 0$, the section ring $R_{\mathcal{L}}$ is the semigroup algebra R_P . For $a < 0$, it is known that $H^i(X, \mathcal{L}^{\otimes a}) = 0$ for $i < \dim P$ [Ful93, pg. 74]. Theorem 6.12 then implies that R_P is Cohen–Macaulay, recovering Proposition 6.9.

The following partial converse to Theorem 6.12 will be useful.

Proposition 6.16. *Let X be a projective scheme, and let \mathcal{L} be a globally generated line bundle on X . Let $f: X \rightarrow Y$ be the surjective map induced by the complete linear series on X . Assume that $R_{\mathcal{L}}$ is Cohen–Macaulay and that $R^i f_* \mathcal{O}_X = 0$ for $i > 0$. Then $H^i(X, \mathcal{L}^{\otimes a}) = 0$ for all $a \in \mathbb{Z}$ and all $0 < i < \dim Y$.*

For example, if \mathcal{L} is ample, then f is a finite map, and so the hypothesis of Proposition 6.16 is satisfied.

Proof of Proposition 6.16. The hypothesis on the vanishing on the higher direct images implies that the Leray spectral sequence computing the cohomology of $\mathcal{L}^{\otimes a}$ on X by pushing forward along f degenerates, and so $H^i(X, \mathcal{L}^{\otimes a}) \cong H^i(Y, f_* \mathcal{O}_X \otimes \mathcal{O}(a))$ for each i . By Proposition 6.14 and the Cohen–Macaulayness assumption, we know that $H^i(Y, f_* \mathcal{O}_X \otimes \mathcal{O}(a))$ vanishes for all $a \in \mathbb{Z}$ and any $0 < i < \dim Y$, giving the result. \square

Remark 6.17. Proposition 6.16 does not hold without the assumption that $R^i f_* \mathcal{O}_X = 0$ for $i > 0$. For example, if $\mathcal{L} = \mathcal{O}_X$ and X is geometrically integral, then $R_{\mathcal{L}} = k[t, t^{-1}]$ is always Cohen–Macaulay, but $H^i(X, \mathcal{O}_X)$ may be non-zero for all i . See [Kov] for an example where f is birational.

7. FURTHER PROPERTIES OF THE g -POLYNOMIAL

We now discuss some properties and alternative descriptions of the g -polynomial.

7.1. An intersection-theoretic description of the g -polynomial. We describe a formula for the g -polynomial in terms of intersection theory in the Chow ring of a matroid. This formula was conjectured in [LdMRS20, Conjecture 1] and proved in [BEST23, Theorem 10.12].

Theorem 7.1. *Let M be a loopless and coloopless matroid of rank r . Then*

$$g_M(t) = (-1)^{r+c(M)} \sum_{i=0}^r \deg_M(c(\mathcal{Q}_M^\vee) \cdot c_{r-i}(\mathcal{S}_M) \cdot (1 + \alpha + \cdots + \alpha^{r-1})) t^i.$$

Proof. Applying [LLPP24, Proposition 5.1] and [EHL23, Proposition 6.4], which help one compute ζ for certain classes, we have

$$\sum_{a=0}^{\infty} \zeta(\text{Sym}^a[\mathcal{Q}_M]) t^a = \frac{1}{(1-t)^{n-r}} \left(c(\mathcal{Q}_M^\vee) \cdot \left(1 + \frac{c_1(\mathcal{S}_M)}{t-1} + \frac{c_2(\mathcal{S}_M)}{(t-1)^2} + \cdots \right) \right)$$

as a power series with coefficients in $A^\bullet(M)$. Using Proposition 5.14, we deduce that

$$\begin{aligned} \sum_{a=0}^{\infty} \chi(M, \text{Sym}^a[\mathcal{Q}_M]) t^a &= \frac{1}{(1-t)^{n-r}} \deg_M \left(c(\mathcal{Q}_M^\vee) \cdot \left(1 + \frac{c_1(\mathcal{S}_M)}{t-1} + \frac{c_2(\mathcal{S}_M)}{(t-1)^2} + \cdots \right) \cdot (1 + \alpha + \cdots + \alpha^{r-1}) \right) \\ &= \frac{\deg_M(c(\mathcal{Q}_M^\vee) \cdot ((1-t)^r - c_1(\mathcal{S}_M)(1-t)^{r-1} + \cdots) \cdot (1 + \alpha + \cdots + \alpha^{r-1}))}{(1-t)^n} \end{aligned}$$

Let $c = c(M)$, and write $g_M(t) = g_c t^c + \cdots + g_r t^r$. Comparing with (18), we see that

$$\frac{\deg_M(c(\mathcal{Q}_M^\vee) \cdot ((1-t)^r - c_1(\mathcal{S}_M)(1-t)^{r-1} + \cdots) \cdot (1 + \alpha + \cdots + \alpha^{r-1}))}{(1-t)^c} = g_c + g_{c+1}(t-1) + \cdots + g_r(t-1)^{r-c}.$$

Multiplying by $(1-t)^c$ and comparing the coefficients of $(t-1)^i$ on each side gives the result. \square

7.2. The g -polynomial in terms of Euler characteristics of tautological bundles. Next, we discuss a formula for the g -polynomial which appeared in [FS12]. This formula is key to the approach of [BF24] to Theorem 5.38.

Theorem 7.2. [FS12, Theorem 6.1] *Let M be a loopless and coloopless matroid of rank r . Then*

$$g_M(t) = (-1)^{c(M)} \sum_{i=0}^r \chi(X_n, \wedge^i[\mathcal{S}_M] \otimes \wedge^i[\mathcal{Q}_M^\vee]) (t+1)^i.$$

By considering the t^r coefficient and using the fact that $\wedge^r[\mathcal{S}_M] = [\mathcal{L}_{M^\perp}^{-1}] = \wedge^{n-r}[\mathcal{Q}_M^\vee]$, we have the following corollary, which is critical in the proof of Theorem 5.38 in [BF24].

Proposition 7.3. *Let M be a loopless and coloopless matroid of rank r on $[n]$. Then*

$$\omega(M) = (-1)^{c(M)} \chi(X_n, \wedge^r[\mathcal{Q}_M^\vee] \otimes \wedge^{n-r}[\mathcal{Q}_M^\vee]).$$

The key fact used in the proof of Theorem 7.2 is the following result.

Proposition 7.4. [FS12, Lemma 6.2] *Let M be a loopless and coloopless matroid. Then, for any $p, q \geq 0$, $\chi(X_n, \wedge^p[\mathcal{S}_M] \otimes \wedge^q[\mathcal{Q}_M^\vee]) = 0$ unless $p = q$.*

Fink and Speyer also show that if $p = q$, then the T -equivariant Euler characteristic of $\wedge^p[\mathcal{S}_M] \otimes \wedge^q[\mathcal{Q}_M^\vee]$ is an integer. The proof of Proposition 7.4 is based on a clever combinatorial analysis of the formula one obtains from Theorem 4.15. See Section 9.4 for a discussion of this and related results.

Proof of Theorem 7.2. For convenience, we prove the result when M is realizable; the general case follows from Corollary 4.11, using Proposition 5.20 and Proposition 5.13. Let L be a realization of M . Consider $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$ and its universal line bundle $\mathcal{O}(1)$. It follows from Definition 5.2 and the discussion in Section 5.3 that

$$g_M(t) = (-1)^{c(M)} \sum_{i=0}^r t^i (-1)^i \chi(\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee), [\mathcal{O}(-1)] \cdot (1 - [\mathcal{O}(-1)])^{n-i-1}).$$

Let $\pi: \mathbb{P}_{W_L}(\mathcal{Q}_L^\vee) \rightarrow W_L$ be the natural map. We can compute the Euler characteristic of a sheaf on $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$ by first pushing forward along π . It follows from the fact that $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee) = \text{Proj}_{W_L} \text{Sym } \mathcal{Q}_L$ that, for $a \geq 0$, we have $\pi_* \mathcal{O}(a) = \text{Sym}^a \mathcal{Q}_L$. Furthermore, we have $R^i \pi_* \mathcal{O}(a) = 0$ for $i > 0$, $a \geq 0$, for example by the theorem on cohomology and base change [Vak25, Theorem 25.1.6]. Grothendieck duality implies that $R^i \pi_* \mathcal{O}(a) = 0$ if $i < n - r - 1$ and $a < 0$, or if $i = n - r - 1$ and $-(n - r) < a < 0$. Grothendieck duality also gives that $R^{n-r-1} \pi_* \mathcal{O}(-a) = \det \mathcal{Q}_L^\vee \otimes \text{Sym}^{a-n+r} \mathcal{Q}_L^\vee$ for $a \geq n - r$. As $\det \mathcal{Q}_L^\vee \cong \mathcal{L}_{M^\perp}^{-1}$, we have

$$(20) \quad g_M(t) = (-1)^{c(M)} \sum_{i=0}^r t^i (-1)^i \sum_{j=0}^{r-i} (-1)^j \binom{n-i-1}{n-r-1+j} \chi(W_L, \mathcal{L}_{M^\perp}^{-1} \otimes \text{Sym}^j \mathcal{Q}_L^\vee).$$

For any $j \geq 0$, the exact sequence $0 \rightarrow \mathcal{S}_L \rightarrow \mathcal{O}^{\oplus n} \rightarrow \mathcal{Q}_L \rightarrow 0$ gives rise to an exact sequence of sheaves on W_L

$$0 \rightarrow \wedge^j \mathcal{S}_L \rightarrow \mathcal{O}^{\oplus n} \otimes \wedge^{j-1} \mathcal{S}_L \rightarrow \text{Sym}^2 \mathcal{O}^{\oplus n} \otimes \wedge^{j-2} \mathcal{S}_L \rightarrow \dots \rightarrow \text{Sym}^{j-1} \mathcal{O}^{\oplus n} \otimes \mathcal{S}_L \rightarrow \text{Sym}^j \mathcal{O}^{\oplus n} \rightarrow \text{Sym}^j \mathcal{Q}_L \rightarrow 0.$$

Dualizing this and considering the corresponding identity in K -theory, we deduce the following equality:

$$[\text{Sym}^j \mathcal{Q}_L^\vee] = \sum_{s=0}^j (-1)^s \binom{n+j-s-1}{j-s} \wedge^s [\mathcal{S}_L^\vee].$$

For any vector bundle \mathcal{E} , there is an isomorphism $\wedge^s \mathcal{E} \cong \wedge^{\text{rk}(\mathcal{E})-s} \mathcal{E}^\vee \otimes \det \mathcal{E}$. As $\det \mathcal{S}_L \cong \mathcal{L}_{M^\perp}^{-1}$, we have

$$[\text{Sym}^j \mathcal{Q}_L^\vee] \otimes [\mathcal{L}_{M^\perp}^{-1}] = \sum_{s=0}^j (-1)^s \binom{n+j-s-1}{j-s} \wedge^{r-s} [\mathcal{S}_L].$$

Substituting this into (20), we compute

$$\begin{aligned} g_M(t) &= (-1)^{c(M)} \sum_{i=0}^r t^i (-1)^i \sum_{j=0}^{r-i} (-1)^j \binom{n-i-1}{n-r-1+j} \sum_{s=0}^j (-1)^s \binom{n+j-s-1}{j-s} \chi(M, \wedge^{r-s} [\mathcal{S}_M]) \\ &= (-1)^{c(M)} \sum_{s=0}^r (-1)^s \chi(M, \wedge^{r-s} [\mathcal{S}_M]) \sum_{i=0}^{r-s} t^i (-1)^i \sum_{j=s}^{r-i} (-1)^j \binom{n-i-1}{n-r-1+j} \binom{n+j-s-1}{j-s} \\ &= (-1)^{c(M)} \sum_{s=0}^r (-1)^{r-s} \chi(M, \wedge^{r-s} [\mathcal{S}_M]) \sum_{i=0}^{r-s} t^i \binom{r-s}{i} \\ &= (-1)^{c(M)} \sum_{s=0}^r (-1)^{r-s} \chi(M, \wedge^{r-s} [\mathcal{S}_M]) (t+1)^{r-s}. \end{aligned}$$

The result follows from the following computation, which uses (13) and Proposition 7.4:

$$(-1)^i \chi(M, \wedge^i [\mathcal{S}_M]) = \sum_{j=0}^{n-r} (-1)^j \chi(X_n, \wedge^i [\mathcal{S}_M] \otimes \wedge^j [\mathcal{Q}_M^\vee]) = \chi(X_n, \wedge^i [\mathcal{S}_M] \otimes \wedge^i [\mathcal{Q}_M^\vee]). \quad \square$$

Remark 7.5. In particular, the above argument shows that $\frac{d}{dt}g_M(t)|_{t=-1}$ is equal to $(-1)^{1+c(M)}\chi(M, [\mathcal{S}_M])$. An explicit combinatorial formula for $\frac{d}{dt}g_M(t)|_{t=-1}$ is given in [Pan25, Proposition 1.5]. He observes that if M is a connected graphic matroid, then $\chi(M, [\mathcal{S}_M]) = 1$; see [Pan25, Theorem 1.8].

7.3. Sections of tautological bundles on wonderful varieties. As discussed in Section 5.3, the g -polynomial of a realizable matroid can be computed in terms of the sections of symmetric powers of \mathcal{Q}_L on W_L . In this section, we use some deformation theory to prove some related results about the cohomology of \mathcal{Q}_L on W_L .

Let M be a matroid of rank r on $[n]$. Let Gr_M be the subset of $\text{Gr}(r, n)$ (over k) given by the locally closed locus where the Plücker coordinates corresponding to the non-bases of M vanish, and the Plücker coordinates corresponding to the bases of M do not vanish. This has a natural scheme structure, and it is called the *thin Schubert cell* corresponding to M .

Thin Schubert cells are affine schemes. The corresponding ring has a simple presentation as the quotient of $k[p_B^{\pm 1}]_{B \text{ basis}}$ by relations obtained by setting $p_N = 0$ in the Plücker relations for each non-basis N . However, their geometry is very complicated. For example, Gr_M is non-empty if and only if M is realizable over k . Moreover, the Mnëv universality theorem [Mnë88] states that for any affine scheme X of finite type over the prime subfield of k , there is a rank 3 matroid M and a natural number N such that Gr_M is isomorphic to an open subscheme of $X \times \mathbb{A}^N$ whose projection to X is surjective. In particular, thin Schubert cells can be “arbitrarily singular.” See [Laf03, Section 1.8] for a modern proof.

For a projective variety X , the *Hilbert scheme* of X is a scheme $\text{Hilb } X$ whose points parametrize subschemes of X , i.e., a subscheme Z of X defines a point $[Z]$ of $\text{Hilb } X$. Giving a map from a k -scheme Y to $\text{Hilb } X$ is the same thing as giving a subscheme \mathcal{Z} of $X \times Y$ which is flat over Y . The Hilbert scheme can be used to analyze deformations of subschemes in X . See [FGI⁺05, Chapter 5].

Theorem 7.6. *Let L be a linear subspace of k^n realizing a loopless matroid M . Then there is an open subset U of the Hilbert scheme of X_n containing $[W_L]$ such that there is an isomorphism from U to Gr_M which sends $[W_L]$ to $[L]$.*

Before proving Theorem 7.6, we discuss some applications. Because \mathcal{Q}_L has a section which transversely cuts out W_L inside of X_n , the restriction of \mathcal{Q}_L to W_L is the normal bundle of W_L in X_n . Standard deformation theory [FGI⁺05, Proposition 6.5.2] identifies $H^0(W_L, \mathcal{Q}_L)$ with $T_{[W_L]} \text{Hilb } X_n$, the tangent space to $[W_L]$ in $\text{Hilb } X_n$, which, by Theorem 7.6, is the tangent space to $[L]$ in Gr_M . Furthermore, $H^1(W_L, \mathcal{Q}_L)$ is the obstruction space to extending infinitesimal deformations of W_L in X_n . Computing the obstruction space is difficult, but if $[L]$ is a singular point of Gr_M , then $H^1(W_L, \mathcal{Q}_L) \neq 0$ because there is an infinitesimal deformation that cannot be extended past some order.

There is a map from $k^n \cong H^0(W_L, \mathcal{O}^{\oplus n})$ to $H^0(W_L, \mathcal{Q}_L)$. The corresponding map from k^n to the tangent space to $[W_L]$ in $\text{Hilb } X_n$ comes from the torus action: the action map $T \times \text{Hilb } X_n \rightarrow \text{Hilb } X_n$ induces a map from the Lie algebra of T to the tangent space to $[W_L]$ in $\text{Hilb } X_n$. By Proposition 1.5, if the matroid of L has c components, then the image of this map has dimension $n - c$, and $\dim H^0(W_L, \mathcal{S}_L) = c$, as $H^0(W_L, \mathcal{S}_L)$ is the kernel of the map $k^n \rightarrow H^0(W_L, \mathcal{Q}_L)$.

The Euler characteristic $\chi(W_L, \mathcal{Q}_L)$ is encoded in the g -polynomial of M . When L is 3-dimensional, we can completely compute the cohomology of \mathcal{Q}_L in terms of this Euler characteristic and the geometry of Gr_M . Because W_L is 2-dimensional, we just need to compute $H^1(W_L, \mathcal{Q}_L)$ and $H^2(W_L, \mathcal{Q}_L)$. Consider the long exact sequence on cohomology associated to the short exact sequence $0 \rightarrow \mathcal{S}_L \rightarrow \mathcal{O}^{\oplus n} \rightarrow \mathcal{Q}_L \rightarrow 0$ of sheaves on W_L . Note that $H^i(W_L, \mathcal{O}^{\oplus n}) = 0$ for $i > 0$, so we see that $H^2(W_L, \mathcal{Q}_L) = 0$. This gives that $\dim H^1(W_L, \mathcal{Q}_L) = \dim T_{[L]} \text{Gr}_M - \chi(W_L, \mathcal{Q}_L)$.

Example 7.7. Let L be a realization of the Fano matroid F_7 . Note that F_7 is connected, and it is known that T acts transitively on the realizations of F_7 , so Gr_{F_7} is identified with $\mathbb{G}_m^7/\mathbb{G}_m$. In particular, $\dim H^0(W_L, \mathcal{Q}_L) = 6$. Using Example 6.6, we see that $\chi(W_L, \mathcal{Q}_L) = 5$, so $\dim H^1(W_L, \mathcal{Q}_L) = 1$.

Using Example 6.6, one can show that if L is a realization of F_7 , then the ring $\bigoplus_{a \geq 0} H^0(W_L, \text{Sym}^a \mathcal{Q}_L)$ is not Cohen–Macaulay. The nonvanishing of $H^1(W_L, \mathcal{Q}_L)$ makes this unsurprising.

Example 7.8. Let L be a realization of the non-Fano matroid F_7^+ . Note that F_7^+ is connected, and it is known that $\text{Gr}_{F_7^+}$ is an irreducible variety of dimension 7. Using Example 6.8, we see that

$$\chi(W_L, \text{Sym}^a \mathcal{Q}_L) = 4 \binom{a+5}{5} - 6 \binom{a+4}{4} + 3 \binom{a+3}{3},$$

so $\chi(W_L, \mathcal{Q}_L) = 6$. This implies that $\dim H^1(W_L, \mathcal{Q}_L) = 1$.

Example 7.9. Let $k = \mathbb{C}$, and let L be the row span of the matrix

$$\begin{pmatrix} 1 & 0 & \frac{-1+\sqrt{5}}{2} & 1 & \frac{-1+\sqrt{5}}{2} & 0 & 0 & 1 & 1 \\ 0 & 1 & \frac{-3+\sqrt{5}}{2} & 1 & \frac{-3+\sqrt{5}}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\sqrt{5}}{2} & 0 & \frac{-3+\sqrt{5}}{2} & 1 & 1 & \frac{-3+\sqrt{5}}{2} & -1 \end{pmatrix}.$$

Let M be the matroid realized by L . Then M is known as the *Perles matroid*, and it is related to the regular pentagon. Using Example 5.4, one can compute that $g_M(t) = 10t + 18t^2 + 9t^3$, which implies that $\chi(M, \mathcal{Q}_M) = 8$. Over \mathbb{C} , Gr_M consists of two 8-dimensional torus-orbits, so $\dim H^0(W_L, \mathcal{Q}_L) = 8$ and all other cohomology vanishes. Now suppose that k has characteristic 5, and let L' be the row span of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 2 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 0 & 2 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then L' is also a realization of M . In characteristic 5, Gr_M consists of a single torus-orbit which is generically non-reduced with multiplicity 2. This implies that $\dim H^0(W_{L'}, \mathcal{Q}_{L'}) = 9$ and $\dim H^1(W_{L'}, \mathcal{Q}_{L'}) = 1$.

Example 7.10. Let $k = \mathbb{C}$. In [CL25, Theorem 4.7], there is an example of a rank 3 matroid M on [12] with the property that Gr_M is 12-dimensional, and it has two singular points whose tangent space is 13-dimensional. Using Example 5.4, one can check that $g_M(t) = 27t + 51t^2 + 25t^3$, which implies that $\chi(M, \mathcal{Q}_M) = 12$. If L is a singular point of Gr_M , then $\dim H^0(W_L, \mathcal{Q}_L) = 13$ and $\dim H^1(W_L, \mathcal{Q}_L) = 1$. If L is a smooth point, then $\dim H^0(W_L, \mathcal{Q}_L) = 12$ and $\dim H^1(W_L, \mathcal{Q}_L) = 0$.

In particular, the dimensions of the cohomology groups $H^i(W_L, \mathcal{Q}_L)$ are not determined by the matroid of L . The g -polynomial is controlled by the cohomology of negative twists of $\mathcal{O}(1)$ on $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$. As the cohomology of $\mathcal{O}(1)$ on $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$ is the same as the cohomology of \mathcal{Q}_L on W_L , this makes it difficult to control the cohomology of negative twists of $\mathcal{O}(1)$ by analyzing the positive twists.

Remark 7.11. Let \mathcal{L} be a line bundle on X_n , and let $L \subseteq k^n$ be a realization of a loopless matroid. In general, the cohomology of the restriction of \mathcal{L} to W_L is not determined by the matroid of L . For example, suppose we choose 3 distinct points in \mathbb{P}^2 , and we choose two general lines through each of the points. This gives a realization L of $U_{3,6}$. Let E_i be the exceptional divisor corresponding to the i th point, and consider the line bundle $\mathcal{L} = \mathcal{O}(\alpha - E_1 - E_2 - E_3)$. Then $H^0(W_L, \mathcal{L})$ is the space of linear forms on L which vanish on the 3 points. If the 3 points are general, then we will have $H^0(W_L, \mathcal{L}) = 0$, but if the 3 points lie on a line, then we will have $H^0(W_L, \mathcal{L}) \neq 0$.

To make it easier to compute $\chi(M, [\mathcal{Q}_M])$, we relate it to the *expected dimension* of the thin Schubert cell of M . This is a combinatorially-defined quantity introduced by Ford in [For15] which estimates the dimension of Gr_M . This estimate can be wrong, for example when the conditions that the matroid imposes on $\text{Gr}(r, n)$ fail to be independent. However, Ford has shown that, in many cases, the dimension of Gr_M is equal to its expected dimension; see [For15, Theorem 4.7].

Proposition 7.12. *Let M be a loopless and coloopless matroid. Then the expected dimension of Gr_M is equal to $\chi(M, [\mathcal{Q}_M])$.*

Proof. In [For15, Theorem 5.3], Ford has shown that the expected dimension is a valuative invariant of matroids. Proposition 5.13 and (13) implies that the function which assigns a matroid M to the integer $\chi(M, [\mathcal{Q}_M])$ is valuative, so it suffices to prove that it is equal to the expected dimension for a collection of matroids which generate $\text{Val}_n^{\text{ncf}}$; see Proposition 5.1. If M is a quasi series-parallel matroid with c components, then $g_M(t) = t^c$, so $\chi(M, [\mathcal{Q}_M]) = n - c$. This is equal to the dimension of the thin Schubert cell: quasi series-parallel matroids are binary, which implies that the thin Schubert cell consists of a single torus-orbit; see [Oxl11, Proposition 6.6.5]. In [For15, Theorem 4.7], Ford shows that the expected dimension of a matroid which is a *positroid* is equal to the dimension of its thin Schubert cell. Not all quasi series-parallel matroids are positroids, but there is a family of quasi series-parallel matroids called *snake matroids* which are positroids. Snake matroids generate $\text{Val}_n^{\text{ncf}}$; see, e.g., [FS24, Appendix A], so this implies the result. \square

Example 7.13. Let M be a loopless and coloopless Schubert matroid. Then it follows from [For15, Theorem 4.7] that $\chi(M, [\mathcal{Q}_M])$ is equal to the dimension of the corresponding Schubert variety. This implies that if L is a realization of a loopless and coloopless Schubert matroid, then $\dim H^0(W_L, \mathcal{Q}_L) = \chi(W_L, \mathcal{Q}_L)$. If M has rank 3, then this implies that $H^i(W_L, \mathcal{Q}_L) = 0$ for $i > 0$.

Example 7.14. Let M be a connected graphic matroid on n elements. As $\chi(M, [\mathcal{S}_M]) + \chi(M, [\mathcal{Q}_M]) = n$, Remark 7.5 implies that $\chi(M, [\mathcal{Q}_M])$ is equal to $n - 1$. This is equal to the dimension of the thin Schubert cell: graphic matroids are binary, which implies that the thin Schubert cell consists of a single torus-orbit; see [Oxl11, Proposition 6.6.5]. In particular, for graphic matroids, the expected dimension of Gr_M is equal to its dimension.

Example 7.15. Let M denote the matroid R_{10} ; for any field k , this is the connected matroid of rank 5 realized by the row span of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

This matroid is regular, i.e., it is realizable over any field. One can compute that $g_M(t) = 10t + 35t^2 + 45t^3 + 20t^4 + t^5$, for example, by using the algorithm in [Fer23, Section 5]. This implies that $\chi(M, [\mathcal{Q}_M])$ is equal to 10. But M is binary, so the dimension of Gr_M is 9. Note that $g_M(t-1)/(t-1) = 1 + t - 9t^2 + 16t^3 + t^4$.

In Example 7.15, the expected dimension of Gr_M exceeds the dimension of Gr_M . This cannot happen for matroids of rank 3: if L is a realization of M , then because $H^2(W_L, \mathcal{Q}_L) = 0$, we have $\dim H^0(W_L, \mathcal{Q}_L) \geq \chi(W_L, \mathcal{Q}_L)$. Theorem 7.6 implies that $\dim H^0(W_L, \mathcal{Q}_L)$ is the dimension of the tangent space to Gr_M at $[L]$, which is greater than or equal to the dimension of Gr_M at $[L]$.

In [Fer23, Corollary 4.4], Ferroni showed that if M is a loopless and coloopless Schubert matroid, then the polynomial $g_M(t-1)/(t-1)$ has non-negative coefficients; note that a loopless and coloopless Schubert matroid is connected. This implies that the numerator of the series $\sum_{a \geq 0} \chi(M, \text{Sym}^a[\mathcal{Q}_M])$ has non-negative coefficients. Comparing with Proposition 6.1, this suggests the following conjecture.

Conjecture 7.16. *Let L be a realization of a loopless and coloopless Schubert matroid M . Then the ring $\bigoplus_{a \geq 0} H^0(W_L, \text{Sym}^a \mathcal{Q}_L)$ is Cohen–Macaulay and generated in degree 1, and its Hilbert function is the polynomial $a \mapsto \chi(M, \text{Sym}^a[\mathcal{Q}_M])$.*

Example 7.13 also gives some evidence for this conjecture. It is reasonable to hope that the conclusion will hold for some generalizations of Schubert matroids, such as transversal matroids and positroids. In [Pan25, Section 4.1], Panzer gives examples of graphic matroids M where the coefficients of $g_M(t-1)/(t-1)$ are negative, so Conjecture 7.16 cannot be extended to graphic matroids, even if we omit the “generated in degree 1” condition.

Example 7.17. Let L be a realization of a series-parallel matroid on $[n]$. Then $\beta(M) = 1$, so the map m from $\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee)$ to \mathbb{P}^{n-1} is birational onto its image H , the hyperplane where the sum of all of the coordinates is 0; see Proposition 10.9. Because m is a birational map between smooth varieties of the same dimension, we have $m_*\mathcal{O} \cong \mathcal{O}$ and $R^i m_*\mathcal{O} = 0$ for $i > 0$. This implies that the pullback map $H^0(H, \mathcal{O}(a)) \rightarrow H^0(\mathbb{P}_{W_L}(\mathcal{Q}_L^\vee), \mathcal{O}(a))$ is an isomorphism, so we have

$$\bigoplus_a H^0(W_L, \text{Sym}^a \mathcal{Q}_L) \cong \text{Sym } k^n / k.$$

It is not clear how to compute the ring $\bigoplus_{a \geq 0} H^0(W_L, \text{Sym}^a \mathcal{Q}_L)$ in more examples. It would be interesting to compute the ring $\bigoplus_{a \geq 0} H^0(W_L, \text{Sym}^a \mathcal{Q}_L)$ when L is a realization of the Fano matroid.

We now begin the proof of Theorem 7.6. The following result will allow us to control deformations of wonderful varieties.

Proposition 7.18. [Huh14, Theorem 46(ii)] *Let L be a realization of a loopless matroid M , and let Y be an integral subvariety of X_n such that $[W_L] = [Y]$ in the Chow ring of X_n . Then $Y = W_{L'}$ for some realization L' of M .*

Proof. We indicate the idea, and we refer to [Huh14] for more details. First, one studies the class $[W_L]$ in the Chow ring of X_n . This class, which is called the *Bergman class*, depends only on the matroid of L , and the matroid can be easily recovered from the Minkowski weight corresponding to it. Furthermore, the Bergman class is not pushed forward from an irreducible torus-invariant divisor, so we may assume that Y is not contained in any irreducible torus-invariant divisor, i.e., it meets the open torus $\mathbb{G}_m^n / \mathbb{G}_m$ embedded in X_n .

Let π denote the map $X_n \rightarrow \mathbb{P}^{n-1}$ induced by the complete linear system of $\mathcal{O}(\alpha)$, and let $r = \dim L$. Then $\pi(W_L) = \mathbb{P}L$, and so $\pi_*[W_L]$ is the class of a linear subspace of \mathbb{P}^{n-1} of dimension $r-1$. This implies that $\pi_*[Y] = [\pi(Y)]$ is the class of a linear subspace of dimension $r-1$, so $\pi(Y)$ is a linear subspace $\mathbb{P}L'$ of \mathbb{P}^{n-1} , with $\dim L' = r$. Note that π is an isomorphism over the open torus $\mathbb{G}_m^n / \mathbb{G}_m$ in X_n , so $\mathbb{P}L'$ intersects $\mathbb{G}_m^n / \mathbb{G}_m$. Therefore Y contains the closure of $\mathbb{P}L' \cap \mathbb{G}_m^n / \mathbb{G}_m$. Because Y is integral and $\dim Y = \dim \mathbb{P}L'$, this implies that Y is equal to $W_{L'}$. Because the Bergman class determines the underlying matroid, L' must be a realization of M . \square

To prove Theorem 7.6, we will need to use tautological bundles associated to families of realizations of M . On $X_n \times \text{Gr}_M$, there are vector bundles $\mathcal{S}^{\text{univ}}$ and $\mathcal{Q}^{\text{univ}}$ which fit into a short exact sequence

$$0 \rightarrow \mathcal{S}^{\text{univ}} \rightarrow \mathcal{O}^{\oplus n} \rightarrow \mathcal{Q}^{\text{univ}} \rightarrow 0,$$

and for any realization L of M , the restriction of this sequence to $X_n \times \{[L]\}$ is the sequence $0 \rightarrow \mathcal{S}_L \rightarrow \mathcal{O}^{\oplus n} \rightarrow \mathcal{Q}_L \rightarrow 0$. The bundles $\mathcal{S}^{\text{univ}}$ and $\mathcal{Q}^{\text{univ}}$ are pulled back from a T -equivariant morphism $X_n \times \text{Gr}_M \rightarrow \text{Gr}(r, n) \times \text{Gr}_M$. One way to construct these vector bundles is to use a variant of Klyachko’s description of toric vector bundles [Kly95] in terms of certain filtrations on the fiber over the identity that works in families; see [Pay08, Section 3.3]. Note that \mathcal{S}_L and \mathcal{Q}_L are *not* naturally toric vector bundles, because the torus embedded in X_n does not act on them naturally, so the classification must be modified slightly.

Proposition 7.19. *Let M be a loopless matroid on $[n]$. Then there is a subscheme \mathcal{W} on $X_n \times \mathrm{Gr}_M$ which is flat over Gr_M , and whose fiber over a point $[L]$ of Gr_M is the wonderful variety W_L .*

Proof. Let s denote the section of $\mathcal{Q}^{\mathrm{univ}}$ corresponding to the image of the section $(1, \dots, 1)$ from $\mathcal{O}^{\oplus n}$. As described in Section 5.3, for each point $[L]$ of Gr_M , the vanishing locus of s transversely cuts out W_L in the fiber over $[L]$. As $X_n \times \mathrm{Gr}_M$ is flat over Gr_M , [Sta25, 00MF] implies that the vanishing locus of s is flat over Gr_M , giving the desired subscheme. \square

Proof of Theorem 7.6. Proposition 7.19 produces a map from Gr_M to $\mathrm{Hilb} X_n$, where a point $[L]$ of Gr_M maps to the point $[W_L]$ of $\mathrm{Hilb} X_n$. We will show that this map is an open embedding by constructing an inverse near $[L]$.

Let Y be an open subset of $\mathrm{Hilb} X_n$ containing $[W_L]$, and let \mathcal{Z} be the universal subscheme in $X_n \times Y$, which is flat over Y . Let $\pi: \mathcal{Z} \rightarrow Y$ be the projection, and let y be the point $[W_L]$ of Y , so W_L is the fiber of π over y . As W_L is geometrically integral, by [Sta25, 0559, 0578], we can replace Y by an open subset such that the fibers are all geometrically integral. By Proposition 7.18, the fibers are all wonderful varieties of realizations of M .

On X_n , the line bundle corresponding to the divisor class α is globally generated, and its complete linear system induces the map $X_n \rightarrow \mathbb{P}^{n-1}$. Let \mathcal{L} denote the pullback of this line bundle to $X_n \times \mathrm{Gr}_M$. There is a surjection $\mathcal{O}^{\oplus n} \rightarrow \mathcal{L}$ which induces the map $X_n \rightarrow \mathbb{P}^{n-1}$, i.e., the restriction of \mathcal{L} to a fiber is isomorphic to $\mathcal{O}(\alpha)$.

Let K be the kernel of the map $\mathcal{O}^{\oplus n} \rightarrow \mathcal{L}$. By inspecting the long exact sequence on cohomology, the restriction of K to any fiber of π has no higher cohomology. The theorem on cohomology and base change [Vak25, Theorem 25.1.6] implies that $R^1\pi_*K = 0$ and that $\pi_*\mathcal{L}$ is a vector bundle of rank r . As each fiber is geometrically integral, $\pi_*\mathcal{O} \cong \mathcal{O}$, so the surjection $\mathcal{O}^{\oplus n} \rightarrow \pi_*\mathcal{L}$ induces a map from Y to $\mathrm{Gr}(r, n)$. We will show that the image of this map is contained in Gr_M , and so it is a local inverse to the map from Gr_M to $\mathrm{Hilb} X_n$ constructed previously.

By Proposition 7.18, the image of any point in Y under this map lies in Gr_M , so we may reduce to the case when Y is a (possibly non-reduced) point. If the image of Y is not (scheme-theoretically) contained in Gr_M , then there is some element S of $\binom{[n]}{r}$ such that the Plücker coordinate p_S vanishes on $[L]$, but does not vanish on the image of Y .

Because S is not a basis of M , L^S , the image of L under the coordinate projection to k^S , is a subspace of k^S of dimension t , with $t < r$. There is a line bundle \mathcal{L}_S on X_n with the property that the complete linear series of its restriction to W_L induces the map $W_L \rightarrow \mathbb{P}L^S$. There is a surjection $\mathcal{O}^{\oplus r} \rightarrow \mathcal{L}_S$ on X_n ; let K_S denote the kernel of this map.

It is known that the higher cohomology of the restriction of \mathcal{L}_S to W_L vanishes; for example, this is a special case of Theorem 9.3, although it is much more elementary. Also, the long exact sequence on cohomology implies that the higher cohomology of the restriction of K_S to W_L vanishes. The theorem on cohomology and base change gives a short exact sequence

$$0 \rightarrow \pi_*K_S \rightarrow \mathcal{O}_Y^{\oplus r} \rightarrow \pi_*\mathcal{L}_S \rightarrow 0,$$

and shows that π_*K_S and $\pi_*\mathcal{L}_S$ are vector bundles. The fiber of π_*K_S over $[L]$ is identified with the kernel of $L \rightarrow L^S$, so π_*K_S is a vector bundle of rank $r - t > 0$. But this contradicts that p_S does not vanish on the image of Y . \square

If \mathcal{E} is a vector bundle on a projective variety X , then $H^1(X, \mathcal{E} \otimes \mathcal{E}^\vee)$ is identified with the space of deformations of \mathcal{E} , see [FGI⁺05, Proposition 6.5.1]. It is reasonable to expect that all equivariant deformations of the vector bundle \mathcal{Q}_L on X_n are obtained by deforming L to another realization of the same matroid. This motivates the following conjecture.

Conjecture 7.20. *Let L be a realization of a matroid M on $[n]$. Then the subspace of $H^1(X_n, \mathcal{Q}_L \otimes \mathcal{Q}_L^\vee)$ which is fixed by the action of T is isomorphic to $T_{[L]} \text{Gr}_M$.*

Together with an appropriate version of the upper semicontinuity of dimensions of cohomology groups for families of equivariant vector bundles, Conjecture 7.20 and Example 7.10 would imply that $\dim H^1(X_n, \mathcal{Q}_L \otimes \mathcal{Q}_L^\vee)$ is not determined by the matroid realized by L . This should be compared with Corollary 9.41.

8. KINDRED SUBSCHEMES OF A PRODUCT OF PROJECTIVE LINES

All approaches to proving Theorem 5.38 rely on proving some vanishing theorems. These vanishing theorems come from the remarkable properties of a family of subschemes of a product of projective lines. These subschemes were first defined in [EFL25], but they originate in work of Brion [Bri01, Bri03] and are closely related to work of Conca, de Negri, and Gorla on Cartwright–Sturmfels ideals [CDNG15, CDNG20, CDNG22]. The material in this section is mostly based on [EFL25].

We first recall a description of the K -ring of a product of projective lines. There is an affine paving of $(\mathbb{P}^1)^\ell$ induced by the decomposition $\mathbb{P}^1 = \mathbb{A}^1 \sqcup \{\infty\}$. Given a subset S of $[\ell]$, the corresponding *coordinate subspace* of $(\mathbb{P}^1)^\ell$ is $Y_S := \{(x_1, \dots, x_\ell) : x_i = \infty \text{ if } i \notin S\}$. Proposition 4.2 implies that $K((\mathbb{P}^1)^\ell)$ has a basis given by the structure sheaves of the coordinate subspaces. It will be more convenient to use a different description. Given a coherent sheaf \mathcal{F} on $(\mathbb{P}^1)^\ell$ (or a class $[\mathcal{F}] \in K((\mathbb{P}^1)^\ell)$), its *Snapper polynomial* is the function $p \in \mathbb{Q}[a_1, \dots, a_\ell]$ which satisfies

$$p(a_1, \dots, a_\ell) = \chi((\mathbb{P}^1)^\ell, \mathcal{F} \otimes \mathcal{O}(a_1, \dots, a_\ell)) \text{ for all } (a_1, \dots, a_\ell) \in \mathbb{Z}^\ell.$$

If \mathcal{F} is the sheaf corresponding to a \mathbb{Z}^ℓ -graded module over the coordinate ring of $(\mathbb{P}^1)^\ell$, i.e., the ring $\bigoplus_{(a_1, \dots, a_\ell)} H^0((\mathbb{P}^1)^\ell, \mathcal{O}(a_1, \dots, a_\ell))$, then the Snapper polynomial of \mathcal{F} is the multigraded Hilbert polynomial of that module, i.e., it agrees with the dimension of the (a_1, \dots, a_ℓ) th graded piece if (a_1, \dots, a_ℓ) is sufficiently deep into the positive orthant. The Snapper polynomial of a class in $K((\mathbb{P}^1)^\ell)$ is a squarefree polynomial in (a_1, \dots, a_ℓ) (i.e., every monomial with non-zero coefficient in the Snapper polynomial has no squares), and the Snapper polynomial of a class $[\mathcal{F}]$ determines the class in $K((\mathbb{P}^1)^\ell)$. Given a matroid on a set E , we say that a subset I of E is *independent* if it is a subset of a basis.

Definition 8.1. [EFL25, Definition 2.1] *Let X be a closed subscheme of $(\mathbb{P}^1)^\ell$ for some ℓ . Then X is kindred if it is empty, or if there is a matroid M on $[\ell]$ such that, for all $(a_1, \dots, a_\ell) \in \mathbb{Z}^\ell$, we have*

$$\chi(X, \mathcal{O}(a_1, \dots, a_\ell)) = \sum_{I \text{ independent in } M} \prod_{i \in I} a_i.$$

The matroid M associated to a kindred subscheme X is called the *progenitor matroid* of X . As we will see, kindred subschemes have a very close relationship with their degenerations in $(\mathbb{P}^1)^\ell$.

Clearly, whether a subscheme of $(\mathbb{P}^1)^\ell$ is kindred is determined by the Snapper polynomial of its structure sheaf, and so it is determined by the class $[\mathcal{O}_X] \in K((\mathbb{P}^1)^\ell)$. In particular, by Example 3.7, any flat degeneration Y in $(\mathbb{P}^1)^\ell$ of a kindred subscheme X has the same K -class, and so is kindred.

The group of 2×2 upper triangular matrices acts on \mathbb{P}^1 by Möbius transformations, with unique fixed point ∞ . Let B be the product of ℓ copies of this group, acting on $(\mathbb{P}^1)^\ell$ with unique fixed point (∞, \dots, ∞) . The B -orbit closures in $(\mathbb{P}^1)^\ell$ are exactly the coordinate subspaces Y_S .

Lemma 8.2. *Every subscheme X of $(\mathbb{P}^1)^\ell$ admits a flat degeneration to a B -fixed subscheme of $(\mathbb{P}^1)^\ell$.*

Proof. Algebraically, this amounts to taking the multigraded generic initial ideal of the ideal of X in the homogeneous coordinate ring of $(\mathbb{P}^1)^\ell$; see [CCRC23, Definition 5.4]. We give a geometric argument.

The action of B on $(\mathbb{P}^1)^\ell$ induces an action of B on the Hilbert scheme of $(\mathbb{P}^1)^\ell$, which is a countable union of projective varieties that parametrize subschemes of $(\mathbb{P}^1)^\ell$. The algebraic group B is solvable, i.e., it has

a filtration by subgroups whose subquotients are isomorphic to \mathbb{G}_m or \mathbb{G}_a . The Borel fixed point theorem then implies that orbit closures of the action of B on the Hilbert scheme contain a B -fixed point.

Consider the B -orbit closure of the point $[X]$ in the Hilbert scheme of $(\mathbb{P}^1)^\ell$. As B is connected, there is a curve connecting the point $[\mathcal{O}_X]$ to a B -fixed point. The universal family over this curve gives a flat degeneration of X to a B -fixed subscheme. \square

In particular, every kindred subscheme degenerates to a B -fixed kindred subscheme. We now classify B -fixed kindred subschemes of $(\mathbb{P}^1)^\ell$. For a matroid M on $[\ell]$, set

$$Y_M := \bigcup_{S \text{ basis of } M} Y_S.$$

Proposition 8.3. *Let X be a non-empty B -fixed kindred subscheme of $(\mathbb{P}^1)^\ell$ with progenitor matroid M . Then $X = Y_M$.*

Proof. We first check that Y_M is kindred. The homogeneous coordinate ring of Y_M is the quotient of $k[x_1, \dots, x_\ell, y_1, \dots, y_\ell]$ by the ideal $(\prod_{i \in D} y_i : D \text{ not independent in } M)$. The multigraded Hilbert function of this ring is given by the formula in Definition 8.1, as for each $(a_1, \dots, a_\ell) \in \mathbb{N}^\ell$, the elements $\{x_1^{a_1 - b_1} x_2^{a_2 - b_2} \dots y_1^{b_1} \dots y_\ell^{b_\ell} : \{i : b_i > 0\} \text{ is independent in } M\}$ form a basis for the degree (a_1, \dots, a_ℓ) part of this ring. As the Snapper polynomial agrees with the Hilbert function on points sufficiently deep in the positive orthant, this implies the desired formula for the Snapper polynomial.

The Chow ring of $(\mathbb{P}^1)^\ell$ has a basis given by the classes of coordinate subspaces. The relationship between the Chow ring and the K -ring of $(\mathbb{P}^1)^\ell$ given in (8) implies that

$$[X] = \sum_{S \text{ basis of } M} [Y_S] \text{ in } A^\bullet((\mathbb{P}^1)^\ell).$$

Because X is B -fixed, it is a union of B -orbits. This implies that X contains Y_S for every basis S of M , so X contains Y_M . There is therefore a short exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{Y_M} \rightarrow 0,$$

where \mathcal{I} is the ideal sheaf of Y_M in X . By assumption, $[\mathcal{O}_X] = [\mathcal{O}_{Y_M}]$, so $[\mathcal{I}] = 0$ in $K((\mathbb{P}^1)^\ell)$. But if \mathcal{L} is a sufficiently ample line bundle, then $H^i((\mathbb{P}^1)^\ell, \mathcal{I} \otimes \mathcal{L}) = 0$ for $i > 0$ and $\mathcal{I} \otimes \mathcal{L}$ is globally generated. As $\chi((\mathbb{P}^1)^\ell, \mathcal{I} \otimes \mathcal{L}) = 0$, this implies that $H^0((\mathbb{P}^1)^\ell, \mathcal{I} \otimes \mathcal{L}) = 0$. This shows that $\mathcal{I} = 0$ as sheaves, so $X = Y_M$. \square

Corollary 8.4. *Kindred subschemes of $(\mathbb{P}^1)^\ell$ are Cohen–Macaulay, pure dimensional, geometrically reduced, and geometrically connected.*

Proof. It is clear that Y_M is pure dimensional, geometrically reduced, and geometrically connected. It is known that Y_M is Cohen–Macaulay: its coordinate ring is the extension of the Stanley–Reisner ring of the independence complex of M by some extra variables, and it is known that independence complexes of matroids are Cohen–Macaulay [PB80, Theorem 3.2.1]. It follows from Lemma 8.2 and Proposition 8.3 that every kindred subscheme X has a flat degeneration to Y_M for some matroid M . It is known that this implies that X has these properties as well [Sta25, 045U, 02NM, 0C0D, 055J]. \square

Remark 8.5. If Y is a non-empty union of coordinate subspaces of $(\mathbb{P}^1)^n$ such that the image of Y under any coordinate projection is pure dimensional, then Y must be the union of coordinate subspaces corresponding to the bases of a matroid; see [Oxl11, Exercise 1.1.3]. This motivates the appearance of matroids in the definition of kindred subschemes.

In particular, because degenerations of kindred subschemes are kindred, Corollary 8.4 implies that every degeneration of a kindred subscheme in $(\mathbb{P}^1)^\ell$ satisfies the properties in Corollary 8.4. This is useful, because it is often difficult to explicitly verify that a degeneration is reduced.

Another corollary of Proposition 8.3 will be very useful. Corollary 8.7 below is deduced from the technique of *Frobenius splitting*, which was introduced in the 1980s by Mehta and Ramanathan [MR85]. A thorough reference for Frobenius splitting techniques is [BK05, Chapter 1].

If X is a scheme over a field k of characteristic $p > 0$, then it is equipped with a map $F: X \rightarrow X$, the (absolute) Frobenius. This map is the identity on the underlying topological space of X , and on an affine open in X it raises all functions to their p th power. There is an induced map $\mathcal{O}_X \rightarrow F_*\mathcal{O}_X$. A *Frobenius splitting* of X is a morphism $\varphi: F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$ of \mathcal{O}_X -modules which splits this map.

A subscheme Y of X is *compatibly split* if $\varphi(F_*\mathcal{I}_Y) \subseteq \mathcal{I}_Y$, where \mathcal{I}_Y is the ideal sheaf of Y in X . It is easy to see that φ induces a Frobenius splitting of Y . Also, an intersection or union of compatibly split subschemes is compatibly split.

One of the applications of Frobenius splitting is that it gives a very powerful strengthening of Serre vanishing.

Proposition 8.6. *Let X be a Frobenius split scheme, and let \mathcal{L} be an ample line bundle on X . Then*

- (1) $H^i(X, \mathcal{L}) = 0$ for any $i > 0$.
- (2) If X is Cohen–Macaulay, then $H^i(X, \mathcal{L}^{-1}) = 0$ for any $i < \dim X$.
- (3) If Y is a compatibly split subscheme of X , then the restriction map $H^0(X, \mathcal{L}) \rightarrow H^0(Y, \mathcal{L})$ is surjective.

Proof. For any line bundle \mathcal{L} , there is an isomorphism $F^*\mathcal{L} \cong \mathcal{L}^{\otimes p}$ [BK05, Lemma 1.2.6]. Because F is a finite and therefore affine map, the natural map $H^i(X, F_*\mathcal{F}) \rightarrow H^i(X, \mathcal{F})$ is an isomorphism for each i . We therefore have isomorphisms

$$H^i(X, \mathcal{L}^{\otimes p}) \cong H^i(X, F_*(F^*\mathcal{L})) \cong H^i(X, \mathcal{L} \otimes F_*\mathcal{O}_X),$$

where the last isomorphism is by the projection formula. The Frobenius splitting implies that the inclusion $\mathcal{L} \otimes \mathcal{O}_X \rightarrow \mathcal{L} \otimes F_*\mathcal{O}_X$ is split, which gives an injection $H^i(X, \mathcal{L}) \rightarrow H^i(X, \mathcal{L}^{\otimes p})$. If $i > 0$ and \mathcal{L} is ample, then Serre vanishing implies that $H^i(X, \mathcal{L}^{\otimes p^v}) = 0$ for v sufficiently large, proving (1). If X is Cohen–Macaulay with dualizing sheaf ω_X , then Serre duality states that $H^i(X, \mathcal{L}^{-1}) \cong H^{\dim X - i}(X, \omega_X \otimes \mathcal{L})^*$. In particular, if $i < \dim X$, then $H^i(X, \mathcal{L}^{-p^v}) = 0$ for v sufficiently large. As $H^i(X, \mathcal{L}^{-1})$ injects into $H^i(X, \mathcal{L}^{-p^v})$ for all $v > 0$, this proves (2). To prove (3), one proves a vanishing theorem for twists of the ideal sheaf of Y in X ; see [BK05, Theorem 1.2.8]. \square

Viewing $(\mathbb{P}^1)^\ell$ as a quotient of $(GL_2)^\ell$, the group B is a Borel subgroup, and the coordinate subspaces of $(\mathbb{P}^1)^\ell$ are exactly the Schubert varieties. Very generally, it is known that there is a splitting of any homogeneous space (such as a partial flag variety) for which all of the Schubert varieties are compatibly split [BK05, Theorem 2.3.10]. This splitting can be easily constructed directly in the case of $(\mathbb{P}^1)^\ell$. In any case, this implies that, over a field of positive characteristic, each Y_M is a compatibly split subvariety of $(\mathbb{P}^1)^\ell$. This implies the following result.

Corollary 8.7. *Let X be a kindred subscheme of $(\mathbb{P}^1)^\ell$. For any $a_1, \dots, a_\ell \geq 0$, we have $H^i(X, \mathcal{O}(a_1, \dots, a_\ell)) = 0$ for all $i > 0$. For any $a_1, \dots, a_\ell < 0$, we have $H^i(X, \mathcal{O}(a_1, \dots, a_\ell)) = 0$ for all $i < \dim X$.*

Proof. By Lemma 8.2 and Proposition 8.3, X has a degeneration to Y_M for some matroid M . By upper semicontinuity [Sta25, 0BDN], it suffices to prove these results for Y_M . First suppose that the characteristic of k is positive. Then the vanishing of the cohomology below the top degree for negative twists is a special case of Proposition 8.6(2), and the vanishing of the higher cohomology of $\mathcal{O}(a_1, \dots, a_\ell)$ follows

from Proposition 8.6(1) if all of the a_i are positive. If the a_i are only assumed to be non-negative, then the vanishing of the higher cohomology can be deduced using the theory of Frobenius splitting relative to a divisor [BK05, Theorem 1.4.8].

If k has characteristic 0, then we will deduce the desired vanishing theorem for Y_M from the case when k has positive characteristic. Note that Y_M is defined over $\text{Spec } \mathbb{Z}$, and that any cohomology group of the restriction of any line bundle on $(\mathbb{P}^1)^\ell$ to Y_M over k has the same dimension as the corresponding cohomology group of the restriction over $\text{Spec } \mathbb{Q}$. By upper semicontinuity [Sta25, 0BDN], for any prime p , the dimension of this cohomology group is upper bounded by the dimension of the same cohomology group for the reduction of Y_M modulo p . The Frobenius splitting argument above shows that the dimensions of the cohomology groups in the statement of Corollary 8.7 are 0 in characteristic p , giving the result. \square

The next result is crucial for applications to Theorem 5.38.

Proposition 8.8. [EFL25, Proposition 2.9] *Let X be a kindred subscheme of $(\mathbb{P}^1)^\ell$, and let $p: (\mathbb{P}^1)^\ell \rightarrow (\mathbb{P}^1)^m$ be a coordinate projection. Then $p(X)$ is a kindred subscheme of $(\mathbb{P}^1)^m$, the natural map from $\mathcal{O}_{p(X)}$ to $p_*\mathcal{O}_X$ is an isomorphism, and $R^i p_*\mathcal{O}_X = 0$ for $i > 0$.*

Proof. The vanishing of the higher direct images of \mathcal{O}_X is an immediate consequence of Proposition 6.13 and Corollary 8.7. This implies that $p_*[\mathcal{O}_X] = [p_*\mathcal{O}_X]$ in $K((\mathbb{P}^1)^m)$, and so we have a formula for the Snapper polynomial of $p_*\mathcal{O}_X$. The rest of the argument is similar to the proof of Proposition 8.3; see [EFL25]. \square

The following result allows us to give many examples of kindred subschemes. If X is a subscheme of $(\mathbb{P}^1)^\ell$ which is pure of dimension d , then we say that the multidegree of X is *multiplicity-free* if all of the coefficients are either 0 or 1 in the expansion of $[X] \in A_d((\mathbb{P}^1)^\ell)$ into the basis given by classes of coordinate subspaces.

Concretely, it follows from a version of Bertini's theorem [Ful98, B.9.1] that X is multiplicity-free if there is an open subset U of X such that, for every subset S of $[\ell]$ of size $\ell - d$, the intersection of U with a general translate of Y_S is either empty or consists of a reduced point.

Proposition 8.9. [Bri03] *Let X be an integral subvariety of $(\mathbb{P}^1)^\ell$ whose multidegree is multiplicity-free. Then X is kindred.*

In particular, Proposition 8.9 implies that every irreducible component of a kindred subscheme is kindred. We give a new proof of Proposition 8.9. A somewhat similar proof was given in [CCRC23, Section 6]. Recall that a scheme over a field K is said to *geometrically* have a property if its base change to any field containing K has that property.

Lemma 8.10. *Let K be a field, and let X be an integral subscheme of $(\mathbb{P}^1)^n$ over K whose multidegree is multiplicity-free. Then X is geometrically integral.*

Proof. Let K' be an extension of K . Note that the multidegree of $X_{K'}$, viewed as a subscheme of $(\mathbb{P}^1)^n$ over K' , is multiplicity-free. There is some finite Galois subextension of K' such that the components are defined over that extension, and the Galois group acts transitively on the components; see [Sta25, Section 0364]. In particular, the fundamental class of each of the components is the same. But this implies that each coefficient of the multidegree is divisible by the number of components, so $X_{K'}$ is irreducible.

By [Sta25, 04KS], if $X_{K'}$ is not reduced, then $X_{K'}$ is generically non-reduced. This contradicts that the multidegree of $X_{K'}$ is multiplicity-free. \square

Proof of Proposition 8.9. First, note that by [BH20, Corollary 4.7] or [CCRL⁺20], the set of subsets S of $[\ell]$ where the coefficient of $[Y_S]$ is positive in the multidegree of X is the set of bases of a matroid M .

If $\dim X = 0$, then X is a reduced point and so is kindred. We may therefore assume that $\dim X \geq 1$. We first check that, for each i , the intersection of X with a general translate of the coordinate hyperplane

$Y_{[\ell] \setminus i}$ is either empty, or is an integral variety whose multidegree is multiplicity-free, and so is kindred by induction. For ease of notation, we take $i = \ell$.

Let $\mathcal{U} \subseteq X \times (\mathbb{A}^2 \setminus \{0\})$ be the scheme given by $\{(p, u, v) : p \in X, p = ([x_1, y_1], \dots), ux_\ell + vy_\ell = 0\}$, with projection $\pi : \mathcal{U} \rightarrow (\mathbb{A}^2 \setminus \{0\})$. The fibers of π are the intersections of X with translates of the coordinate hyperplane $Y_{[\ell-1]}$. Because $Y_{[\ell-1]}$ is a Cartier divisor, the class of a fiber of π in the Chow ring of $(\mathbb{P}^1)^\ell$ is the product of the class of X with the class of $Y_{[\ell-1]}$. In particular, the multidegree of a fiber of π is multiplicity-free. The set of subsets S of $[\ell-1]$ where the coefficient of $[Y_S]$ is positive in the multidegree of a fiber in $(\mathbb{P}^1)^{\ell-1}$ is the set of bases of the matroid M/ℓ .

We will check that the generic fiber of π is integral, using an argument based on [FOV99, Theorem 1.5.4]. Consider the copy of $(\mathbb{P}^1)^{\ell-1} \times \mathbb{A}^1$ obtained by removing $Y_{[\ell-1]}$. On this open set, y_ℓ is equal to 1, and the generic fiber of π is cut out by the equation $ux_\ell + v = 0$, and so the generic fiber of π is integral. By doing a linear change of coordinates on \mathbb{A}^2 , we can repeat this argument to see that the intersection of the generic fiber with the complement of any translate of $Y_{[\ell-1]}$ is integral. This implies that the generic fiber of π is integral: if it has multiple components, then it would still have multiple components if we remove a translate of $Y_{[\ell-1]}$ that does not pass through the generic point of any of the components.

As the multidegree of the generic fiber of π is multiplicity-free, the generic fiber of π is geometrically integral by Lemma 8.10. It then follows from [Sta25, 0559 and 0578] that the general fiber of π is integral. We see that the general fiber of π is kindred, with progenitor matroid M/ℓ .

We can now use induction to control all of the coefficients of the Snapper polynomial of X except for the constant term. For each $i \in [\ell]$, let H_i be a general translate of the coordinate hyperplane $Y_{[\ell] \setminus i}$. Using the exact sequence $0 \rightarrow \mathcal{O}_X(-H_i) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X \cap H_i} \rightarrow 0$ and induction, we see that

$$\sum_{J \text{ independent in } M/i} \prod_{j \in J} a_j = \chi(X, \mathcal{O}(a_1, \dots, a_\ell)) - \chi(X, \mathcal{O}(a_1, \dots, a_i - 1, \dots, a_\ell)).$$

Using this for all i , we see that

$$\chi(X, \mathcal{O}(a_1, \dots, a_\ell)) = \chi(X, \mathcal{O}) + \sum_{\emptyset \neq I \text{ independent in } M} \prod_{i \in I} a_i.$$

It therefore suffices to show that $\chi(X, \mathcal{O}) = 1$. Let Y be a degeneration of X to a B -fixed subscheme of $(\mathbb{P}^1)^\ell$; see Lemma 8.2. Then Y contains Y_M , and there is a short exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{Y_M} \rightarrow 0,$$

where \mathcal{I} is the ideal sheaf of Y_M in Y . Because the difference between the Snapper polynomials of Y and Y_M is a constant, the support of \mathcal{I} is 0-dimensional, and in particular the support is affine. This gives that $H^i((\mathbb{P}^1)^\ell, \mathcal{I}) = 0$ for $i > 0$. As $H^i(Y_M, \mathcal{O}) = 0$ for $i > 0$ by Corollary 8.7, we see that $H^i(Y, \mathcal{O}) = 0$ for $i > 0$, so $\dim H^i(X, \mathcal{O}) = 0$ for $i > 0$ by upper semicontinuity [Sta25, 0BDN]. We see that $\dim H^0(X, \mathcal{O}) = \chi(X, \mathcal{O})$. As X is geometrically integral, $\dim H^0(X, \mathcal{O}) = 1$ by [Sta25, 0BUG]. \square

We now give several examples of kindred subschemes by verifying the hypothesis of Proposition 8.9.

Example 8.11. Let $L \subseteq k^n$ be a linear subspace, and let Y_L be the closure of L in $(\mathbb{P}^1)^n$. This variety, which is usually called the matroid Schubert variety or arrangement Schubert variety, was introduced and studied in [AB16]. Clearly Y_L is integral, and we will check that its multidegree is multiplicity-free, proving that Y_L is kindred. Let S be a subset of $[n]$ of size $n - \dim L$. The intersection of Y_L with a general translate of Y_S is identified with the preimage of a general point of the coordinate projection $L \hookrightarrow k^n \rightarrow k^{S^c}$. This is either empty or a single reduced point, depending on whether S^c is a basis of the matroid of L .

Importantly for us, wonderful varieties can be realized as kindred subschemes. We first describe how wonderful varieties are embedded into a product of projective lines. Given a subspace $L \subseteq k^n$ which is not

contained in any coordinate hyperplane, for each $\{i, j\}$ there is a map $L \rightarrow k^2$ given by projecting onto the coordinates labeled by i and j . This induces a rational map $\mathbb{P}L \dashrightarrow \mathbb{P}^1$. Combining all of these maps together, we have a rational map $\mathbb{P}L \dashrightarrow (\mathbb{P}^1)^{\binom{n}{2}}$. The wonderful variety W_L is the closure of the image of this rational map. To check this, it suffices to check in the case of the permutohedral toric variety X_n , when it is a straightforward toric variety computation. See [EFL25, Proposition 3.9].

Example 8.12. The embedding of W_L into $(\mathbb{P}^1)^{\binom{n}{2}}$ realizes it as a kindred subscheme. In [LLPP24, Corollary 7.5], the Snapper polynomial of the wonderful variety is directly computed, from which one can verify that it is kindred; see [EFL25, Proof of Corollary 3.7]. We can also check that the multidegree of W_L is multiplicity-free. This is a special case of the main result of [Li18], but it can easily be verified directly. Indeed, let S be a subset of $\binom{[n]}{2}$ of size $\binom{n}{2} - \dim L + 1$, and let U be the projectivized hyperplane arrangement complement in W_L . Then the intersection of U with a general translate of Y_S can be identified with the intersection of U with $\dim L - 1$ hyperplanes, each hyperplane obtained as the inverse image of a general line in k^2 under a coordinate projection $k^n \rightarrow k^2$. The intersection of these hyperplanes in $\mathbb{P}L$ consists of a single reduced point. The intersection of these hyperplanes in U is therefore either empty or a single reduced point.

Example 8.13. Let L_1 and L_2 be subspaces of k^n , with L_2 not contained in any coordinate hyperplane. Then the *semi-inverted Hadamard product* is the closure of the subset $\{v/w : v \in L_1, w \in L_2 \cap \mathbb{G}_m^n\}$ of \mathbb{A}^n , where the division is coordinate-wise. The semi-inverted Hadamard product is the closure of the image of a rational map $L_1 \times L_2 \dashrightarrow \mathbb{A}^n$, so it is irreducible. The closure of a semi-inverted Hadamard product in $(\mathbb{P}^1)^n$ is a kindred subscheme; see Remark 9.19.

Example 8.14. Let $m \leq n$, and identify \mathbb{A}^{mn} with the space of $m \times n$ matrices. Let $X \subseteq \mathbb{A}^{mn}$ be either the space of matrices of rank at most 1 or the space of matrices of rank at most $m - 1$. Then it is observed in [HL24] that the closure of X in $(\mathbb{P}^1)^{mn}$ is kindred, and that this also holds for several other determinantal varieties [HL24, Corollary 1.6].

In [HL24, Remark 2.9], it is shown that if $X \subset \mathbb{A}^n$ is an integral subvariety, then X has a universal Gröbner basis consisting of squarefree polynomials if and only if the closure of X in $(\mathbb{P}^1)^n$ is kindred.

We illustrate the utility of Proposition 8.8 by giving a simple proof of Lemma 5.11.

Proof of Lemma 5.11. By Example 8.12, the embedding of W_L into $(\mathbb{P}^1)^{\binom{n}{2}}$ is kindred. Consider the coordinate projection to $(\mathbb{P}^1)^{\binom{[s_1]}{2}} \times \cdots \times (\mathbb{P}^1)^{\binom{[s_c]}{2}}$, where we forget the factors labeled by $\{a, b\}$ unless $\{a, b\} \subseteq S_i$ for some i . We have $L = \bigoplus_{i=1}^c L \cap k^{S_i}$ by Proposition 1.5, so the image of W_L under this coordinate projection is $W_{L^{s_1}} \times \cdots \times W_{L^{s_c}}$. The result then follows from Proposition 8.8. \square

We introduce a concept which will be useful in the sequel. Let $X \subseteq \mathbb{A}^n$ be an integral subvariety of dimension r . The *algebraic matroid* of X is the matroid on $[n]$ whose bases are the subsets S of $[n]$ such that the coordinate projection from X to \mathbb{A}^S is dominant. See [Oxl11, Section 6.7].

The algebraic matroid of X can be computed in terms of the fundamental class of the closure \overline{X} of X in $(\mathbb{P}^1)^n$. If we write

$$[\overline{X}] = \sum_{S \in \binom{[n]}{r}} c_S [Y_S]$$

in terms of the basis for $A_r((\mathbb{P}^1)^n)$ given by classes of coordinate subspaces, then $\{S : c_S > 0\}$ is the set of bases of the algebraic matroid of X . Indeed, c_S is the degree of the map $X \rightarrow \mathbb{A}^S$.

If X is an integral subscheme of \mathbb{A}^n such that the multidegree of the closure of X in $(\mathbb{P}^1)^n$ is multiplicity-free, then the progenitor matroid of the closure of X is the algebraic matroid of X .

The following result will also be useful. Given $X \subseteq (\mathbb{P}^1)^\ell$, its *homogeneous coordinate ring* is the \mathbb{Z}^ℓ -graded ring $\bigoplus_{(a_1, \dots, a_\ell) \in \mathbb{N}^\ell} H^0(X, \mathcal{O}(a_1, \dots, a_\ell))$.

Proposition 8.15. *Let Y be a connected variety, and let $\mathcal{X} \subseteq (\mathbb{P}^1)^\ell \times Y$ be a flat family such that the fiber over some $p \in Y$ is kindred. Then the homogeneous coordinate rings of the fibers of \mathcal{X} also form a flat family over Y .*

Proof. Let $\pi: \mathcal{X} \rightarrow Y$ be the projection. Because being kindred is a condition on the class of the structure sheaf in $K((\mathbb{P}^1)^\ell)$, the fiber over any point in Y is kindred; see Example 3.7. By Corollary 8.7, for any (a_1, \dots, a_ℓ) and any $q \in Y$, $H^i(\pi^{-1}(q), \mathcal{O}(a_1, \dots, a_\ell)) = 0$ for all $i > 0$. The theorem on cohomology and base change [Vak25, Theorem 25.1.6] implies that $\pi_* \mathcal{O}(a_1, \dots, a_\ell)$ is a locally free sheaf on Y whose fiber over q is identified with $H^0(\pi^{-1}(q), \mathcal{O}(a_1, \dots, a_\ell))$. The sheaf of algebras $\bigoplus_{(a_1, \dots, a_\ell) \in \mathbb{N}^\ell} \pi_* \mathcal{O}(a_1, \dots, a_\ell)$ on Y is flat, and its relative spectrum is the family of homogeneous coordinate rings. \square

Finally, we prove some additional results of Brion on integral kindred subschemes.

Proposition 8.16. [Bri02] *Let X be an integral kindred subscheme. Then X is normal, and, if k has characteristic 0, X has rational singularities.*

Proposition 8.16 plays an important role in the work of Berget and Fink [BF22, BF24], and it is the reason for the assumption that the underlying field has characteristic 0 that appears in the geometric statements in those works. However, as we discuss below, the use of Proposition 8.16 can be replaced by Proposition 8.8, which holds over any field. To prove Proposition 8.16, we will need the following two lemmas.

Lemma 8.17. *Let X be a proper kindred subscheme of $(\mathbb{P}^1)^\ell$, and let \mathcal{F} be a coherent sheaf on X . For each i , let $p_i: (\mathbb{P}^1)^\ell \rightarrow (\mathbb{P}^1)^{\ell-1}$ be the projection forgetting the i th coordinate. Suppose that $p_{i*} \mathcal{F} = 0$ for all i . Then $\mathcal{F} = 0$.*

Proof. Suppose that \mathcal{F} is nonzero, and let Z be an irreducible component of the support of \mathcal{F} . Let \mathcal{F}' be the subsheaf of \mathcal{F} consisting of sections supported on Z . It suffices to show that $\mathcal{F}' = 0$.

Because X is a proper subscheme of $(\mathbb{P}^1)^\ell$, Z is also a proper subscheme of $(\mathbb{P}^1)^\ell$. This implies that there is some i such that the restriction of p_i to Z is generically finite onto its image: we may choose any i which is not a coloop of the algebraic matroid of Z . This implies that $\text{Supp } p_{i*} \mathcal{F}' = p_i(Z)$. But $p_{i*} \mathcal{F}'$ is a subsheaf of $p_{i*} \mathcal{F}$, a contradiction. \square

Lemma 8.18. *Let $\pi: X \rightarrow Y$ be a surjective map between integral varieties, with X normal. If the natural map $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_X$ is an isomorphism, then Y is normal.*

Proof. Let $\nu: Y^\nu \rightarrow Y$ be the normalization. As X is normal, the universal property of normalization [Vak25, Section 10.7] gives a map $\bar{\pi}: X \rightarrow Y^\nu$ such that $\pi = \nu \circ \bar{\pi}$. The map $\mathcal{O}_Y \rightarrow \pi_* \mathcal{O}_X$ factors through $\nu_* \mathcal{O}_{Y^\nu}$. As the maps $\mathcal{O}_Y \rightarrow \nu_* \mathcal{O}_{Y^\nu}$ and $\nu_* \mathcal{O}_{Y^\nu} \rightarrow \pi_* \mathcal{O}_X$ are injective because ν and $\bar{\pi}$ are surjective, we see that the map $\mathcal{O}_Y \rightarrow \nu_* \mathcal{O}_{Y^\nu}$ is an isomorphism. Because ν is affine, this implies that ν is an isomorphism. \square

Proof of Proposition 8.16. Let M be the progenitor matroid of X . If M has a coloop, then X can be identified with the product of a kindred subscheme of $(\mathbb{P}^1)^{\ell-1}$ with \mathbb{P}^1 ; see Lemma 9.29 for a more general statement. We can therefore reduce to the case when M has no coloops. This implies that the restriction of the coordinate projection p_i to X is birational for each i .

We first prove that X is normal. Let $\nu: X^\nu \rightarrow X$ be the normalization of X . Let $\mathcal{F} = \nu_* \mathcal{O}_{X^\nu} / \mathcal{O}_X$, so there is a short exact sequence of sheaves $0 \rightarrow \mathcal{O}_X \rightarrow \nu_* \mathcal{O}_{X^\nu} \rightarrow \mathcal{F} \rightarrow 0$. Applying p_{i*} to this sequence for some i and using the $Rp_{i*} \mathcal{O}_X = \mathcal{O}_{p_i(X)}$ by Proposition 8.8, we have a short exact sequence $0 \rightarrow \mathcal{O}_{p_i(X)} \rightarrow p_{i*} \nu_* \mathcal{O}_{X^\nu} \rightarrow p_{i*} \mathcal{F} \rightarrow 0$. By induction, $p_i(X)$ is normal, so Zariski's main theorem [Vak25, Exercise 28.4.B]

implies that $p_{i*}\nu_*\mathcal{O}_{X^\nu} = \mathcal{O}_{p_i(X)}$. This implies that $p_{i*}\mathcal{F} = 0$. As this holds for all i , $\mathcal{F} = 0$ by Lemma 8.17, and so X is normal.

Now suppose that k has characteristic 0, and let $\pi: \tilde{X} \rightarrow X$ be a resolution. For each i , $p_i \circ \pi$ is a resolution of $p_i(X)$. As $p_i(X)$ has rational singularities by induction, we see that $R(p_i \circ \pi)_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_{p_i(X)}$. Note that the fibers of p_i have dimension at most 1, so for any coherent sheaf \mathcal{G} on X , we have $R^j p_{i*}\mathcal{G} = 0$ for $j \geq 2$. The E_2 page of the associated Leray spectral sequence therefore looks like the following

$$\begin{array}{c|cccc}
 \vdots & \vdots & \vdots & \dots & \vdots \\
 2 & p_{i*}R^2\pi_*\mathcal{O}_{\tilde{X}} & R^1p_{i*}R^2\pi_*\mathcal{O}_{\tilde{X}} & 0 & \dots \\
 1 & p_{i*}R^1\pi_*\mathcal{O}_{\tilde{X}} & R^1p_{i*}R^1\pi_*\mathcal{O}_{\tilde{X}} & 0 & \dots \\
 0 & p_{i*}\pi_*\mathcal{O}_{\tilde{X}} & R^1p_{i*}\pi_*\mathcal{O}_{\tilde{X}} & 0 & \dots \\
 \hline
 & 0 & 1 & 2 & \dots
 \end{array}$$

All of the differentials are 0, so each term except for $p_{i*}\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_{p_i(X)}$ on the E_2 page must vanish. In particular, we see that $p_{i*}R^j\pi_*\mathcal{O}_{\tilde{X}} = 0$ for each $j > 0$ and each i , so Lemma 8.17 implies that $R^j p_{i*}\mathcal{G} = 0$. \square

Remark 8.19. There is a version of the above results for subschemes of a product of projective spaces $\mathbb{P}^{a_1} \times \dots \times \mathbb{P}^{a_\ell}$. For each polymatroid P (see Definition 9.2) on $[\ell]$ with $\text{rk}_P(i) \leq a_i$ for each i , there is a subscheme Y_P of $\mathbb{P}^{a_1} \times \dots \times \mathbb{P}^{a_\ell}$ described in the introduction to [EL26]. Viewing $\mathbb{P}^{a_1} \times \dots \times \mathbb{P}^{a_\ell}$ as a quotient of $GL_{a_1+1} \times \dots \times GL_{a_\ell+1}$, Y_P is a union of orbits of a Borel subgroup. One could define a subscheme X of $\mathbb{P}^{a_1} \times \dots \times \mathbb{P}^{a_\ell}$ to be kindred if it is empty, or if $[\mathcal{O}_X] = [\mathcal{O}_{Y_P}]$ in $K(\mathbb{P}^{a_1} \times \dots \times \mathbb{P}^{a_\ell})$ for some polymatroid P . With this definition, analogues of Corollary 8.4, Corollary 8.7, Proposition 8.8, and Proposition 8.9 all hold.

9. THREE APPROACHES TO OMEGA POSITIVITY

We now discuss the proof of Theorem 5.38. We give three approaches to proving the non-negativity of the omega invariant. The first approach is the one taken in [EFL25], which uses a formula for the omega invariant as the Euler characteristic of a line bundle on the wonderful variety (Proposition 5.23). The authors of [EFL25] prove the Cohen–Macaulayness of a section ring on the wonderful variety using Frobenius splitting techniques. This implies that the cohomology of certain line bundles is concentrated in one degree, allowing one to control the sign of their Euler characteristics. Weaker results using this approach were previously obtained in [EL26].

The second approach is based on an idea originating in [BF22]. This approach proves the Cohen–Macaulayness of section rings on certain projective bundles over the wonderful variety, which translates into vanishing theorems on the wonderful variety. These vanishing theorems are different than those proved by the first approach, but they also include the vanishing theorem that is needed to prove the non-negativity of omega (at least for realizable matroids).

The third approach is the one taken in [BF24]. Unlike the other two approaches, this one uses a formula for omega as the Euler characteristic of a vector bundle on the permutohedral toric variety (Proposition 7.3). Using results of Weyman [Wey03], the cohomology of this vector bundle is related to the minimal multigraded free resolution of a certain affine subvariety. Using a degeneration argument, one can understand this free

resolution well enough to prove that the cohomology of this vector bundle is concentrated in one degree, allowing one to control the sign of the Euler characteristic of this vector bundle.

We mostly focus on the case of realizable matroids, as dealing with non-realizable matroids can be quite technical. However, we do not assume that k has characteristic 0 in any of these arguments.

Before proceeding, we mention one approach to Theorem 5.38 which has so far *not* succeeded, even for matroids realizable in positive characteristic.

In [AHK18, Definition 4.2], Adiprasito, Huh, and Katz introduced a cone of classes in $A^1(M)$ which is sometimes called the *combinatorially nef* cone. When M is realized by L , each class in $A^1(M)$ is the first Chern class of a unique line bundle on W_L . The line bundles whose first Chern classes are combinatorially nef are in fact nef on W_L [Che25, Proposition 5.4], even though the nef cone of W_L is in general *not* determined by the matroid realized by L , as is shown by examples like the one in Remark 7.11. The authors of [AHK18] show that, for all matroids, the action of combinatorially nef elements on $A^\bullet(M)$ has properties similar to the action of nef elements on the cohomology ring of a smooth projective variety. It is natural to conjecture that a version of Corollary 4.7 holds for combinatorially nef line bundles.

Conjecture 9.1. *Let M be a loopless matroid of rank r , and let \mathcal{L} be a combinatorially nef line bundle which has $\deg_M(c_1(\mathcal{L})^{r-1}) > 0$. Then $(-1)^{r-1}\chi(M, \mathcal{L}^{-1}) \geq 0$.*

Conjecture 9.1 holds for matroids which are realizable over a field of characteristic 0 by Theorem 4.6. It is shown in [Xie10] that Kawamata–Viehweg vanishing holds for rational surfaces, so Conjecture 9.1 holds for all realizable matroids of rank 3. If the line bundle \mathcal{L} is the line bundle corresponding to a generalized permutohedron, then Conjecture 9.1 follows from [EFL25, Theorem B]. See [Che25, Section 5] for further discussion and some additional cases.

Let Q be a lattice polytope in \mathbb{R}^n which is contained in the hyperplane $x_1 + \cdots + x_n = c$ for some integer c . Then Q defines an ample line bundle on the toric variety of the normal fan of Q . We can pull this line bundle back to a smooth projective toric variety \tilde{X} which is a blow-up of the permutohedral toric variety X_n . If L is a linear subspace of k^n which is not contained in any coordinate hyperplane, then this produces a line bundle on the strict transform \tilde{W}_L of W_L in \tilde{X} , i.e., the closure of the intersection of $\mathbb{P}L$ with the torus in X_n . In [CCKR26], Chan, Clader, Klivans, and Ross show that the Euler characteristic of the restriction of this line bundle to \tilde{W}_L only depends on the matroid of L , and they explain how to define this Euler characteristic for any loopless matroid, extending the definition in [LLPP24]. It is natural to extend Conjecture 9.1 to this setting.

9.1. Omega positivity via section rings on the wonderful variety. Following [EFL25], we will prove vanishing theorems for the restrictions of nef line bundles on the permutohedral toric variety X_n to a wonderful variety. We begin by describing nef line bundles on X_n .

Definition 9.2. *A polymatroid P on E is a function $\text{rk}_P: 2^E \rightarrow \mathbb{Z}$ which satisfies the following properties:*

- (1) $\text{rk}_P(\emptyset) = 0$ (normalization),
- (2) if $A \subset B$, then $\text{rk}_P(A) \leq \text{rk}_P(B)$ (monotonicity), and
- (3) for any $A, B \subset E$, we have $\text{rk}_P(A) + \text{rk}_P(B) \geq \text{rk}_P(A \cap B) + \text{rk}_P(A \cup B)$ (submodularity).

For example, each matroid M gives rise to a polymatroid, by considering its rank function $\text{rk}_M(S) = \max_{B \text{ basis}} |B \cap S|$. See Definition 9.13 for further examples. Given a polymatroid P , its *basis polytope* $B(P)$ is the polytope

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i \in S} x_i \leq \text{rk}_P(S) \text{ for all } S, \text{ and } x_1 + \cdots + x_n = \text{rk}(P)\}.$$

When P comes from a matroid, $B(P)$ is the basis polytope of that matroid. A polymatroid is determined by its basis polytope. As X_n is a toric variety, nef line bundles on X_n correspond to lattice polytopes whose normal fan is a coarsening of the fan of X_n . It is known that, up to translation, these polytopes are exactly the basis polytopes of polymatroids [Edm70], see [BEST23, Section 2.7]. In particular, each polymatroid gives rise to a nef line bundle \mathcal{L}_P on X_n , and every nef line bundle arises in this way. The line bundle \mathcal{L}_{M^\perp} on X_n which appears in Proposition 5.23 is the line bundle corresponding to the dual matroid to M .

On a toric variety, every nef line bundle is globally generated. Each polymatroid induces a map $f_P: X_n \rightarrow X_P$ from X_n to the toric variety of the normal fan of $B(P)$. It is known that this map is induced by the complete linear system of \mathcal{L}_P . This follows from a generalization of Theorem 1.16 to polymatroids; see [EHL23, Proposition 3.10].

Theorem 9.3. [EFL25, Theorem A] *Let P be a polymatroid, and let L be a linear subspace of k^n which is not contained in any coordinate hyperplane. Then:*

- (1) $H^i(W_L, \mathcal{L}_P^{\otimes a}) = 0$ unless $i = 0$ and $a \geq 0$, or $i = \dim f_P(W_L)$ and $a < 0$.
- (2) For any $a \geq 0$, the restriction map $H^0(X_n, \mathcal{L}_P^{\otimes a}) \rightarrow H^0(W_L, \mathcal{L}_P^{\otimes a})$ is surjective.
- (3) We have $R^i f_{P*} \mathcal{O}_{W_L} = 0$ for each $i > 0$, and the natural map $\mathcal{O}_{f_P(W_L)} \rightarrow f_{P*} \mathcal{O}_{W_L}$ is an isomorphism.

There are examples given in [EFL25, Section 5] that show that the hypotheses of Theorem 9.3 cannot be relaxed too much to include more general families of line bundles.

Corollary 9.4. *Let P be a polymatroid, and let L be a linear subspace of k^n which is not contained in any coordinate hyperplane. Then the section ring $R_P := \bigoplus_{a \geq 0} H^0(W_L, \mathcal{L}_P^{\otimes a})$ is normal, Cohen–Macaulay, and generated in degree 1.*

Proof. That R_P is Cohen–Macaulay is an immediate consequence of Theorem 6.12 and Theorem 9.3(1). To see that it is generated in degree 1, it suffices to note that $\bigoplus_{a \geq 0} H^0(X_n, \mathcal{L}_P^{\otimes a})$ is generated in degree 1 by [Wel76, Chapter 18.6, Theorem 3] and use Theorem 9.3(2). Because W_L is smooth (and in particular normal), Theorem 9.3(3) implies that $f_P(W_L)$ is normal by Lemma 8.18. Theorem 9.3(2) then implies that $f_P(W_L)$ is projectively normal, i.e., R_P is normal. \square

The ring R_P in Corollary 9.4 can be described concretely. The coordinate hyperplanes define linear forms $x_1, \dots, x_n \in \text{Sym } L^*$. Then R_P is the subring of $\text{Sym } L^*$ generated by $\{x_1^{a_1} \cdots x_n^{a_n} : (a_1, \dots, a_n) \in B(P)\}$. Theorem 9.3(1) implies that the dimension of the span of this set is equal to $\chi(M, \mathcal{L}_P)$, where M is the matroid realized by L , and in particular the dimension of the span depends only on M . For a general subset S of \mathbb{N}^n , the dimension of the span of $\{x_1^{a_1} \cdots x_n^{a_n} : (a_1, \dots, a_n) \in S\}$ depends on L and not just on the matroid M .

Furthermore, Theorem 9.3 implies that the Hilbert function of R_P agrees with the polynomial $a \mapsto \chi(W_L, \mathcal{L}_P^{\otimes a})$. Proposition 6.1 then gives the following inequalities.

Corollary 9.5. *Let P be a polymatroid, and let L be a realization of a loopless matroid M . Let d be the degree of the polynomial $a \mapsto \chi(W_L, \mathcal{L}_P^{\otimes a})$. Define $h^*(M, \mathcal{L}_P) = (h_0^*, h_1^*, \dots, h_d^*)$ via the formula*

$$\sum_{a \geq 0} \chi(W_L, \mathcal{L}_P^{\otimes a}) t^a = \frac{h_0^* + h_1^* t + \cdots + h_d^* t^d}{(1-t)^{d+1}}.$$

Then $h^(M, \mathcal{L}_P)$ is a Macaulay vector. In particular, it is non-negative.*

Example 9.6. Let $\mathcal{O}(\beta)$ denote the line bundle on W_L corresponding to $P(U_{n-1, n})$. The image of W_L under the complete linear system of $\mathcal{O}(\beta)$ can be identified with the closure in \mathbb{P}^{n-1} of the image of $\mathbb{P}L \cap \mathbb{G}_m^n / \mathbb{G}_m$ under the inversion map. This variety is called the *projective reciprocal plane*, and its coordinate ring is called the *Orlik–Terao algebra*. It follows from Theorem 9.3 that the projective reciprocal plane is projectively

normal and arithmetically Cohen–Macaulay, i.e., that the Orlik–Terao algebra is Cohen–Macaulay. This fact was originally proven in [PS06] by constructing a Gröbner degeneration of the projective reciprocal plane to the Stanley–Reisner scheme of the *broken circuit complex* of a matroid, which is known to be Cohen–Macaulay; see [Bjö92]. Using [LLPP24, Lemma 8.5], one can show that

$$\sum_{a \geq 0} \chi(W_L, \mathcal{O}(a\beta)) t^a = \frac{t^r}{(1-t)^r} T_M(t^{-1}, 0),$$

where M is the matroid realized by L , and $T_M(x, y)$ is the Tutte polynomial of M . As $\chi(W_L, \mathcal{O}(a\beta)) = \dim H^0(W_L, \mathcal{O}(a\beta))$ is the dimension of the a th graded piece of the Orlik–Terao algebra, this recovers a result of Terao [Ter02]. This implies that the β invariant of M is equal to $\dim H^{r-1}(W_L, \mathcal{O}(-\beta))$. This interpretation of the β invariant appears to be unrelated to the interpretation in Section 5.5.

Example 9.7. When P is the basis polytope of a uniform matroid, then the section ring $\bigoplus_{a \geq 0} H^0(W_L, \mathcal{L}_P^{\otimes a})$ is called the *higher order Orlik–Terao algebra* in the commutative algebra literature; see [TX21, GST18, Bur23]. Corollary 9.4 confirms the conjecture of Tohăneanu that higher order Orlik–Terao algebras are Cohen–Macaulay [Toh21].

Example 9.8. The visible contours variety \mathcal{V}_L is the image of W_L under the complete linear system of \mathcal{L}_{M^\perp} . Theorem 9.3 implies that it is projectively normal and arithmetically Cohen–Macaulay. However, there is no explicit generating set known for the ideal of \mathcal{V}_L in $\mathbb{P}^{\binom{n}{r}-1}$. It is possible to use Proposition 5.14 to obtain a formula for the Hilbert series of \mathcal{V}_L in terms of intersection numbers in the Chow ring of M , but a simple formula for these intersection numbers is not known.

We now give some examples which show that some plausible extensions of Theorem 9.3 do not hold. Let Q be a lattice polytope in \mathbb{R}^n which is contained in the hyperplane $x_1 + \cdots + x_n = c$ for some $c \in \mathbb{Z}_{\geq 0}$. Choose a smooth projective fan which refines both the permutohedral fan and the normal fan of Q . Let \tilde{W}_L denote the strict transform of $\mathbb{P}L$ in the corresponding toric variety; this is a smooth projective variety which is equipped with a nef line bundle \mathcal{L}_Q pulled back from the toric variety of Q . The line bundle \mathcal{L}_Q has a basepoint free linear system pulled back from the sections on the toric variety of Q . This linear system has a generating set labeled by lattice points. As mentioned previously, the results of [CCKR26] shows that $\chi(\tilde{W}_L, \mathcal{L}_Q)$ only depends on the matroid of L .

Let R_Q be the subring of the section ring of \mathcal{L}_Q generated by this linear series. The ring R_Q can be described concretely. Let x_i be a linear form on L cutting out the intersection of L with the i th coordinate hyperplane. View x_i as an element of $\text{Sym } L^*$. Then R_Q is the subring of $\text{Sym } L^*$ generated by the monomials in x_1, \dots, x_n corresponding to lattice points of Q . Let f_Q denote the map from \tilde{W}_L to the toric variety of Q . Then R_Q is the homogeneous coordinate ring of $f_Q(\tilde{W}_L)$.

When Q is a polymatroid, Corollary 9.4 states that R_Q enjoys various nice properties. Also, when $L = k^n$, toric geometry gives nice properties of R_Q and \mathcal{L}_Q : for example, the higher cohomology of any non-negative twist of \mathcal{L}_Q vanishes, the cohomology of any negative twist of \mathcal{L}_Q is concentrated in one degree, and the normalization of R_Q is generated in degree 1. If $L = k^n$ and, for each a , every lattice point in the dilate aQ is a sum of a lattice points in Q , then R_Q is normal, $f_{Q*} \mathcal{O}_{\tilde{W}_L} \cong \mathcal{O}_{f_Q(\tilde{W}_L)}$, and $R^i f_{Q*} \mathcal{O}_{\tilde{W}_L} = 0$ for $i > 0$. In general, the ring R_Q can be quite badly behaved.

Example 9.9. Suppose that $n = 2m$, and let Q be the polytope given by the convex hull of $e_1 + e_{m+1}, e_2 + e_{m+2}, \dots, e_m + e_{2m}$. Suppose that $L = L_1 \oplus L_2^\perp$, where L_1 and L_2 are subspaces of k^m . Then the map induced by the linear series on \mathcal{L}_Q factors through a resolution of singularities of $\mathbb{P}L_1 \times \mathbb{P}L_2^\perp$. It follows that $H^i(\tilde{W}_L, \mathcal{L}_Q^{\otimes a}) = H^i(\mathbb{P}_{W_{L_1}}(\mathcal{Q}_{L_2}^\vee), \mathcal{O}(a))$. The results in Section 7.3 then give examples where the higher cohomology of \mathcal{L}_Q may not vanish, and the dimension of $H^0(\tilde{W}_L, \mathcal{L}_Q)$ may not be combinatorially determined.

We give one further example. Suppose that k is a field of characteristic 3, and let L_1 and L_2 be the row span of the matrices

$$\begin{pmatrix} 2 & 0 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \\ 0 & 2 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 1 & 1 & 2 & 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 0 & 0 & 2 & 0 & 0 & 2 \\ 2 & 2 & 0 & 2 & 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 2 & 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \end{pmatrix},$$

respectively. Then one can compute that

$$(21) \quad \sum_{a=0}^{\infty} \chi(\tilde{W}_L, \mathcal{L}_Q^a) t^a = \frac{1-t+7t^2+t^3}{(1-t)^7}.$$

The image of f_Q is contained in \mathbb{P}^8 , and the product of the degree of the induced map $\mathbb{P}L_1 \times \mathbb{P}L_2^\perp \dashrightarrow \mathbb{P}^8$ and the degree of the image is 8, the sum of the coordinates of the numerator of (21). Using a computer algebra system, one can check that $f_Q(\tilde{W}_L)$ is a hypersurface of degree 8 inside a hyperplane of \mathbb{P}^8 . In particular, the map $\mathbb{P}L_1 \times \mathbb{P}L_2^\perp \dashrightarrow \mathbb{P}^8$ is birational onto its image. Furthermore, the image of f_Q is not normal. If one supposes that the higher direct images of $\mathcal{O}_{\tilde{W}_L}$ vanish and that the normalization of R_Q is Cohen–Macaulay, then one can use (21) to contradict Proposition 6.1. One therefore deduces that either $R^i f_{Q*} \mathcal{O}_{\tilde{W}_L} \neq 0$ for some $i > 0$, or the normalization of R_Q is not Cohen–Macaulay.

Example 9.10. Let k be a field of characteristic 0, and let $Q \subseteq \mathbb{R}^9$ be the convex hull of the set

$$\{e_1 + e_5 + e_6, e_6 + e_7 + e_8, e_4 + e_6 + e_8, e_2 + e_5 + e_6, e_2 + e_7 + e_8, e_1 + e_3 + e_7, e_1 + e_2 + e_7, e_4 + e_5 + e_9, e_1 + e_4 + e_5\}.$$

Then Q is a unimodular simplex. Let L_1 and L_2 be the row spans of the matrices

$$\begin{pmatrix} 0 & 3 & 21 & 84 & 240 & 0 & 0 & 12 & 60 \\ 1 & 4 & 10 & 19 & 32 & 20 & 7 & 13 & 23 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & -1 & 1 & -7 & 0 & 0 & 3 & -3 \\ 0 & 0 & 1 & -2 & 5 & 1 & -1 & 0 & 2 \\ 1 & 0 & 0 & 1 & -1 & -3 & 1 & -1 & 0 \end{pmatrix},$$

respectively. The matroids represented by L_1 and L_2 are the same. The Hilbert function of the ring R_Q defined using L_1 begins $(1, 9, 27, 54, 90, 135, \dots)$. The Hilbert function of the ring R_Q defined using L_2 begins $(1, 8, 27, 54, 90, 135, \dots)$.

Example 9.11. If one has an example where R_Q exhibits some bad behavior, such as not being Cohen–Macaulay or normal, then one can produce an example where the image of f_Q exhibits the same behavior through the following procedure. Suppose that Q is contained in the hyperplane $x_1 + \dots + x_n = c$. Then let $\tilde{L} = L \oplus k \subseteq k^{n+1}$, and consider the polytope \tilde{Q} given by taking the convex hull of the result of adding e_{n+1} to each lattice point of Q and also the lattice point $(0, \dots, 0, c+1)$. Then the image of $\tilde{W}_{\tilde{L}}$ under $f_{\tilde{Q}}$ has a dense open set which is isomorphic to the spectrum of R_Q .

See [EFL25, Section 5.1] for more examples, including an example of an ample line bundle on a wonderful variety with non-vanishing higher cohomology (and, indeed, negative Euler characteristic), and an example of a line bundle \mathcal{L} on a wonderful variety with the property that \mathcal{L} is not pulled back from a nef line bundle on the permutohedral toric variety, but $\mathcal{L}^{\otimes 2}$ is of the form \mathcal{L}_P for a polymatroid P .

We now give an extension of Corollary 9.5 to all matroids. Let P be a polymatroid. Because of the map $K(X_n) \rightarrow K(M)$, the line bundle \mathcal{L}_P defines a class in $K(M)$. Furthermore, $\chi(W_L, \mathcal{L}_P^{\otimes a}) = \chi(M, \mathcal{L}_P^{\otimes a})$. The inequalities proven in Corollary 9.5 can be extended to all matroids.

Theorem 9.12. [EFL25, Theorem B] *Let M be a loopless matroid on E , and let P be a polymatroid on E . Let d be the degree of the polynomial $a \mapsto \chi(M, \mathcal{L}_P^{\otimes a})$. Define $h^*(M, \mathcal{L}_P) = (h_0^*, h_1^*, \dots, h_d^*)$ via the formula*

$$\sum_{a \geq 0} \chi(M, \mathcal{L}_P^{\otimes a}) t^a = \frac{h_0^* + h_1^* t + \dots + h_d^* t^d}{(1-t)^{d+1}}.$$

Then $h^(M, \mathcal{L}_P)$ is a Macaulay vector. In particular, it is non-negative.*

This result was conjectured in [EL26]. We give an explicit, somewhat complicated combinatorial formula for the degree of the polynomial $a \mapsto \chi(M, \mathcal{L}_P^{\otimes a})$. Just like a matroid induces a partition of $[n]$ into its connected components, a polymatroid induces a partition $[n] = S_1 \sqcup \dots \sqcup S_c$. Then the degree $d(M, P)$ of this polynomial is

$$(22) \quad d(M, P) = \min_{[n]=T_1 \sqcup \dots \sqcup T_\ell} \sum_{i=1}^{\ell} (\text{rk}_M(T_i) - 1),$$

where the partition $[n] = T_1 \sqcup \dots \sqcup T_\ell$ coarsens the partition $[n] = S_1 \sqcup \dots \sqcup S_c$. See [EFL25, Section 4].

When M is realized by L , then $d(M, P)$ is the dimension of $f_P(W_L)$. If $B(P)$ has dimension $n-1$, then it follows from (22) that $d(M, P) = r-1$; see [EL26, Theorem 4.10] for an easy proof that the degree of the polynomial $a \mapsto \chi(M, \mathcal{L}_P^{\otimes a})$ is $r-1$ in this case. This is unsurprising: if M is realized by L and $B(P)$ is full-dimensional, then the map $f_P: X_n \rightarrow X_P$ is an isomorphism over the torus in X_n , and so the restriction of f_P to W_L is birational.

Proof of Theorem 5.38. The result is clear if the ground set has size 1, so we may assume that M is loopless and coloopless. By Proposition 5.10 and Corollary 5.22, we have $\omega(M_1 \oplus M_2) = \omega(M_1) \cdot \omega(M_2)$, so we may assume that M is connected. In this case, Proposition 5.23 gives that $\omega(M) = (-1)^{r-1} \chi(M, \mathcal{L}_{M^\perp}^{-1})$. Because M is connected, $B(M^\perp)$ has dimension $n-1$. It follows from (15) that $(-1)^{r-1} \chi(M, \mathcal{L}_{M^\perp}^{-1}) = h_{r-1}^*(M, \mathcal{L}_{M^\perp})$, so this is a special case of Theorem 9.12. \square

Proof of Theorem 9.3. We prove this in the case when \mathcal{L}_P is ample on X_n , and then we indicate the additional steps that are needed to do the general case. When \mathcal{L}_P is ample, Theorem 9.3(3) is vacuous, because an ample line bundle on a smooth projective toric variety is very ample [CLS11, Theorem 6.1.15], so f_P is an isomorphism onto its image. The first step is to degenerate W_L to a torus-invariant subscheme of X_n .

The torus T acts on the Hilbert scheme of X_n . As T is solvable, by the Borel fixed point theorem, the torus-orbit closure of the point of the Hilbert scheme corresponding to W_L contains a torus-fixed point. Pulling back the universal family to a curve connecting the point corresponding to W_L to a torus-fixed point gives the desired degeneration. In fact, we can arrange for this degeneration to be induced by a 1-parameter subgroup, i.e., a Gröbner degeneration. If Y is the torus-orbit closure in the Hilbert scheme of the point corresponding to W_L and H is a sufficiently general 1-parameter subgroup of T , then every H -fixed point of Y will be a T -fixed point as well. We can take this degeneration to be the flat limit of $\{t \cdot W_L : t \in H\}$. Call this degeneration $\text{in}(W_L)$.

By Example 8.12, the embedding of W_L into $(\mathbb{P}^1)^{\binom{n}{2}}$ is kindred. This embedding factors through the inclusion of W_L into X_n . Therefore the inclusion of any degeneration of W_L inside of X_n into $(\mathbb{P}^1)^{\binom{n}{2}}$ is kindred as well, so Corollary 8.4 implies that $\text{in}(W_L)$ is reduced and Cohen–Macaulay.

As $\text{in}(W_L)$ is a reduced torus-fixed subscheme of X_n , it is a union of torus-invariant subvarieties. As such, it is defined over $\text{Spec } \mathbb{Z}$. On any normal toric variety, there is a standard Frobenius splitting for which all torus-invariant subvarieties are compatibly split; see, e.g., [Pay09, Proposition 3.2]. In particular, over any field of positive characteristic, $\text{in}(W_L)$ is a compatibly split subvariety of X_n . Proposition 8.6 then implies

the desired vanishing results and surjectivity of global sections. As in the proof of Corollary 8.7, the same results hold for W_L by upper semicontinuity [Sta25, 0BDN].

Now we discuss the case when \mathcal{L}_P is not ample. Let $f_P: X_n \rightarrow X_P$ be the map induced by the complete linear series of \mathcal{L}_P . Note that $f_P(\text{in}(W_L))$ is a reduced union of torus-invariant subvarieties of X_P . We will show that $f_P(\text{in}(W_L))$ is Cohen–Macaulay, and so we can apply Proposition 8.6 to prove vanishing theorems on $f_P(\text{in}(W_L))$. We will also show that $f_{P*}\mathcal{O}_{\text{in}(W_L)} = \mathcal{O}_{f_P(\text{in}(W_L))}$ and that $R^i f_{P*}\mathcal{O}_{\text{in}(W_L)} = 0$ for $i > 0$. The Leray spectral sequence then relates the cohomology of the very ample line bundle on $f_P(\text{in}(W_L))$ to the cohomology of \mathcal{L}_P on $\text{in}(W_L)$, proving the desired vanishing.

The strategy taken in [EFL25, Proposition 3.10] is to show that the restriction of f_P to $\text{in}(W_L)$ is “locally a coordinate projection of a kindred subscheme.” There is an open cover $\{U_\sigma\}$ of X_P indexed by maximal cones in the fan of X_P . For each σ , one can construct an embedding of U_σ into $(\mathbb{P}^1)^\ell$ such that the restriction of f_P to $f_P^{-1}(U_\sigma)$ is the restriction of a coordinate projection $(\mathbb{P}^1)^{\binom{n}{2}} \rightarrow (\mathbb{P}^1)^\ell$. As $\text{in}(W_L)$ is a kindred subscheme of $(\mathbb{P}^1)^{\binom{n}{2}}$ and the properties that we need to verify are local on X_P , the result follows from Proposition 8.8. \square

We briefly comment on the proof of Theorem 9.12. For each polymatroid P and realizable loopless matroid M , the proof of Theorem 9.3 constructs a Cohen–Macaulay union of toric subvarieties of X_P and an ample line bundle such that the Hilbert polynomial of this ample line bundle is given by $a \mapsto \chi(M, \mathcal{L}_P^{\otimes a})$. The Cohen–Macaulayness and Frobenius splitting imply that the homogeneous coordinate ring of this union of toric varieties is Cohen–Macaulay, giving the desired inequalities.

Using a result of Katz [Kat09, Theorem 10.1], one can give a tropical description of this union of toric subvarieties. This description makes sense for any loopless matroid. From this description and a result of Knutson [Knu], it is possible to show that the function which associates a loopless matroid to the class of the structure sheaf of this union of toric subvarieties in $K_{\text{co}}(X_P)$ is valuative. Using Corollary 4.11, one shows that the Hilbert polynomial of this union of subvarieties is given by $a \mapsto \chi(M, \mathcal{L}_P^{\otimes a})$. An argument similar to the end of the proof of Theorem 9.3 can be used to prove that this union of toric subvarieties is Cohen–Macaulay for any loopless matroid M , so a Frobenius splitting argument and Theorem 6.12 imply that its section ring is Cohen–Macaulay, and its Hilbert function agrees with its Hilbert polynomial.

9.2. Omega positivity via section rings on projective bundles. We now describe an approach to proving vanishing theorems on wonderful varieties using tautological bundles. This approach was introduced in [BF22] and extended in [Liu25]. In those works, the characteristic of k is assumed to be 0 because they make crucial use of Proposition 8.16. We can remove this assumption by using Proposition 8.8, and so the results given here are somewhat stronger.

For this approach, it will be convenient to introduce tautological bundles associated to realizations of polymatroids. Subspace arrangements give rise to polymatroids in the same way that hyperplane arrangements give rise to matroids. We will consider essential subspace arrangements, i.e., collections of subspaces W_1, \dots, W_n in a vector space L with $\cap W_i = 0$. For each i , choose a set S_i of hyperplanes whose intersection is W_i . Then we can view this subspace arrangement as arising from an embedding $L \hookrightarrow k^{S_1} \oplus \dots \oplus k^{S_n}$, where W_i is the intersection of L with the subspace of $k^{S_1} \oplus \dots \oplus k^{S_n}$ where the coordinates labeled by S_i are 0.

Definition 9.13. *A polymatroid P on $[n]$ is said to be realizable over k if there are sets S_1, \dots, S_n and a linear subspace L of $k^{S_1} \oplus \dots \oplus k^{S_n}$ where, for every subset T of $[n]$, $\text{rk}_P(T)$ is the codimension in L of the intersection of L with the subspace of $k^{S_1} \oplus \dots \oplus k^{S_n}$ where the coordinates labeled by elements of $\cup_{i \in T} S_i$ are 0.*

For any subspace L of $k^{S_1} \oplus \cdots \oplus k^{S_n}$, the function described in Definition 9.13 is a polymatroid. Matroids can be described as polymatroids where $\text{rk}_P(i) \in \{0, 1\}$ for each i , and a matroid is realizable over k if and only if it is realizable over k when viewed as a polymatroid.

Let $L \subseteq k^{S_1} \oplus \cdots \oplus k^{S_n}$ be a subspace arrangement, with $\dim L = r$ and $|S_1| + \cdots + |S_n| = N$, so L defines a point $[L]$ of $\text{Gr}(r, N)$. Let P be the polymatroid realized by L . Consider the action of $T = \mathbb{G}_m^n$ on $\text{Gr}(r, N)$, where the i th copy of \mathbb{G}_m scales the coordinates labeled by S_i .

Lemma 9.14. *The moment polytope of $\overline{T \cdot [L]}$ is the basis polytope of P .*

Proof. Let $\pi: \mathbb{R}^N \rightarrow \mathbb{R}^n$ be the linear map which sends a standard basis vector e_i of \mathbb{R}^N to the standard basis vector of \mathbb{R}^n corresponding to the part of the partition $[N] = S_1 \sqcup \cdots \sqcup S_n$ which contains i . Let M be the matroid on $[N]$ realized by L . By the discussion in Section 1.2, the moment polytope of $\overline{T \cdot [L]}$ is $\pi(B(M))$. If we choose the basis for each k^{S_i} generically, which may be assumed, then that $\pi(B(M))$ is the basis polytope of P follows from [EL24, Lemma 3.2]. \square

In particular, there is a T -equivariant map from X_n to $\text{Gr}(r, N)$ which sends the identity of the torus in X_n to $[L]$. Using Theorem 1.16, one can show that $\overline{T \cdot [L]}$ is in fact normal, although this is not important to what follows.

Given a subspace arrangement $L \subseteq k^{S_1} \oplus \cdots \oplus k^{S_n}$, the *dual* subspace arrangement is the orthogonal complement L^\perp , also viewed as a subspace of $k^{S_1} \oplus \cdots \oplus k^{S_n}$. Note that the polymatroid realized by the dual subspace arrangement depends on the choice of realization of L as a subspace of $k^{S_1} \oplus \cdots \oplus k^{S_n}$, not just on the subspace arrangement in L .

Definition 9.15. *Let $L \subseteq k^{S_1} \oplus \cdots \oplus k^{S_n}$ be a subspace arrangement, with $\dim L = r$ and $|S_1| + \cdots + |S_n| = N$. The tautological bundles \mathcal{S}_L and \mathcal{Q}_L on X_n are the duals of the pullbacks of the universal quotient bundle and universal subbundle on $\text{Gr}(N - r, N)$ along the T -equivariant map which sends the identity of the torus in X_n to $[L^\perp]$.*

Over a point $t \in \mathbb{G}_m^n / \mathbb{G}_m \subset X_n$, the fiber of \mathcal{S}_L is $t^{-1}L$, where the action of T is as described above. By construction, there is a short exact sequence

$$0 \rightarrow \mathcal{S}_L \rightarrow \mathcal{O}^{\oplus N} \rightarrow \mathcal{Q}_L \rightarrow 0.$$

When $L \subseteq k^n$ is a realization of a matroid, the vector bundles defined in Definition 9.15 agree with the vector bundles defined in Section 5.3. Many of the nice properties and formulas are special to the matroid case, however.

Example 9.16. Let L_1 and L_2 be two subspaces of k^n . We can view $L_1 \oplus L_2$ as a subspace of $k^2 \oplus \cdots \oplus k^2$ in a natural way. Then $\mathcal{S}_{L_1} \oplus \mathcal{S}_{L_2}$ is the tautological bundle associated to the polymatroid $L_1 \oplus L_2 \subseteq k^2 \oplus \cdots \oplus k^2$. The basis polytope of the polymatroid realized by this subspace arrangement is the Minkowski sum of the matroid polytopes of the matroids realized by L_1 and L_2 .

Let $L_1 \subseteq k^{S_1} \oplus \cdots \oplus k^{S_n}$ be a subspace arrangement, and let $L_2 \subseteq k^n$ be a hyperplane arrangement realizing a loopless matroid M . We will consider the restriction of \mathcal{S}_{L_1} to W_{L_2} . We view the total space $\text{Tot}(\mathcal{S}_{L_1})$ as a subscheme of $W_{L_2} \times \mathbb{A}^N$, where $N = |S_1| + \cdots + |S_n|$. Combining the embedding of W_{L_2} into $(\mathbb{P}^1)^{\binom{2}{2}}$ from Example 8.12 with the natural embedding $\mathbb{A}^N \hookrightarrow (\mathbb{P}^1)^N$ realizes $\text{Tot}(\mathcal{S}_{L_1})$ as a subscheme of $(\mathbb{P}^1)^{\binom{2}{2}} \times (\mathbb{P}^1)^N$.

Proposition 9.17. *The closure of $\text{Tot}(\mathcal{S}_{L_1})$ in $(\mathbb{P}^1)^{\binom{2}{2}} \times (\mathbb{P}^1)^N$ is kindred.*

Proof. We will use Proposition 8.9. Choose elements a_1, \dots, a_n of k such that L_2 is not contained in the hyperplane $a_1x_1 + \dots + a_nx_n = 0$. Let H denote the affine hyperplane $\{(v_1, \dots, v_n) \in L_2 : a_1v_1 + \dots + a_nv_n = 1\}$ in L_2 . Then the following set is a dense open subset of $\text{Tot}(\mathcal{S}_{L_1})$:

$$(23) \quad \{(v, v^{-1} \cdot w) : v \in H \cap \mathbb{G}_m^n, w \in L_1\}.$$

Here $v^{-1} \cdot w$ means that we scale the coordinates of w labeled by S_i by v_i^{-1} . The open set in (23) is isomorphic to an open subset of $H \times L_1$, via the map which sends an element (x, y) of the above open set to $(x, x \cdot y) \in H \times L_1$.

In order to check that the multidegree is multiplicity-free, we choose a subset T of $\binom{[n]}{2} \sqcup [N]$ of size equal to the dimension of $\text{Tot}(\mathcal{S}_{L_1})$, and we intersect the set in (23) with general translates of coordinate hyperplanes corresponding to the elements of T . We know that the conditions are general enough that the intersection will be a finite, possibly empty, set.

Intersecting with a translate of the coordinate hyperplane corresponding to $\{i, j\} \in \binom{[n]}{2}$ imposes the condition $v_i = \lambda v_j$ for some general $\lambda \in k$. Intersecting with a translate of a coordinate hyperplane corresponding to $s \in [N]$, where $s \in S_\ell$, imposes the condition $v_\ell^{-1} w_s = \lambda$ for some general $\lambda \in k$.

In terms of the identification of the open set in (23) with an open subset of $H \times L_1$, both of these types of conditions are linear conditions. An intersection of the appropriate number of sufficiently general linear conditions with an open subset of an affine space is either empty or a reduced point. \square

Corollary 9.18. *Let $p: \text{Tot}(\mathcal{S}_{L_1}) \rightarrow \mathbb{A}^N$ be the natural projection. Then $p_* \mathcal{O}_{\text{Tot}(\mathcal{S}_{L_1})} = \mathcal{O}_{p(\text{Tot}(\mathcal{S}_{L_1}))}$ and $R^i p_* \mathcal{O}_{\text{Tot}(\mathcal{S}_{L_1})} = 0$ for $i > 0$. Also, $p(\text{Tot}(\mathcal{S}_{L_1}))$ is Cohen–Macaulay.*

Proof. Let Y denote the closure of $\text{Tot}(\mathcal{S}_{L_1})$ in $W_{L_2} \times (\mathbb{P}^1)^N$, and let \tilde{p} denote the projection $W_{L_2} \times (\mathbb{P}^1)^N \rightarrow (\mathbb{P}^1)^N$. Proposition 8.8 implies that $\tilde{p}_* \mathcal{O}_Y = \mathcal{O}_{\tilde{p}(Y)}$, $R^i \tilde{p}_* \mathcal{O}_Y = 0$ for $i > 0$, and $\tilde{p}(Y)$ is Cohen–Macaulay. Then $\text{Tot}(\mathcal{S}_{L_1})$ is the preimage of the open subset \mathbb{A}^N of $(\mathbb{P}^1)^N$, which implies the statement about higher direct images. As $p(\text{Tot}(\mathcal{S}_{L_1}))$ is an open subset of $\tilde{p}(Y)$, it is Cohen–Macaulay. \square

Remark 9.19. Suppose that $|S_i| = 1$ for all i , i.e., L_1 is a subspace of k^n . Then $p(\text{Tot}(\mathcal{S}_{L_1}))$ is the semi-inverted Hadamard product defined in Example 8.13. By Proposition 9.17 and Proposition 8.8, the closure of the semi-inverted Hadamard product in $(\mathbb{P}^1)^n$ is kindred. That semi-inverted Hadamard products are Cohen–Macaulay was previously proven in [BF24, Corollary 5.7] using a related argument.

Corollary 9.20. *For each $a \geq 0$, the restriction map $H^0(W_{L_2}, \text{Sym}^a \mathcal{O}^{\oplus N}) \rightarrow H^0(W_{L_2}, \text{Sym}^a \mathcal{S}_{L_1}^\vee)$ is surjective, and $H^i(W_{L_2}, \text{Sym}^a \mathcal{S}_{L_1}^\vee) = 0$ for any $i > 0$.*

Proof. Let $\pi: \text{Tot}(\mathcal{S}_{L_1}) \rightarrow W_{L_2}$ be the natural projection. Because $\text{Tot}(\mathcal{S}_{L_1}) = \text{Spec}_{W_{L_2}}(\text{Sym} \mathcal{S}_{L_1}^\vee)$, π is an affine map. This implies that $\pi_* \mathcal{O}_{\text{Tot}(\mathcal{S}_{L_1})} = \bigoplus_{a \geq 0} \text{Sym}^a \mathcal{S}_{L_1}^\vee$ and $R^i \pi_* \mathcal{O}_{\text{Tot}(\mathcal{S}_{L_1})} = 0$ for $i > 0$. When we further push this forward to a point, we get the same thing as if we push forward along p and then push forward to a point. By Corollary 9.18, if we push forward along p and then to a point the higher direct images are 0, which implies that the higher cohomology of $\pi_* \mathcal{O}_{\text{Tot}(\mathcal{S}_{L_1})} = \bigoplus_{a \geq 0} \text{Sym}^a \mathcal{S}_{L_1}^\vee$ vanishes.

The coordinate ring of $p(\text{Tot}(\mathcal{S}_{L_1}))$ is $\bigoplus_{a \geq 0} H^0(W_{L_2}, \text{Sym}^a \mathcal{S}_{L_1}^\vee)$. Because p is proper, $p(\text{Tot}(\mathcal{S}_{L_1}))$ is a closed subscheme of \mathbb{A}^N , so the coordinate ring of \mathbb{A}^N surjects onto its coordinate ring. This surjection is identified with the map $\bigoplus_{a \geq 0} H^0(W_{L_2}, \text{Sym}^a \mathcal{O}^{\oplus N}) \rightarrow \bigoplus_a H^0(W_{L_2}, \text{Sym}^a \mathcal{S}_{L_1}^\vee)$, proving the result. \square

Let d_{L_1} denote the dimension of the image of $\text{Tot}(\mathcal{S}_{L_1})$ in \mathbb{A}^N , and let r be the dimension of L_1 .

Theorem 9.21. *For any $a \geq 0$, $H^i(W_{L_2}, \det \mathcal{S}_{L_1} \otimes \text{Sym}^a \mathcal{S}_{L_1}) = 0$ unless $i = d_{L_1} - r$.*

Proof. Let $\bar{p}: \mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1}) \rightarrow \mathbb{P}^{N-1}$ be the natural map. Note that \bar{p} is induced by the complete linear system of the globally generated line bundle $\mathcal{O}(1)$ on $\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1})$. For $a \geq 0$ and any i , there is a natural identification

of $H^i(W_{L_2}, \text{Sym}^a \mathcal{S}_{L_1}^\vee)$ with $H^i(\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1}), \mathcal{O}(a))$. It follows from Corollary 9.20 and Proposition 6.13 that $R^i \bar{\pi}_* \mathcal{O}_{\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1})} = 0$ for $i > 0$.

By Corollary 9.20, $\bigoplus_a H^0(\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1}), \mathcal{O}(a))$ is the coordinate ring of $p(\text{Tot}(\mathcal{S}_{L_1}))$, so Corollary 9.18 implies that this ring is Cohen–Macaulay. It follows from Proposition 6.16 that for any $a < 0$, we have $H^j(\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1}), \mathcal{O}(a)) = 0$ unless $j = d_{L_1} - 1$, as $d_{L_1} - 1$ is the dimension of the image of $\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1})$ under the complete linear system of $\mathcal{O}(1)$.

Let $\bar{\pi}: \mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1}) \rightarrow W_{L_2}$ be the natural projection. For any $a \geq 0$, we have $R^{r-1} \bar{\pi}_* \mathcal{O}(-a-r) \cong \det \mathcal{S}_{L_1} \otimes \text{Sym}^a \mathcal{S}_{L_1}$, and $R^i \bar{\pi}_* \mathcal{O}(-a-r) = 0$ for $i \neq r-1$. The conclusion follows, as the Leray spectral sequence gives an isomorphism

$$H^i(W_{L_2}, \det \mathcal{S}_{L_1} \otimes \text{Sym}^a \mathcal{S}_{L_1}) \cong H^{i+r-1}(\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1}), \mathcal{O}(-a-r)). \quad \square$$

We now give several more concrete applications of Corollary 9.20 and Theorem 9.21. We first give a numerical consequence.

Corollary 9.22. *Let Y be the image of $\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1})$ in \mathbb{P}^{N-1} , and let $p(a)$ denote the Hilbert function of Y . Then $p(a)$ is a polynomial of degree $d_{L_1} - 1$. If we write*

$$\sum_{a \geq 0} p(a) t^a = \frac{h_0^* + h_1^* t + \cdots + h_{d_{L_1}-1}^* t^{d_{L_1}-1}}{(1-t)^{d_{L_1}}},$$

then $h_q^* \geq 0$ for each q . Furthermore, $h_q^* = 0$ if $q > d_{L_1} - r$, and $h_{d_{L_1}-r}^* = (-1)^{d_{L_1}-r} \chi(W_{L_2}, \det \mathcal{S}_{L_1})$.

Proof. By Corollary 9.18, the coordinate ring of Y is $\bigoplus_{a \geq 0} H^0(\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1}), \mathcal{O}(a))$, and the higher cohomology of $\mathcal{O}(a)$ vanishes for $a \geq 0$. This implies that the Hilbert function of Y is a polynomial of degree equal to the dimension of Y , which is $d_{L_1} - 1$. As the coordinate ring of Y is Cohen–Macaulay by Corollary 9.18, Proposition 6.1 implies that $h_q^* \geq 0$ for each q . From (15), we have

$$(24) \quad p(a) = \sum_{q=0}^{d_{L_1}-1} h_q^* \binom{a + d_{L_1} - 1 - q}{d_{L_1} - 1}.$$

Let $\bar{\pi}: \mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1}) \rightarrow W_{L_2}$ be the natural map. For $-r < a < 0$, we have $R^i \bar{\pi}_* \mathcal{O}(a) = 0$ for all i . This implies that $\chi(\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1}), \mathcal{O}(a)) = p(a) = 0$ for $-r < a < 0$. Evaluating (24) at $a = -1, -2, \dots, -r+1$, we see that $h_q^* = 0$ for $q > d_{L_1} - r$. Evaluating at $a = -r$, we see that $h_{d_{L_1}-r}^* = (-1)^{d_{L_1}-1} \chi(\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1}), \mathcal{O}(-r))$. As in the proof of Theorem 9.21, pushing forward $\mathcal{O}(-r)$ to W_{L_2} shows that $\chi(\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1}), \mathcal{O}(-r)) = (-1)^{r-1} \chi(W_{L_2}, \det \mathcal{S}_{L_1})$, so $h_{d_{L_1}-r}^* = (-1)^{d_{L_1}-r} \chi(W_{L_2}, \det \mathcal{S}_{L_1})$. \square

Proposition 9.23. *Let $L \subseteq k^n$ be a realization of a loopless matroid, and let P be a polymatroid on $[n]$ which is realizable over k . Then for any $a < 0$, the cohomology of $\mathcal{L}_P^{\otimes a}$ on W_L is concentrated in a single degree.*

Because k is algebraically closed, it follows from the Nullstellensatz that P is realizable over k if it is realizable over some field with the same characteristic as k . As stated, Proposition 9.23 is a special case of Theorem 9.3. We will compare them further below.

Proof of Proposition 9.23. For $b > 0$, the line bundle $\mathcal{L}_P^{\otimes b}$ is isomorphic to the line bundle \mathcal{L}_{bP} , where bP is the polymatroid with rank function equal to b times the rank function of P . Suppose that P is realized by a subspace $V \subseteq k^{S_1} \oplus \cdots \oplus k^{S_n}$. Then bP is realized by $V^{\oplus b} \subseteq (k^{S_1})^{\oplus b} \oplus \cdots \oplus (k^{S_n})^{\oplus b}$. So we may reduce to the case $a = -1$.

Let \mathcal{S}_{V^\perp} denote the tautological bundle of the dual polymatroid to P . We claim that $\det \mathcal{S}_{V^\perp} \cong \mathcal{L}_P^{-1}$. Indeed, $\mathcal{S}_{V^\perp}^\vee$ is the pullback of the tautological quotient bundle on a Grassmannian. The determinant of the tautological quotient bundle on a Grassmannian is the line bundle $\mathcal{O}(1)$ which induces the Plücker embedding of the Grassmannian. The pullback of an ample line bundle to a toric variety is the line bundle corresponding to the moment polytope, which is the basis polytope of the dual of P in this case. See Lemma 9.14.

It then follows from Theorem 9.21 that the cohomology of $\det \mathcal{S}_{V^\perp}$ on W_L is concentrated in one degree, proving the theorem. \square

In order to use Theorem 9.21, it is helpful to understand the integer d_{L_1} appearing there. We will restrict our attention to hyperplane arrangements. Let $L_1 \subseteq k^n$ be a realization of a matroid M_1 of rank r_1 , and suppose that L_2 is a realization of a loopless matroid M_2 of rank r_2 .

The *diagonal Dilworth truncation* $D(M_1, M_2)$ of M_1 and M_2 is a matroid whose independent sets are subsets I of $[n]$ such that, for any $j \in [n]$, $I \cup j$ can be written as a disjoint union of an independent set of M_1 and an independent set of M_2 . This construction dates back to Edmonds [Edm70, Theorem 8], and it has appeared several times in the literature; see [Ber22, Li18]. See [BF24, Section 4] for a discussion of the combinatorics of diagonal Dilworth truncations. The significance of the diagonal Dilworth truncation for us comes from the following result.

Proposition 9.24. *The algebraic matroid of the closure of the semi-inverted Hadamard product $\{v/w : v \in L_1, w \in L_2 \cap \mathbb{G}_m^n\}$ in \mathbb{A}^n is $D(M_1, M_2)$.*

Proof. The closure of the semi-inverted Hadamard product in $(\mathbb{P}^1)^n$ can be described as the image of the rational map $\mathbb{P}(L_1 \oplus L_2) \dashrightarrow (\mathbb{P}^1)^n$ given by sending the line spanned by the vector (v, w) in $L_1 \oplus L_2$ to $([v_1, w_1], [v_2, w_2], \dots, [v_n, w_n])$, where $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$. The multidegree of the closure is computed in [Li18, Theorem 1.1]; see also [BES24, Theorem 5.2.4] and [Lar25, Theorem 1.2(2)]. \square

In particular, Proposition 9.24 implies that the dimension of the semi-inverted Hadamard product is equal to the rank of $D(M_1, M_2)$. Note that the rank of $D(M_1, M_2)$ is at most $\text{rk}(M_1) + \text{rk}(M_2) - 1$. We say that $D(M_1, M_2)$ has the *expected rank* if its rank is equal to $\text{rk}(M_1) + \text{rk}(M_2) - 1$.

As mentioned before, the image of the total space of \mathcal{S}_{L_1} in \mathbb{A}^n is the semi-inverted Hadamard product of L_1 and L_2 . In particular, the number d_{L_1} occurring in Theorem 9.21 is equal to $\text{rk}(D(M_1, M_2))$. This proves the following.

Corollary 9.25. *Let $L_2 \subseteq k^n$ be a realization of a loopless matroid M_2 , and let M_1 be a matroid which is realizable over k . Then $H^i(W_{L_2}, \mathcal{L}_{M_1^\perp}^{-1}) = 0$ unless $i = \text{rk}(D(M_1, M_2)) - \text{rk}(M_1)$.*

Interestingly, Corollary 9.25 often says something different from Theorem 9.3. For example, if M_1 is connected, then Theorem 9.3 states that $H^i(W_{L_2}, \mathcal{L}_{M_1^\perp}^{-1}) = 0$ unless $i = \text{rk}(M_2) - 1$. If $D(M_1, M_2)$ does not have the expected rank, then combining this with Corollary 9.25 implies that all cohomology of $\mathcal{L}_{M_1^\perp}^{-1}$ vanishes. For example, this holds if $\text{rk}(M_1) + \text{rk}(M_2) > n + 1$.

Conjecture 9.26. *Let $L_2 \subseteq k^n$ be a realization of a loopless matroid M_2 , and let M_1 be a matroid. Then $H^i(W_{L_2}, \mathcal{L}_{M_1^\perp}^{-1}) = 0$ unless $i = \text{rk}(D(M_1, M_2)) - \text{rk}(M_1)$.*

Theorem 9.3 shows that the cohomology of $\mathcal{L}_{M_1^\perp}^{-1}$ on W_{L_2} is concentrated in degree $\dim f_{M_1^\perp}(W_{L_2})$, which is equal to the quantity $d(M_2, M_1)$ in (22). The content of the above conjecture is that if $d(M_2, M_1)$ is not equal to $\text{rk}(D(M_1, M_2)) - \text{rk}(M_1)$, then $\chi(M_2, \mathcal{L}_{M_1^\perp}^{-1}) = 0$.

Example 9.27. If $\text{rk}(D(M_1, M_2)) = n$, then the semi-inverted Hadamard product is all of \mathbb{A}^n . Corollary 9.20 implies that the map $\text{Sym } k^n \rightarrow \bigoplus_{a \geq 0} H^0(W_{L_2}, \text{Sym}^a \mathcal{S}_{L_1}^\vee)$ is an isomorphism.

In some cases, we can use the above techniques to prove that $H^i(W_{L_2}, \mathcal{L}_{M_1^\perp}^{-1}) = 0$ for all i .

Proposition 9.28. *Let $L_2 \subseteq k^n$ be a realization of a loopless matroid M_2 , and let M_1 be a matroid which is realizable over k . If there is a connected component of $D(M_1, M_2)$ which is not a union of connected components of M_1 , then $H^i(W_{L_2}, \mathcal{L}_{M_1^\perp}^{-1}) = 0$ for all i .*

Lemma 9.29. *Let X be an integral closed subvariety of \mathbb{A}^n , and suppose that the algebraic matroid M of X is disconnected, with components S_1, \dots, S_c . For $S \subseteq [n]$, let p_S be the coordinate projection from \mathbb{A}^n to \mathbb{A}^S . Then the natural map from X to $p_{S_1}(X) \times p_{S_2}(X) \times \dots \times p_{S_c}(X)$ is an isomorphism.*

Proof. The map is clearly injective and injective on tangent vectors because this is true when $X = \mathbb{A}^n$, so to prove that it is an isomorphism it suffices to show that it is surjective. Because $p_{S_1}(X) \times \dots \times p_{S_c}(X)$ is irreducible, it suffices to check that $\dim X = \dim p_{S_1}(X) \times \dots \times p_{S_c}(X)$. The dimension of $p_S(X)$ is $\text{rk}_M(S)$. Because the S_i are connected components of M , we have

$$\dim X = \text{rk}(M) = \text{rk}_M(S_1) + \dots + \text{rk}_M(S_c). \quad \square$$

Proof of Proposition 9.28. Suppose that $D(M_1, M_2)$ has connected components S_1, \dots, S_c , with $S_1 \sqcup \dots \sqcup S_c = [n]$. Let $d = \text{rk}(D(M_1, M_2))$, and let d_i be the rank of the component with vertex set S_i . Let r be the rank of M_1 , and let $r_i = \text{rk}_{M_1}(S_i)$.

Let X denote the semi-inverted Hadamard product of L_1 and L_2 , the closure of $\{v/w : v \in L_1, w \in L_2 \cap \mathbb{G}_m^n\}$ in \mathbb{A}^n . By Lemma 9.29 and Proposition 9.24, the natural map from X to $p_{S_1}(X) \times \dots \times p_{S_c}(X)$ is an isomorphism. Note that $p_{S_i}(X)$ is the semi-inverted Hadamard product of $p_{S_i}(L_1)$ and $p_{S_i}(L_2)$. The coordinate ring of X is the tensor product of the coordinate rings of these semi-inverted Hadamard products. Let $L_1^{S_i} = p_{S_i}(L_1)$, and define $L_2^{S_i}$ similarly. Note that $\dim L_1^{S_i} = r_i$.

As discussed previously, X is the cone over the image of $\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1})$ in \mathbb{P}^{n-1} , and the semi-inverted Hadamard product of $L_1^{S_i}$ and $L_2^{S_i}$ is the cone over the image of $\mathbb{P}_{W_{L_2^{S_i}}}(\mathcal{S}_{L_1^{S_i}})$ in $\mathbb{P}^{|S_i|-1}$. Let $p(a)$ denote the Hilbert polynomial of $\mathcal{O}(1)$ on $\mathbb{P}_{W_{L_2}}(\mathcal{S}_{L_1})$, and let $q_i(a)$ denote the Hilbert polynomial of $\mathcal{O}(1)$ on $\mathbb{P}_{W_{L_2^{S_i}}}(\mathcal{S}_{L_1^{S_i}})$ for each i . We have

$$(25) \quad \sum_{a \geq 0} p(a)t^a = \prod_{i=1}^c \sum_{a \geq 0} q_i(a)t^a.$$

Let $h_0^*(L_1) + h_1^*(L_1)t + \dots + h_{d-r}^*(L_1)t^{d-r} = (1-t)^d \sum_{a \geq 0} p(a)t^a$, and let $h_0^*(L_1^{S_i}) + h_1^*(L_1^{S_i})t + \dots + h_{d_i-r_i}^*(L_1^{S_i})t^{d_i-r_i} = (1-t)^{d_i} \sum_{a \geq 0} q_i(a)t^a$; that these are polynomials of the specified form follows from Corollary 9.22. By (25), we see that

$$h_0^*(L_1) + h_1^*(L_1)t + \dots + h_{d-r}^*(L_1)t^{d-r} = \prod_{i=1}^c (h_0^*(L_1^{S_i}) + h_1^*(L_1^{S_i})t + \dots + h_{d_i-r_i}^*(L_1^{S_i})t^{d_i-r_i}).$$

The right-hand side has degree at most $d_1 + \dots + d_c - r_1 - \dots - r_c = d - \text{rk}_{M_1}(S_1) - \dots - \text{rk}_{M_1}(S_c)$. If some S_i is not a union of connected components of M_1 , then $\text{rk}_{M_1}(S_1) + \dots + \text{rk}_{M_1}(S_c) > r$. This implies that $h_{d-r}^*(L_1) = 0$. By Corollary 9.22, $h_{d-r}^*(L_1) = (-1)^{d-r} \chi(W_{L_2}, \mathcal{L}_{M_1^\perp}^{-1}) = 0$. By Proposition 9.23, the cohomology of $\mathcal{L}_{M_1^\perp}^{-1}$ is concentrated in a single degree, so this shows that all cohomology vanishes. \square

Example 9.30. Proposition 9.28 does not characterize when all cohomology of $\mathcal{L}_{M_1^\perp}^{-1}$ on W_{L_2} vanishes. For example, let k be a field of characteristic 5, and let

$$L_1 = \text{row span} \begin{pmatrix} 0 & 0 & 1 & 4 & 3 & 2 & 0 \\ 2 & 0 & 2 & 0 & 4 & 2 & 1 \\ 0 & 4 & 3 & 3 & 4 & 1 & 0 \end{pmatrix} \text{ and } L_2 = \text{row span} \begin{pmatrix} 2 & 4 & 0 & 1 & 1 & 2 & 0 \\ 2 & 0 & 3 & 0 & 0 & 2 & 3 \\ 4 & 1 & 0 & 4 & 4 & 4 & 0 \end{pmatrix}.$$

Let M_1 and M_2 be the corresponding matroids. Then $D(M_1, M_2)$ is a connected matroid of rank 5, but $\chi(W_{L_2}, \mathcal{L}_{M_1^\perp}^{-1}) = 0$, so it follows from Theorem 9.3 or Corollary 9.25 that $H^i(W_{L_2}, \mathcal{L}_{M_1^\perp}^{-1}) = 0$ for all i .

Proof of Theorem 5.38 for realizable matroids. By Proposition 5.25 and Corollary 5.22, it suffices to show that if M is a connected matroid of rank r on at least 2 elements, then $\omega(M) \geq 0$. Suppose that M is realized by $L \subseteq k^n$. For such a matroid, Proposition 5.23 states that $\omega(M) = (-1)^{r-1} \chi(W_L, \mathcal{L}_{M^\perp}^{-1})$.

If $D(M, M)$ has rank less than $2r - 1$, then [BF24, Lemma 4.13] shows that $D(M, M)$ is not connected. In this case, Proposition 9.28 implies that $\chi(W_L, \mathcal{L}_{M^\perp}^{-1}) = 0$, so $\omega(M) = 0$. If $D(M, M)$ has rank equal to $2r - 1$, then Corollary 9.25 implies that $H^i(W_L, \mathcal{L}_{M^\perp}^{-1}) = 0$ unless $i = r - 1$, so $(-1)^{r-1} \chi(W_L, \mathcal{L}_{M^\perp}^{-1}) \geq 0$. \square

Remark 9.31. If k is a field of characteristic 0, then a trick used in [BF22] (see also [Liu25]) implies vanishing theorems for Schur functors of tautological bundles. If \mathcal{E} is a vector bundle over a variety over a field k of characteristic 0, then one has an isomorphism

$$(26) \quad \text{Sym}^\bullet(\mathcal{E}^{\oplus s}) \cong \bigoplus_{|\lambda| \leq s} S^\lambda k^s \otimes S^\lambda(\mathcal{E}),$$

where S^λ is the Schur functor associated to λ ; see [Wey03, Corollary 2.3.3]. In particular, if \mathcal{S}_L is a tautological subbundle of a subspace arrangement, then so is $\mathcal{S}_L^{\oplus s}$, and so applying Corollary 9.20 implies that the higher cohomology of $S^\lambda \mathcal{S}_L^\vee$ vanishes for any partition λ . If k has positive characteristic, then there is no decomposition like (26) in general.

As discussed in Section 7.3, one cannot expect the same cohomology vanishing results for tautological quotient bundles. On the permutohedral toric variety, however, cohomology vanishing results for tautological quotient bundles can be deduced from those for subbundles.

Recall that X_n is the wonderful variety of the Boolean arrangement $k^n \subseteq k^n$. There is an involution $\text{crem}: X_n \rightarrow X_n$, called the *Cremona involution*, which restricts to inversion on the dense torus in X_n . Let $L \subseteq k^{S_1} \oplus \cdots \oplus k^{S_n}$ be a realization of a polymatroid. We have

$$\text{crem}^* \mathcal{S}_L \cong \mathcal{Q}_{L^\perp}^\vee.$$

Indeed, if $|S_1| + \cdots + |S_n| = N$, then $\text{crem}^* \mathcal{O}^{\oplus N}$ is the trivial bundle linearized so that $t = (t_1, \dots, t_n)$ acts by scaling the coordinates labeled by each S_i by t_i . The bundle $\text{crem}^* \mathcal{S}_L$ is the unique subbundle whose fiber over the identity is L ; $\mathcal{Q}_{L^\perp}^\vee$ can be described in the same way. Corollary 9.20 and Theorem 9.21 then imply the following result.

Corollary 9.32. *For any $a \geq 0$, $H^i(X_n, \text{Sym}^a \mathcal{Q}_L) = 0$ for all $i > 0$, and the map $H^0(X_n, \text{Sym}^a \mathcal{O}^{\oplus N}) \rightarrow H^0(X_n, \text{Sym}^a \mathcal{Q}_L)$ is surjective.*

The analogues of the results in this section for non-realizable matroids have yet to be developed.

9.3. Omega positivity via free resolutions. Finally, we discuss the technique used in [BF24] to prove Theorem 5.38. The statements and proofs in this section are somewhat different from those in [BF24], but all of the ideas are due to Berget and Fink. Let L_1 and L_2 be subspaces of k^n , and consider the vector bundle $\mathcal{S}_{L_1} \oplus \mathcal{S}_{L_2}$ over X_n . Let $p: \text{Tot}(\mathcal{S}_{L_1} \oplus \mathcal{S}_{L_2}) \rightarrow \mathbb{A}^{2n}$ be the map induced by the inclusion $\mathcal{S}_{L_1} \oplus \mathcal{S}_{L_2} \subseteq \mathcal{O}^{\oplus 2n}$.

Set $\hat{Y}_{L_1, L_2} := p(\text{Tot}(\mathcal{S}_{L_1} \oplus \mathcal{S}_{L_2}))$. Because p is proper, this is a closed subvariety of \mathbb{A}^{2n} . It can be described as the closure of the set $\{(t^{-1}v, t^{-1}w) : v \in L_1, w \in L_2, t \in T\}$. We will use coordinates $x_1, \dots, x_n, y_1, \dots, y_n$ on \mathbb{A}^{2n} , with the x coordinates corresponding to the $\mathcal{O}^{\oplus n}$ summand which contains \mathcal{S}_{L_1} . If the matroids realized by L_1 and L_2 have no loops in common, then let Y_{L_1, L_2} denote the quotient of \hat{Y}_{L_1, L_2} by the T -action, i.e., the multiprojectivization. This is a subvariety of $(\mathbb{P}^1)^n$. If L_1 and L_2 do have a loop in common, then \hat{Y}_{L_1, L_2} is contained in $V(x_i, y_i)$ for some i , and so it does not define a subscheme of $(\mathbb{P}^1)^n$. Note that Y_{L_1, L_2} is the closure of the semi-inverted Hadamard product in $(\mathbb{P}^1)^n$, and that \hat{Y}_{L_1, L_2} is the spectrum of the homogeneous coordinate ring of Y_{L_1, L_2} .

We will use an action of $\mathbb{G}_m^2 \times T$ on \mathbb{A}^{2n} , where $T = \mathbb{G}_m^n$. Here the \mathbb{G}_m^2 factor acts by $(s_1, s_2) \cdot (x_1, \dots, y_n) = (s_1 x_1, \dots, s_1 x_n, s_2 y_1, \dots, s_2 y_n)$, and the action of the T factor is given by $(t_1, \dots, t_n) \cdot (x_1, \dots, y_n) = (t_1 x_1, \dots, t_n x_n, t_1 y_1, \dots, t_n y_n)$. The subvariety \hat{Y}_{L_1, L_2} is preserved by this action, and so this induces a $\mathbb{Z}^2 \times \mathbb{Z}^n$ grading on the ideal I_{L_1, L_2} of \hat{Y}_{L_1, L_2} in the coordinate ring $A = k[x_1, \dots, y_n]$ of \mathbb{A}^{2n} . Note that the quotient of \hat{Y}_{L_1, L_2} by the T -action is the closure of the semi-inverted Hadamard product (see Example 8.13) of L_1 and L_2 in $(\mathbb{P}^1)^n$.

For $(m, p) \in \mathbb{Z}^2 \times \mathbb{Z}^n$, let $A[-m, -p]$ denote the rank 1 free A -module with generator in degree (m, p) . We will be interested in the minimal $\mathbb{Z}^2 \times \mathbb{Z}^n$ -graded free resolution of $A/I_{L_1, L_2}$. This is an exact sequence

$$(27) \quad 0 \rightarrow \bigoplus_{(m, q)} A[-m, -q]^{\beta_{N, m, q}} \rightarrow \dots \rightarrow \bigoplus_{(m, q)} A[-m, -q]^{\beta_{1, m, q}} \rightarrow \bigoplus_{(m, q)} A[-m, -q]^{\beta_{0, m, q}} \rightarrow A/I_{L_1, L_2} \rightarrow 0,$$

where the differentials respect the $\mathbb{Z}^2 \times \mathbb{Z}^n$ -grading. That the complex is *minimal* means that after we tensor this complex over A with k , the differentials are all 0. The $\{\beta_{i, m, q}\}$ are called the multigraded *Betti numbers* of $A/I_{L_1, L_2}$; the indexing is set up so that $\beta_{i, m, q} = 0$ if m or q has a negative coordinate. We will also consider the \mathbb{Z}^2 - and \mathbb{Z}^n -graded Betti numbers of $A/I_{L_1, L_2}$. For $m \in \mathbb{Z}^2$, we will use $\beta_{i, m, *}$ to denote the i th \mathbb{Z}^2 -graded Betti number, and similarly use $\beta_{i, *, q}$ to denote the i th \mathbb{Z}^n -graded Betti number. We have

$$(28) \quad \beta_{i, m, *} = \sum_{q \in \mathbb{Z}^n} \beta_{i, m, q} \text{ and } \beta_{i, *, q} = \sum_{m \in \mathbb{Z}^2} \beta_{i, m, q}.$$

Using the $\mathbb{Z}^2 \times \mathbb{Z}^n$ -graded Betti numbers, we can describe the class of $\mathcal{O}_{\hat{Y}_{L_1, L_2}}$ in $K_{\mathbb{G}_m^2 \times T}(\mathbb{A}^{2n})$. From (27), we have

$$[\mathcal{O}_{\hat{Y}_{L_1, L_2}}] = \sum_{(m, q) \in \mathbb{Z}^2 \times \mathbb{Z}^n} \sum_{i \geq 0} (-1)^i \beta_{i, m, q} [A[-m, -q]].$$

This also determines the class of $\mathcal{O}_{\hat{Y}_{L_1, L_2}}$ in $K_{\mathbb{G}_m^2}(\mathbb{A}^{2n})$ and $K_T(\mathbb{A}^{2n})$. Note that the class of $\mathcal{O}_{\hat{Y}_{L_1, L_2}}$ in $K_T(\mathbb{A}^{2n})$ is closely related to the class of $\mathcal{O}_{Y_{L_1, L_2}}$ in $K((\mathbb{P}^1)^n)$. Indeed, let $Z = V(x_1, y_1) \cup \dots \cup V(x_n, y_n) \subseteq \mathbb{A}^{2n}$, and let $U = \mathbb{A}^{2n} \setminus Z$. Then T acts freely on U with quotient $(\mathbb{P}^1)^n$, and one can identify $K_T(U)$ with $K((\mathbb{P}^1)^n)$. In particular, the class of $\mathcal{O}_{Y_{L_1, L_2}}$ is identified with the pullback of $[\mathcal{O}_{\hat{Y}_{L_1, L_2}}]$ to $K_T(U)$.

In [BF24, Lemma 5.6], Berget and Fink give an explicit universal Gröbner basis for I_{L_1, L_2} . In principle, it is possible to use generators and relations to study the Betti numbers of $A/I_{L_1, L_2}$ explicitly (see Example 9.49), but we will need more powerful techniques. The following result, which is a variant of Weyman's *geometric method* for studying free resolutions [Wey03], will allow us to relate the free resolution to the cohomology of tautological bundles.

Proposition 9.33. [BF24, Theorem 7.17] *Let L_1 and L_2 be subspaces of k^n which do not have a loop in common. For each $i \in \mathbb{Z}$, set*

$$F_i = \bigoplus_{(a,b) \in \mathbb{N}^2} H^{a+b-i}(X_n, \wedge^a \mathcal{Q}_{L_1}^\vee \otimes \wedge^b \mathcal{Q}_{L_2}^\vee) \otimes_k A[(-a, -b), 0],$$

viewed as a $\mathbb{Z}^2 \times \mathbb{Z}^n$ graded free module using the \mathbb{Z}^n grading on the cohomology of $\wedge^a \mathcal{Q}_{L_1}^\vee \otimes \wedge^b \mathcal{Q}_{L_2}^\vee$. Then, for $i \geq 1$, there are differentials $d_i: F_i \rightarrow F_{i-1}$ such that this is the minimal graded free resolution of $A/I_{L_1, L_2}$.

Proof. Note that $\text{Tot}(\mathcal{S}_{L_1} \oplus \mathcal{S}_{L_2})$ is a subscheme of $X_n \times \mathbb{A}^{2n}$. Let π denote the projection of $X_n \times \mathbb{A}^{2n}$ to X_n , and r denote the rank of $\mathcal{S}_{L_1} \oplus \mathcal{S}_{L_2}$. There is a distinguished section of $\pi^*(\mathcal{Q}_{L_1} \oplus \mathcal{Q}_{L_2})$ which transversely cuts out $\text{Tot}(\mathcal{S}_{L_1} \oplus \mathcal{S}_{L_2})$ inside of $X_n \times \mathbb{A}^{2n}$. This gives a Koszul resolution, i.e., the complex

$$\cdots \rightarrow \wedge^2 (\pi^* \mathcal{Q}_{L_1}^\vee \oplus \mathcal{Q}_{L_2}^\vee) \rightarrow \pi^* \mathcal{Q}_{L_1}^\vee \oplus \mathcal{Q}_{L_2}^\vee \rightarrow \mathcal{O} \rightarrow 0.$$

is equal to $\mathcal{O}_{\text{Tot}(\mathcal{S}_{L_1} \oplus \mathcal{S}_{L_2})}$ in the derived category of coherent sheaves on $X_n \times \mathbb{A}^{2n}$. If we push forward $\mathcal{O}_{\text{Tot}(\mathcal{S}_{L_1} \oplus \mathcal{S}_{L_2})}$ to \mathbb{A}^{2n} , then, recalling that X_n is a wonderful variety, Corollary 9.18 implies that we get $\mathcal{O}_{\hat{Y}_{L_1, L_2}}$. If we push forward the above Koszul complex, then a homological algebra argument shows that we get the complex in the statement of the proposition and that it is minimal. See [Wey03, Theorem 5.1.3]. \square

In particular, if $i < 0$, then $H^{a+b-i}(X_n, \wedge^a \mathcal{Q}_{L_1}^\vee \otimes \wedge^b \mathcal{Q}_{L_2}^\vee) = 0$, because the minimal graded free resolution is supported in non-positive cohomological degree. Also, as $F_0 = A$, we see that $H^{a+b}(X_n, \wedge^a \mathcal{Q}_{L_1}^\vee \otimes \wedge^b \mathcal{Q}_{L_2}^\vee) = 0$ unless $a = b = 0$.

We will analyze the $\mathbb{Z}^2 \times \mathbb{Z}^n$ -graded Betti numbers of $A/I_{L_1, L_2}$ by degenerating \hat{Y}_{L_1, L_2} . The key fact is that multigraded Betti numbers are upper semicontinuous under degenerations which preserve the multigrading. That is, let C be a curve, and let X be a subvariety of $\mathbb{A}^{2n} \times C$ which is preserved by the action of $\mathbb{G}_m^2 \times T$ on \mathbb{A}^{2n} and which is flat over C . Then for each i, m , and q , the function which assigns a point of C to the associated $\mathbb{Z}^2 \times \mathbb{Z}^n$ -graded Betti number of the fiber is upper semicontinuous. This means that the graded Betti numbers of \hat{Y}_{L_1, L_2} are bounded above by the graded Betti numbers of any degeneration.

However, the equivariant K -class of $\mathcal{O}_{\hat{Y}_{L_1, L_2}}$ is preserved under degenerations which respect the grading. The modules appearing in the minimal graded free resolution can get bigger when we degenerate, but the new terms must “cancel in consecutive pairs”; see [MS05, Remark 8.30].

For a matroid M on $[n]$, let $T_M(x, y)$ denote the Tutte polynomial of M . The *independence complex* of M is the simplicial complex on $[n]$ whose facets are given by the bases of M .

Proposition 9.34. [Sta77, Theorem 9] *Let M be a matroid on $[n]$, and let $\beta_{i,q}$ denote the \mathbb{Z}^n -graded Betti numbers of the Stanley–Reisner ring of its independence complex. We have $\beta_{i,q} = 0$ unless q is the indicator function of the complement of a flat of M^\perp . If q is the indicator function of the complement of a flat F of M^\perp , then*

$$\beta_{i,q} = \begin{cases} T_{M^\perp/F}(0, 1), & \text{if } i = \text{rk}(M^\perp/F) \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 9.35. *Let M_1 and M_2 be the matroids of L_1 and L_2 , respectively, and assume that they have no loops in common. Then the \mathbb{Z}^n -graded Betti numbers of $A/I_{L_1, L_2}$ are the same as the \mathbb{Z}^n -graded Betti numbers of the Stanley–Reisner ring of the independence complex of $D(M_1, M_2)$.*

Proof. By Example 8.13 and Proposition 9.24, Y_{L_1, L_2} is a kindred subscheme of $(\mathbb{P}^1)^n$ with progenitor matroid $D(M_1, M_2)$. By Lemma 8.2, Y_{L_1, L_2} can be degenerated inside of $(\mathbb{P}^1)^n$ to $Y_{D(M_1, M_2)}$. Proposition 8.15 then gives a flat degeneration of \hat{Y}_{L_1, L_2} to the product of the Stanley–Reisner scheme of the independence complex of $D(M_1, M_2)$ with \mathbb{A}^n , which has the same \mathbb{Z}^n -graded Betti numbers as the Stanley–Reisner scheme. This

flat degeneration respects the \mathbb{Z}^n -grading (but not the \mathbb{Z}^2 -grading). Setting $\beta_{i,q}$ to be the \mathbb{Z}^n -graded Betti numbers of the Stanley–Reisner ring, we have

$$(29) \quad \beta_{i,*,q} \leq \beta_{i,q}.$$

Note that, in $K_T(\mathbb{A}^{2n})$, $[\mathcal{O}_{\hat{Y}_{L_1, L_2}}]$ is equal to the class of the Stanley–Reisner scheme. Inspecting the formula in Proposition 9.34 gives the result, as there is no cancellation in the minimal \mathbb{Z}^n -graded free resolution, and so the Betti numbers can't have jumped. \square

Proposition 9.35 gives good control over the \mathbb{Z}^n -graded Betti numbers of $A/I_{L_1, L_2}$. However, Proposition 9.33 shows that the \mathbb{Z}^2 or $\mathbb{Z}^2 \times \mathbb{Z}^n$ -graded Betti numbers of $A/I_{L_1, L_2}$ are really what is of interest. To analyze those, we will need a different degeneration. We will make use of the \mathbb{G}_m^{2n} -action on \mathbb{A}^{2n} by scaling.

Lemma 9.36. *Assume that L_1 and L_2 have no loops in common. There is a degeneration of \hat{Y}_{L_1, L_2} to a Stanley–Reisner subscheme of \mathbb{A}^{2n} which respects the $\mathbb{Z}^2 \times \mathbb{Z}^n$ -grading. The $\mathbb{Z}^2 \times \mathbb{Z}^n$ graded Betti numbers of $A/I_{L_1, L_2}$ are the same as those of this Stanley–Reisner ring.*

Proof. As in the proof of Theorem 9.3, there is a degeneration of Y_{L_1, L_2} inside of $(\mathbb{P}^1)^n$ to a subscheme X which is fixed by the action of T ; the point is that the torus-orbit closure of the point corresponding to Y_{L_1, L_2} in the Hilbert scheme of $(\mathbb{P}^1)^n$ contains a T -fixed point. By Example 3.7, X is kindred, and so is reduced by Corollary 8.4. The homogeneous coordinate ring of X is fixed by the torus \mathbb{G}_m^{2n} and is reduced, so it is a Stanley–Reisner subscheme of \mathbb{A}^{2n} . Proposition 8.15 gives a degeneration of \hat{Y}_{L_1, L_2} to the spectrum of the homogeneous coordinate ring of X . This degeneration respects the $\mathbb{Z}^2 \times \mathbb{Z}^n$ -grading: the \mathbb{Z}^2 -grading is induced by the diagonal action of \mathbb{G}_m on $(\mathbb{P}^1)^n$, which commutes with the action of T . This Stanley–Reisner scheme can be further degenerated to $Y_{D(M_1, M_2)}$, so an argument similar to the proof of Proposition 9.35 shows that the \mathbb{Z}^n -graded Betti numbers do not jump under the degeneration of \hat{Y}_{L_1, L_2} to this Stanley–Reisner scheme. Then (28) implies that the $\mathbb{Z}^2 \times \mathbb{Z}^n$ -graded Betti numbers also do not jump. \square

Theorem 9.37. *Let L_1 and L_2 be subspaces of k^n which do not have a loop in common. Then $H^j(X_n, \wedge^i \mathcal{Q}_{L_1}^\vee \otimes \wedge^{n-i} \mathcal{Q}_{L_2}^\vee) = 0$ unless $j = \text{rk}(D(M_1, M_2))$.*

Proof. For $p = (p_1, \dots, p_{2n}) \in \mathbb{Z}^{2n}$, let $\beta_{i,p}$ denote the i th \mathbb{Z}^{2n} -graded Betti number of the Stanley–Reisner ring constructed in Lemma 9.36. Recall that for $m = (a, b) \in \mathbb{Z}^2$ and $q = (q_1, \dots, q_n) \in \mathbb{Z}^n$, $\beta_{i,m,q}$ is the i th $\mathbb{Z}^2 \times \mathbb{Z}^n$ -graded Betti number of $A/I_{L_1, L_2}$. We have

$$\beta_{i,m,q} = \sum_{\substack{p_1 + \dots + p_n = a \\ p_{n+1} + \dots + p_{2n} = b \\ p_j + p_{j+n} = q_j}} \beta_{i,p}.$$

By Proposition 9.34, if $p_1 + \dots + p_{2n} = n$, then $\beta_{i,p} = 0$ unless $i = n - \text{rk}(D(M_1, M_2))$. This means that if $m = (a, b)$ with $a + b = n$, then $\beta_{i,m,q} = 0$ unless $i = n - \text{rk}(D(M_1, M_2))$. Comparing with Proposition 9.33 gives the result. \square

In order to prove stronger results, we will need to analyze the degenerations in Lemma 9.36. In [BF22], the authors define and extensively study a simplicial complex called the *external activity complex of a pair of matroids*. This complex generalizes one introduced in [AB16]. The Stanley–Reisner scheme of this complex is the degeneration constructed in Lemma 9.36.

A *circuit* of a matroid is a minimal subset of the ground set which is not independent. Assume that M_1 and M_2 have no loop in common. In [BF24, Proposition 4.5], Berget and Fink show that if C is a circuit of $D(M_1, M_2)$, then C can be written as $I_1 \sqcup I_2$, where I_1 is independent in M_1 and I_2 is independent in M_2 . There are usually several such decompositions.

Given $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$, we say that w is *sufficiently generic* if for every circuit C of $D(M_1, M_2)$, there is a unique decomposition $C = I_1 \sqcup I_2$, with I_1 independent in M_1 and I_2 independent in M_2 , with the property that $\sum_{i \in I_1} w_i > \sum_{i \in I'_1} w_i$ for every other decomposition $C = I'_1 \sqcup I'_2$. For each basis B of $D(M_1, M_2)$ and $j \notin B$, there is a unique circuit C contained in $B \cup \{j\}$. Let $C = I_1 \sqcup I_2$ be the decomposition of C into independent sets of M_1 and M_2 with $\sum_{i \in I_1} w_i$ maximal. We say that j is *externally active* if $j \in I_1$, and we say that j is *externally passive* if $j \in I_2$. Let $EA(B)$ denote the set of externally active elements, i.e., $EA(B)$ consists of the $j \in E \setminus B$ such that j lies in I_1 in the decomposition of the unique circuit contained in $B \cup \{j\}$. Let $EP(B)$ denote the set of externally passive elements.

Example 9.38. Let $M_1 = U_{1,n}$, and let M_2 be a matroid. Then $D(M_1, M_2)$ is equal to M_2 , and the notion of external activity and passivity introduced above agrees with the classically studied notion; see, e.g., [Bjö92, Section 7.3].

Given a sufficiently generic weight w , let $\Delta_w(M_1, M_2)$ be the complex on $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ whose facets are $\{x_{B \cup EP(B)} y_{B \cup EA(B)} : B \text{ basis}\}$. This complex, which was introduced by Berget and Fink [BF24], is called the *external activity complex* of $D(M_1, M_2)$. Note that it is only defined if M_1 and M_2 have no loop in common, as otherwise the circuit of $D(M_1, M_2)$ corresponding to the loop does not admit a decomposition as a union of independent sets from M_1 and M_2 .

Given $w = (w_1, \dots, w_n) \in \mathbb{Z}^n$, there is an action of \mathbb{G}_m on \mathbb{A}^{2n} given by $t \cdot (x_1, \dots, x_n, y_1, \dots, y_n) = (t^{w_1} x_1, \dots, t^{w_n} x_n, y_1, \dots, y_n)$. The initial degeneration of \hat{Y}_{L_1, L_2} corresponding to w is the flat limit $\lim_{t \rightarrow 0} t \cdot \hat{Y}_{L_1, L_2}$. Because the action of \mathbb{G}_m commutes with the action of $\mathbb{G}_m^2 \times T$ on \mathbb{A}^{2n} , the class in $K_{\mathbb{G}_m^2 \times T}(\mathbb{A}^{2n})$ of the structure sheaf of the initial degeneration is equal to $[\mathcal{O}_{\hat{Y}_{L_1, L_2}}]$.

Proposition 9.39. [BF24, Corollary 5.7] *Let L_1 and L_2 be subspaces of k^n which realize matroids M_1 and M_2 . Assume that M_1 and M_2 do not have a loop in common. If w is sufficiently generic, then the initial degeneration of \hat{Y}_{L_1, L_2} given by w is the Stanley–Reisner scheme of $\Delta_w(M_1, M_2)$.*

Proposition 9.39 is proved using an explicit Gröbner basis for I_{L_1, L_2} , which is given in [Li18, Theorem 1.1]. This Gröbner basis can also be deduced using [HL24, Theorem 2.7]. Combined with Lemma 9.36 and its proof, Proposition 9.39 implies the following result.

Corollary 9.40. *The $\mathbb{Z}^2 \times \mathbb{Z}^n$ -graded Betti numbers of \hat{Y}_{L_1, L_2} are equal to the $\mathbb{Z}^2 \times \mathbb{Z}^n$ -graded Betti numbers of the Stanley–Reisner scheme of $\Delta_w(M_1, M_2)$.*

Combined with Proposition 9.33, this implies the following remarkable result.

Corollary 9.41. *Let L_1 and L_2 be subspaces of k^n realizing matroids M_1 and M_2 , respectively. Then for any a, b , and j , the character of the T -representation $H^j(X_n, \wedge^a \mathcal{Q}_{L_1}^\vee \otimes \wedge^b \mathcal{Q}_{L_2}^\vee)$ depends only on M_1 and M_2 .*

Assume that M_1 and M_2 do not have a loop in common. Using Proposition 9.34, as in the proof of Theorem 9.37, one can show that the T -equivariant Euler characteristic of $\wedge^a \mathcal{Q}_{L_1}^\vee \otimes \wedge^b \mathcal{Q}_{L_2}^\vee$ is “cancellation-free,” in the following sense. Each monomial with non-zero coefficient in $\chi^T(X_n, \wedge^a \mathcal{Q}_{L_1}^\vee \otimes \wedge^b \mathcal{Q}_{L_2}^\vee)$, viewed as an element of $\mathbb{Z}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$, is of the form $T^S := \prod_{i \in S} T_i$ for some subset S of $[n]$. Then $[n] \setminus S$ is a flat of $D(M_1, M_2)^\perp$, and the dimension of the part of $H^i(X_n, \wedge^a \mathcal{Q}_{L_1}^\vee \otimes \wedge^b \mathcal{Q}_{L_2}^\vee)$ where T acts by the character T^S is 0 unless $a + b - i = \text{rk}(D(M_1, M_2)^\perp / ([n] \setminus S))$.

Example 9.42. Let L_1 and L_2 be the row spans of the matrices

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

respectively. Using Theorem 4.15, one can compute that

$$\chi^T(X_6, \wedge^3 \mathcal{Q}_{L_1}^\vee \otimes \wedge^2 \mathcal{Q}_{L_2}^\vee) = T_1 T_2 T_3 T_5 T_6 + T_1 T_3 T_4 T_5 T_6 + T_2 T_3 T_4 T_5 T_6.$$

Then $D(M_1, M_2)^\perp$ is matroid of rank 2 whose non-bases are given by $\{3, 5\}$, $\{3, 6\}$, and $\{5, 6\}$. The $\{1\}$, $\{2\}$, and $\{4\}$ are all flats of $D(M_1, M_2)^\perp$, and the rank of the contraction of $D(M_1, M_2)^\perp$ by any of these flats is 1. This implies that $\dim H^4(X_6, \wedge^3 \mathcal{Q}_{L_1}^\vee \otimes \wedge^2 \mathcal{Q}_{L_2}^\vee) = 3$, and all of the other cohomology groups vanish.

In a technical tour de force, Berget and Fink extend these results to non-realizable matroids. If M_1 and M_2 do not have a loop in common, then the Stanley–Reisner scheme of $\Delta_w(M_1, M_2)$ is not contained in $V(x_i, y_i)$ for any i , so it defines a subscheme of $(\mathbb{P}^1)^n$. For the rest of this section, we will fix a weight $w \in \mathbb{Z}^n$ which is sufficiently generic.

Theorem 9.43. [BF24, Lemma 7.1] *If M_1 and M_2 are matroids on $[n]$ which do not have a loop in common, then for any w which is sufficiently generic, the subscheme of $(\mathbb{P}^1)^n$ defined by $\Delta_w(M_1, M_2)$ is kindred.*

In the realizable case, this is a consequence of Example 8.13 and Proposition 9.39. The proof of Proposition 9.35 then shows that the \mathbb{Z}^n -graded Betti numbers of the ideal of $\Delta_w(M_1, M_2)$ are computed in terms of $D(M_1, M_2)$ by using the formulas in Proposition 9.34. This is useful because of the next theorem.

Theorem 9.44. [BF24, Theorem 7.4] *Let M_1 and M_2 be matroids which do not have a loop in common. The class in $K_{\mathbb{G}_m^2 \times T}(\mathbb{A}^{2n})$ of the structure sheaf of the Stanley–Reisner scheme $\Delta_w(M_1, M_2)$ is equal to*

$$\sum_{(a,b) \in \mathbb{N}^2} (-1)^{a+b} \chi(X_n, \wedge^a [\mathcal{Q}_{M_1}^\vee] \otimes \wedge^b [\mathcal{Q}_{M_2}^\vee]) [A[(-a, -b), 0]].$$

In the realizable case, this follows from Proposition 9.33. Both Theorem 9.43 and Theorem 9.44 are proved by reducing to the realizable case by showing that the external activity complex is “bivaluative” in an appropriate sense, that is, various numerical invariants associated to it define maps from $\text{Val}_n \otimes \text{Val}_n$. The proof of the bivaluativity result is quite complicated; see [BF24, Section 6].

Once these results are known, the proof of Theorem 9.37 can be imitated to give the following positivity result.

Corollary 9.45. *Let M_1 and M_2 be matroids on $[n]$ which do not have a loop in common, then for any i ,*

$$(-1)^{\text{rk}(D(M_1, M_2))} \chi(X_n, \wedge^i [\mathcal{Q}_{M_1}^\vee] \otimes \wedge^{n-i} [\mathcal{Q}_{M_2}^\vee]) \geq 0.$$

In order to prove Theorem 5.38, we will need a vanishing result. The proof is a generalization of the proof of [BF24, Lemma 7.6].

Proposition 9.46. *Let M_1 and M_2 be matroids on $[n]$ which do not have a loop in common, and suppose that there is a component of $D(M_1, M_2)$ which is not a union of components of M_1 . Then*

$$\chi(X_n, \wedge^{n-\text{rk}(M_1)} [\mathcal{Q}_{M_1}^\vee] \otimes \wedge^{\text{rk}(M_1)} [\mathcal{Q}_{M_2}^\vee]) = 0.$$

Proof. Suppose that $D(M_1, M_2)$ has connected components S_1, \dots, S_c , with $S_1 \sqcup \dots \sqcup S_c = [n]$. Let $d = \text{rk}(D(M_1, M_2))$, and let d_i be the rank of the restriction $M_1^{S_i}$ to S_i . Let $r = \text{rk}(M_1)$, and let $r_i = \text{rk}(M_1^{S_i})$.

By [BF24, Lemma 4.23], $\Delta_w(M_1, M_2)$ is the join of the complexes $\Delta_w(M_1^{S_i}, M_2^{S_i})$ for $i = 1, \dots, c$. The class of $[\mathcal{O}_{\Delta_w(M_1, M_2)}]$ in $K_{\mathbb{G}_m^2 \times T}(\mathbb{A}^{2n})$ is the product of the pullbacks of the classes of the structure sheaves of the $\Delta_w(M_1^{S_i}, M_2^{S_i})$. For matroids M, N on a set S , let $V_i(M, N)$ denote the representation of $\mathbb{G}_m^2 \times \mathbb{G}_m^S$ whose character is given by the i th Betti number of $\Delta_w(M, N)$. Then we have

$$V_{n-d}(M_1, M_2) = \bigotimes_{i=1}^c V_{|S_i|-d_i}(M_1^{S_i}, M_2^{S_i}).$$

By Theorem 9.44, $(-1)^d \chi(X_n, \wedge^{n-r}[\mathcal{Q}_{M_1}^\vee] \otimes \wedge^r[\mathcal{Q}_{M_2}^\vee])$ is the dimension of the $(n-r, r)$ th \mathbb{Z}^2 -graded piece of $V_{n-d}(M_1, M_2)$. Applying Theorem 9.44 to the restriction to S_i , we see that $(-1)^{d_i} \chi(X_{|S_i|}, \wedge^{|S_i|-r_i}[\mathcal{Q}_{M_1^{S_i}}^\vee] \otimes \wedge^{r_i}[\mathcal{Q}_{M_2^{S_i}}^\vee])$ is the dimension of the $(|S_i| - r_i, r_i)$ th \mathbb{Z}^2 -graded piece of $V_{|S_i|-d_i}(M_1^{S_i}, M_2^{S_i})$, and the (a, b) th \mathbb{Z}^2 -graded piece is 0 for $b < r_i$. As $r < \sum_{i=1}^c r_i$ because some S_i is not a union of components of M_1 , this implies that the $(n-r, r)$ th \mathbb{Z}^2 -graded piece of $V_{n-d}(M_1, M_2)$ is 0. \square

Proof of Theorem 5.38. By Proposition 5.25 and Corollary 5.22, it suffices to show that if M is a connected matroid of rank r on $n \geq 2$ elements, then $\omega(M) \geq 0$. By Proposition 7.3, we have

$$\omega(M) = -\chi(X_n, \wedge^r[\mathcal{Q}_M^\vee] \otimes \wedge^{n-r}[\mathcal{Q}_M^\vee]).$$

If $D(M, M)$ has the expected rank $2r-1$, then Corollary 9.45 states that $(-1)^{2r-1} \chi(X_n, \wedge^r[\mathcal{Q}_M^\vee] \otimes \wedge^{n-r}[\mathcal{Q}_M^\vee]) \geq 0$, implying the result. If $D(M, M)$ does not have expected rank, then [BF24, Lemma 4.13] states that $D(M, M)$ is disconnected. It follows from Proposition 9.46 that $\omega(M) = 0$. \square

In [FSS24, Section 12], Fink, Shaw, and Speyer study the omega invariants of connected matroids M of rank r on $[n]$ where $n \leq 2r + 2$. Note that if $n < 2r$, then $\omega(M) = 0$ by Corollary 5.22.

Example 9.47. [BF24, Example 7.10] Suppose that $n = 2r$ and M is connected. Suppose that $D(M, M)$ has expected rank and is connected; otherwise $\omega(M) = 0$ by Proposition 9.46. Then $\Delta_w(M, M)$ is a hypersurface in \mathbb{A}^{2n} , so the \mathbb{Z}^2 -graded free resolution of its Stanley–Reisner ring A/I is $0 \rightarrow A[(-a, -b)] \rightarrow A \rightarrow A/I$ for some (a, b) with $a + b = n$. This implies that $(-1)^{n-1} \chi(X_n, \wedge^a[\mathcal{Q}_M^\vee] \otimes \wedge^b[\mathcal{Q}_M^\vee]) = 1$. But $\wedge^i[\mathcal{Q}_M^\vee] = 0$ if $i > n - r = n/2$, so we see that $a = b = n/2$, and that $\omega(M) = 1$ in this case.

Example 9.48. [BF24, Example 7.11] Suppose that $n = 2r + 1$ and M is connected. Suppose that $D(M, M)$ has the expected rank $n - 2$ and is connected; otherwise $\omega(M) = 0$. Then the minimal resolution of the Stanley–Reisner ring of $\Delta_w(M, M)$ has length 2, and the second \mathbb{Z} -graded Betti number is $T_M(0, 1)$. Comparing with Theorem 9.44, we have $T_M(0, 1) = (-1)^{n-2} \chi(X_n, \wedge^r[\mathcal{Q}_M^\vee] \otimes \wedge^{r+1}[\mathcal{Q}_M^\vee]) + (-1)^{n-2} \chi(X_n, \wedge^{r+1}[\mathcal{Q}_M^\vee] \otimes \wedge^r[\mathcal{Q}_M^\vee])$. This implies that $\omega(M) = T_M(0, 1)/2$.

Example 9.49. Suppose that L_2 is the 1-dimensional subspace of k^n spanned by $(1, \dots, 1)$, so $U_{1,n}$ is the matroid of L_2 . Let M_1 be the matroid of L_1 . Then Y_{L_1, L_2} is the matroid Schubert variety of Example 8.11, and $\Delta_w(M_1, U_{1,n})$ is the external activity complex of [AB16]. In this case, Crowley and Proudfoot explicitly computed the $\mathbb{Z}^2 \times \mathbb{Z}^n$ -graded free resolution of \hat{Y}_{L_1, L_2} [CP25].

For a linear subspace L of k^n realizing a matroid M of rank r , let Z_L denote its *internal zonotopal algebra*. This is a quotient of $\text{Sym } L$ by a certain homogeneous ideal; see [CP25, Section 1.1] for a definition and discussion. Its Hilbert series is known to be $q^{n-r} T_M(0, q^{-1})$ [HR11]. In particular, $Z_L = 0$ if L has a coloop. We write $Z_L = Z_L^0 \oplus Z_L^1 \oplus \dots \oplus Z_L^{n-r-1}$ for the decomposition of Z_L into graded pieces. For a subset S of $[n]$, let L^S be the image of L under the coordinate projection $k^n \rightarrow k^S$. Note that $L^{[n] \setminus S}$ is coloopless if and only if S is a flat of M^\perp . If F_i is the i th term of the $\mathbb{Z}^2 \times \mathbb{Z}^n$ -graded free resolution of $A/I_{L_1, L_2}$, then [CP25, Theorem 1.15] gives

$$F_i \cong \bigoplus_{\text{rk}(M_1^\perp/F) = i} \bigoplus_{\ell=0}^{n-|F|-i-1} Z_{L_1^{[n] \setminus F}}^\ell \otimes A[(-n + |F| + i + \ell, -\ell - i), -e_{E \setminus F}],$$

where $e_{E \setminus F} \in \mathbb{Z}^n$ is the indicator function of $E \setminus F$, and F ranges over flats of M_1^\perp . Comparing with Proposition 9.33 gives that

$$H^j(X_n, \wedge^a \mathcal{Q}_{L_1}^\vee \otimes \wedge^b \mathcal{Q}_{L_2}^\vee) \cong \bigoplus_{\substack{\text{rk}(M_1^\perp/F) = a+b-j \\ a+b=n-|F|}} Z_{L_1^{[n] \setminus F}}^{j-a}[-e_{E \setminus F}].$$

For example, suppose that L_1 is coloopless and $n > 1$. Then $\wedge^{n-1}\mathcal{Q}_{L_2}^\vee = \mathcal{L}_{U_{n-1,n}}^{-1}$, which is the line bundle denoted $\mathcal{O}(-\beta)$ in Example 9.6. We have

$$H^j(X_n, \mathcal{Q}_{L_1}^\vee \otimes \mathcal{L}_{U_{n-1,n}}^{-1}) = \begin{cases} Z_{L_1}^{n-r-1}, & \text{if } j = r \\ 0, & \text{otherwise.} \end{cases}$$

If $n > 1$, then the coefficient of y in $T_{M_1}(x, y)$ is $\beta(M_1)$, the beta invariant of M_1 . In particular, we see that $\dim H^r(X_n, \mathcal{Q}_{L_1}^\vee \otimes \mathcal{L}_{U_{n-1,n}}^{-1})$ is equal to $\beta(M_1)$.

From this description, one can see that the cohomology of $\wedge^a \mathcal{Q}_{L_1}^\vee \otimes \wedge^b \mathcal{Q}_{L_2}^\vee$ need not be concentrated in one degree. However, the proof that the dimension of $H^i(X_n, \wedge^a \mathcal{Q}_{L_1}^\vee \otimes \wedge^b \mathcal{Q}_{L_2}^\vee)$ only depends on the matroids of L_1 and L_2 (Corollary 9.41) still proceeds by showing that the cohomology is concentrated in one degree if one considers the grading given by the T -action. In most other cases where the dimension of a cohomology group of a coherent sheaf on some variety associated to a realization of a matroid is shown to be determined by the matroid, the proof also proceeds by showing that the Euler characteristic is an invariant of the matroid and that the cohomology is concentrated in one degree. The only exception that the author is aware of is [CL26, Proposition 3.24].

It is tempting to use a strategy analogous to the one considered here to study the free resolution of semi-inverted Hadamard products. Let L_1 and L_2 be linear subspaces of k^n which realize matroids M_1 and M_2 , respectively, and assume that M_2 is loopless. Recall that the semi-inverted Hadamard product of L_1 and L_2 is the image of the total space of \mathcal{S}_{L_1} over W_{L_2} in \mathbb{A}^n . Let J_{L_1, L_2} denote the ideal of the semi-inverted Hadamard product. Let $F_i = \bigoplus_\ell H^{\ell-i}(W_{L_2}, \wedge^\ell \mathcal{Q}_{L_1}^\vee) \otimes k[x_1, \dots, x_n](-\ell)$. By [Wey03, Theorem 5.1.3], using Corollary 9.18, there is a minimal graded free resolution

$$0 \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow k[x_1, \dots, x_n]/J_{L_1, L_2} \rightarrow 0.$$

Using this, results on the minimal graded free resolution of J_{L_1, L_2} imply results on the cohomology of exterior powers of $\mathcal{Q}_{L_1}^\vee$ on W_{L_2} , and vice versa. For example, because $F_i = 0$ for $i < 0$, we have $H^j(W_{L_2}, \wedge^\ell \mathcal{Q}_{L_1}^\vee) = 0$ if $\ell > j$. Because F_0 has rank 1, this implies that $H^\ell(W_{L_2}, \wedge^\ell \mathcal{Q}_{L_1}^\vee) = 0$ for all $\ell > 0$. Because the semi-inverted Hadamard product is Cohen–Macaulay by Corollary 9.18, the Auslander–Buchsbaum formula implies that $F_i = 0$ for $i > n - \text{rk}(D(M_1, M_2))$, so $H^j(W_{L_2}, \wedge^\ell \mathcal{Q}_{L_1}^\vee) = 0$ if $\ell + \text{rk}(D(M_1, M_2)) - n > j$. However, the lack of an action of T makes it more difficult to control the graded Betti numbers.

One attempt to control the cohomology of $\wedge^\ell \mathcal{Q}_{L_1}^\vee$ on W_{L_2} by using Corollary 9.41. Let r be the dimension of L_2 . Tensoring the Koszul resolution of W_{L_2} (12) with $\wedge^\ell \mathcal{Q}_{L_1}^\vee$, one obtains a complex

$$0 \rightarrow \wedge^\ell \mathcal{Q}_{L_1}^\vee \otimes \wedge^{n-r} \mathcal{Q}_{L_2}^\vee \rightarrow \wedge^\ell \mathcal{Q}_{L_1}^\vee \otimes \wedge^{n-r-1} \mathcal{Q}_{L_2}^\vee \rightarrow \dots \rightarrow \wedge^\ell \mathcal{Q}_{L_1}^\vee \otimes \mathcal{Q}_{L_2}^\vee \rightarrow 0.$$

The homology of this complex is $\wedge^\ell \mathcal{Q}_{L_1}^\vee \otimes \mathcal{O}_{W_{L_2}}$ in degree 0. The associated hypercohomology spectral sequence is

$$(30) \quad E_1^{p,q} = H^q(X_n, \wedge^\ell \mathcal{Q}_{L_1}^\vee \otimes \wedge^{-p} \mathcal{Q}_{L_2}^\vee) \implies H^{p+q}(W_{L_2}, \wedge^\ell \mathcal{Q}_{L_1}^\vee).$$

Corollary 9.41 implies that each term on the E_1 page of this spectral sequence depends only on the matroids of L_1 and L_2 . Each term on the E_1 page has a \mathbb{Z}^n -grading, but the differentials need not respect this grading.

We illustrate this in the case of the Orlik–Terao algebra; see Example 9.6 and Example 9.49. Let L_1 be the 1-dimensional subspace of k^n spanned by $(1, \dots, 1)$. Then the semi-inverted Hadamard product is the closure of the image of L_2 under the coordinate-wise inversion map on the torus $\mathbb{G}_m^n \subset \mathbb{A}^n$, i.e., it is the cone over the reciprocal plane. Its coordinate ring is the Orlik–Terao algebra of L_2 .

Imitating the strategy in the proof of Theorem 9.37, one could attempt to understand the \mathbb{Z} -graded Betti numbers of the Orlik–Terao algebra by Gröbner degenerating it. It is known that Gröbner degenerating

the Orlik–Terao algebra gives the Stanley–Reisner ring of the *broken circuit complex* of the matroid of L_2 [PS06].

This bounds the graded Betti numbers of the Orlik–Terao algebra in terms of those of the broken circuit complex. However, this bound can be strict. Indeed, the broken circuit complex of a matroid depends on the choice of an ordering on the ground set (which corresponds to the choice of weight used in the Gröbner degeneration). In [Toh23, Example 3.2], Tohăneanu gives an example of a matroid where the graded Betti numbers of the broken circuit complex depend on the choice of ordering.

Example 9.50. We illustrate how the spectral sequence (30) works. Let L_2 be the row span of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

I.e., L_2 represents the matroid associated to the complete graph K_4 . Let $A = k[x_1, \dots, x_6]$, so the Orlik–Terao algebra is a quotient of A . Then the minimal graded free resolution of the ideal of the Orlik–Terao algebra of L_2 is

$$0 \rightarrow A[-5]^{\oplus 2} \rightarrow A[-4]^{\oplus 3} \oplus A[-3]^{\oplus 2} \rightarrow A[-2]^{\oplus 4} \rightarrow A \rightarrow 0.$$

Let L_1 be the span of the vector $(1, \dots, 1)$ in k^6 . From the free resolution, one deduces that the following table gives $\dim H^j(W_{L_2}, \wedge^\ell \mathcal{Q}_{L_1}^\vee)$.

$j \setminus \ell$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	0	4	2	0	0
2	0	0	0	0	3	2

On the other hand, the cohomology of $\wedge^\ell \mathcal{Q}_{L_1}^\vee$ on W_{L_2} can be computed using the spectral sequence (30), whose E_1 page has the $H^\bullet(X_6, \wedge^\ell \mathcal{Q}_{L_1}^\vee \otimes \wedge^p \mathcal{Q}_{L_2}^\vee)$ in its $-p$ th row. The E_1 page can be computed using Example 9.49. For example, one can check that $\dim H^3(X_6, \wedge^3 \mathcal{Q}_{L_1}^\vee \otimes \wedge^2 \mathcal{Q}_{L_2}^\vee) = 6$, and all the other cohomology vanishes. Clockwise from the top left, the spectral sequences computing $H^\bullet(W_{L_2}, \wedge^\ell \mathcal{Q}_{L_1}^\vee)$ for $\ell = 2, 3, 4, 5$ are depicted below.

0	0	0	0	5	0	0	0	0	5
0	0	0	0	4	0	0	0	0	4
0	0	0	0	3	k	k^6	k^3	0	3
0	0	k^4	0	2	0	0	0	0	2
0	0	0	0	1	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0
<hr/>	<hr/>	<hr/>	<hr/>	<hr/>	<hr/>	<hr/>	<hr/>	<hr/>	<hr/>
-3	-2	-1	0		-3	-2	-1	0	

$$\begin{array}{cccc|cccc|c}
0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & k^2 & 0 & 3 & 0 & k^3 & k^6 & 0 & 3 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-3 & -2 & -1 & 0 & & -3 & -2 & -1 & 0 &
\end{array}$$

We see that all differentials on the E_1 page have maximal rank, and that $E_2 = E_\infty$.

We now give an example showing that the graded Betti numbers of the Orlik–Terao algebra are *not* combinatorial, i.e., they are not determined by the matroid of L_2 . This shows that the cohomology of $\wedge^\ell \mathcal{Q}_{L_1}^\vee$ on W_{L_2} is not combinatorially determined, complementing the results of Section 7.3.

Example 9.51. Let $k = \mathbb{C}$ and let ζ_3 be a third root of unity. Let L_2 be the row span of the matrix

$$\begin{pmatrix}
1 & 1 & 0 & \zeta_3 & 1 & 0 & -1 & 0 & -\zeta_3 \\
-1 & -\zeta_3 & 1 & 0 & \zeta_3 & 1 & 0 & 1 & 0 \\
0 & 0 & -1 & 1 & 0 & -\zeta_3 & 1 & \zeta_3 & 1
\end{pmatrix}.$$

Then L_2 is known as the *dual Hesse configuration*, which is related to the three-torsion points of an elliptic curve. Let L_1 be the span of the vector $(1, \dots, 1)$ in k^9 . By computing the free resolution of the Orlik–Terao algebra of L_2 , one can check that the following table gives $\dim H^j(W_{L_2}, \wedge^\ell \mathcal{Q}_{L_1}^\vee)$.

$j \setminus \ell$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	0	12	16	3	0	0	0	0
2	0	0	0	0	33	96	100	48	9

Let M be the matroid realized by L_2 . On the other hand, let k be a field of characteristic 3, and let L'_2 be the linear span of the matrix

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 2 & 0 & 1 & 2 & 1 & 2 \\
0 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 0
\end{pmatrix}.$$

Then L'_2 is known as the ternary affine plane, and it is also a realization of M . Then the graded Betti numbers of the Orlik–Terao algebra of L'_2 are different from the graded Betti numbers of the Orlik–Terao algebra of L_2 . The following table gives $\dim H^j(W_{L'_2}, \wedge^\ell \mathcal{Q}_{L_1}^\vee)$.

$j \setminus \ell$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	0	12	16	9	0	0	0	0
2	0	0	0	0	39	96	100	48	9

We see that $\dim H^i(W_{L_2}, \wedge^4 \mathcal{Q}_{L_1}^\vee) \neq \dim H^i(W_{L'_2}, \wedge^4 \mathcal{Q}_{L_1}^\vee)$ for $i = 1, 2$, but that $\chi(W_{L_2}, \wedge^4 \mathcal{Q}_{L_1}^\vee) = \chi(W_{L'_2}, \wedge^4 \mathcal{Q}_{L_1}^\vee)$.

9.4. Comparing the approaches. We now compare the three approaches to the non-negativity of the omega invariant that are discussed above. We begin by comparing the second and third approaches, as the geometry is somewhat similar: both rely on the analysis of a map from the total space of a tautological subbundle to an affine space, and in particular on Corollary 9.18. However, they are genuinely different in that they prove different vanishing statements: see Corollary 9.25 and Theorem 9.37.

There is a possible relationship between special cases of the second and third approaches. We make the following conjecture, which would provide a geometric explanation for Proposition 7.4.

Conjecture 9.52. *Let L_1 and L_2 be linear subspaces of k^n , with L_1 coloopless and L_2 loopless. Then, for any $p, q \geq 0$ with $p \neq q$ and any i , we have $H^i(X_n, \wedge^p \mathcal{S}_{L_1} \otimes \wedge^q \mathcal{Q}_{L_2}^\vee) = 0$.*

Conjecture 9.52 was proved in [Liu25, Theorem B] when $p = 0$ or $q = 0$, and it is also an easy consequence of Proposition 9.33 in this case. Conjecture 9.52 is open even when k has characteristic 0. The main evidence for Conjecture 9.52 is the following strengthening of Proposition 7.4, whose proof uses an analysis of the Atiyah–Bott formula (Theorem 4.15).

Proposition 9.53. [DES26, Proposition A.1] *Let M_1 and M_2 be matroids on $[n]$, with M_1 coloopless and M_2 loopless. Then, for any $p, q \geq 0$ with $p \neq q$, $\chi(X_n, \wedge^p[\mathcal{S}_{M_1}] \otimes \wedge^q[\mathcal{Q}_{M_2}^\vee]) = 0$.*

Proposition 9.53 would be a consequence of Conjecture 9.52: for each p and q , the function which maps (M_1, M_2) to $\chi(X_n, \wedge^p[\mathcal{S}_{M_1}] \otimes \wedge^q[\mathcal{Q}_{M_2}^\vee])$ factors through $\text{Val}_n \otimes \text{Val}_n$ by the proof of [BEST23, Proposition 5.6], so one can use Theorem 4.10 and a version of Proposition 5.1 to reduce Proposition 9.53 to the case of realizable matroids.

If Conjecture 9.52 holds, then we would have the following cohomological consequence. Tensoring the Koszul resolution of the structure sheaf of W_{L_2} with $\wedge^i \mathcal{S}_{L_1}$, we have an exact sequence of sheaves on X_n

$$0 \rightarrow \wedge^i \mathcal{S}_{L_1} \otimes \wedge^{n-r} \mathcal{Q}_{L_2}^\vee \rightarrow \wedge^i \mathcal{S}_{L_1} \otimes \wedge^{n-r-1} \mathcal{Q}_{L_2}^\vee \rightarrow \cdots \rightarrow \wedge^i \mathcal{S}_{L_1} \otimes \mathcal{Q}_{L_2}^\vee \rightarrow \wedge^i \mathcal{S}_{L_1} \rightarrow \wedge^i \mathcal{S}_{L_1}|_{W_{L_2}} \rightarrow 0.$$

If Conjecture 9.52 holds, then for each j , breaking up the long exact sequence into short exact sequences produces an isomorphism

$$(31) \quad H^j(W_{L_2}, \wedge^i \mathcal{S}_{L_1}) \cong H^{i+j}(X_n, \wedge^i \mathcal{S}_{L_1} \otimes \wedge^i \mathcal{Q}_{L_2}^\vee).$$

Let M_1 be the matroid realized by L_1 , and let r_1 denote its rank. Then $\wedge^{r_1} \mathcal{S}_{L_1} \cong \mathcal{L}_{M_1^\perp}^{-1} \cong \wedge^{n-r_1} \mathcal{Q}_{L_1}^\vee$, and (31) gives a connection between the second and third approaches: if L_1 is coloopless and L_2 is loopless, then for each j , we have

$$H^j(W_{L_2}, \mathcal{L}_{M_1^\perp}^{-1}) \cong H^{j+r_1}(X_n, \wedge^{n-r_1} \mathcal{Q}_{L_1}^\vee \otimes \wedge^{r_1} \mathcal{Q}_{L_2}^\vee),$$

so Theorem 9.37 can be compared with Corollary 9.25. Using Proposition 9.53, we can unconditionally derive the consequence of this isomorphism for Euler characteristics.

Proposition 9.54. *Let M_1 be a coloopless matroid of rank r_1 on $[n]$, and let M_2 be a loopless matroid on $[n]$. Then*

$$\chi(M_2, \mathcal{L}_{M_1^\perp}^{-1}) = (-1)^{r_1} \chi(X_n, \wedge^{n-r_1}[\mathcal{Q}_{M_1}^\vee] \otimes \wedge^{r_1}[\mathcal{Q}_{M_2}^\vee]).$$

Proof. By (13) and the fact that $[\mathcal{L}_{M_1^\perp}^{-1}] = \wedge^{n-r_1}[\mathcal{Q}_{M_1}^\vee]$ in $K(X_n)$, we have

$$\chi(M_2, \mathcal{L}_{M_1^\perp}^{-1}) = \sum_{i \geq 0} (-1)^i \chi(X_n, \wedge^{n-r_1}[\mathcal{Q}_{M_1}^\vee] \otimes \wedge^i[\mathcal{Q}_{M_2}^\vee]).$$

As $\wedge^{n-r_1}[\mathcal{Q}_{M_1}^\vee] = \wedge^{r_1}[\mathcal{S}_{M_1}]$ in $K(X_n)$, Proposition 9.53 implies the result. \square

If M_1 has coloops, then let N_1 be the matroid obtained by replacing them with loops. Then $P(M_1^\perp)$ is a translation of $P(N_1^\perp)$, so $\mathcal{L}_{M_1^\perp} \cong \mathcal{L}_{N_1^\perp}$. Proposition 9.54 then gives

$$\chi(M_2, \mathcal{L}_{M_1^\perp}^{-1}) = (-1)^{\text{rk}(N_1)} \chi(X_n, \wedge^{n-\text{rk}(N_1)}[\mathcal{Q}_{N_1}^\vee] \otimes \wedge^{\text{rk}(N_1)}[\mathcal{Q}_{M_2}^\vee]).$$

One can check that $\text{rk}(D(M_1, M_2)) - \text{rk}(D(N_1, M_2))$ is equal to the number of coloops of M_1 . Combining Corollary 9.45 and Proposition 9.54 therefore gives the following result.

Corollary 9.55. *Let M_1 and M_2 be matroids on $[n]$, with M_2 loopless. Then*

$$(-1)^{\text{rk}(D(M_1, M_2)) - \text{rk}(M_1)} \chi(M_2, \mathcal{L}_{M_1^\perp}^{-1}) \geq 0.$$

Example 9.56. The conclusion of Proposition 9.54 may fail if M_1 has coloops. Let k be an algebraically closed field, and let

$$L_1 = \text{row span} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \text{ and } L_2 = \text{row span} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

and let M_1 and M_2 be the matroids on $[5]$ realized by L_1 and L_2 , respectively. Then $D(M_1, M_2)$ is the matroid of rank 4 whose unique circuit is $\{2, 3, 4, 5\}$. Using Corollary 9.25, one can show that

$$H^j(W_{L_2}, \mathcal{L}_{M_1^\perp}^{-1}) \cong \begin{cases} k, & \text{if } j = 2 \\ 0, & \text{otherwise,} \end{cases}$$

so $\chi(W_{L_2}, \mathcal{L}_{M_1^\perp}^{-1}) = 1$. On the other hand, note that $\wedge^2 \mathcal{Q}_{L_2}^\vee = 0$, so $\chi(X_5, \wedge^3 \mathcal{Q}_{L_1}^\vee \otimes \wedge^2 \mathcal{Q}_{L_2}^\vee) = 0$.

However, vanishing theorems like Proposition 9.23 do not appear to be accessible using Theorem 9.37, or even some hypothetical generalization of it to tautological bundles of polymatroids. Indeed, the conclusion of Proposition 9.53 fails badly for tautological bundles of polymatroids. For example, if L is a realization of a polymatroid P , then $(-1)^{\dim B(P)} \chi(X_n, \det \mathcal{S}_L)$ is the number of lattice points in the relative interior of $B(P)$; see Remark 6.11.

On the other hand, we do not know of even a conjectural connection between the first and the other two approaches. Indeed, this is exemplified by the fact that the first approach gives a different formula for the sign of Euler characteristics $\chi(M_2, \mathcal{L}_{M_1^\perp}^{-1})$ from the other two approaches; compare Theorem 9.12 and (22) with Corollary 9.55. This has the following amusing consequence. Let $d(M_2, M_1)$ denote the quantity in (22), and let $e = \text{rk}(D(M_1, M_2)) - \text{rk}(M_1)$. If $d(M_2, M_1) \not\equiv e \pmod{2}$, then $\chi(M_2, \mathcal{L}_{M_1^\perp}^{-1}) = 0$. We conjecture that $\chi(M_2, \mathcal{L}_{M_1^\perp}^{-1}) = 0$ if $d(M_2, M_1) \neq e$; this holds if M_1 and M_2 are realizable over the same field by Theorem 9.3 and Corollary 9.25. This conjecture would imply Conjecture 9.26.

The technique in the first approach is also quite different from the technique in the other two approaches. It is easier to extend the first approach to non-realizable matroids; compare [EFL25, Proof of Proposition 3.4] with [BF24, Section 6]. Additionally, the formula for the sign of the Euler characteristic that results from the first approach is somewhat simpler than the formula resulting from the second and third approaches, especially for line bundles \mathcal{L}_M where M is connected. However, the first approach gives no useful information about the vanishing of Euler characteristics; there is no analogue of Proposition 9.46 in the first approach.

10. CONNECTIONS WITH RIGIDITY THEORY

To conclude, we discuss a connection between the ideas discussed in these notes and graph rigidity in the plane. Let G be a graph on n vertices with no loops or multiple edges. Throughout this section, we assume that $n \geq 3$. We say that G is *generically rigid* in \mathbb{R}^2 if for an open dense subset of $(\mathbb{R}^2)^n$, any motion of the points which preserves the distances between any two points that are connected by an edge in G can be extended to an isometry of \mathbb{R}^2 .

Using some differential topology, one can show that this is equivalent to the following condition. Let $k = \mathbb{C}$, and define the *Cayley–Menger variety* CM_n to be the closure of the image of $(\mathbb{A}^2)^n$ in $\mathbb{A}^{\binom{n}{2}}$ under the map given by $z_{i,j} = (x_i - x_j)^2 + (y_i - y_j)^2$, where $x_1, y_1, x_2, y_2, \dots, x_n, y_n$ are the coordinates on $(\mathbb{A}^2)^n$ and $\{z_{i,j}\}_{i < j}$ are the coordinates on $\mathbb{A}^{\binom{n}{2}}$. Then a graph G with edge set E is generically rigid if the coordinate projection $\text{CM}_n \rightarrow \mathbb{A}^E$ is generically finite. The intuition is that if the distances labeled by E are fixed, then the rest of the distances are determined up to a finite number of choices. If G is embedded in a generic way, then any motion of the vertices of G which preserves the distances in E must preserve all of the distances, and so must come from an isometry. See [AR78].

From the above description, we see that whether a graph is rigid is determined by the algebraic matroid of the Cayley–Menger variety. As we will see, the Cayley–Menger variety is basically a Hadamard product of two linear spaces, which makes it easy to compute this matroid. This description does not generalize to graph rigidity in \mathbb{R}^n for any $n \geq 3$, and graph rigidity is poorly understood for $n \geq 3$.

Recall that if X and Y are subvarieties of \mathbb{A}^n , then the Hadamard product of X and Y is the closure of $\{(x_1 y_1, \dots, x_n y_n) : (x_1, \dots, x_n) \in X, (y_1, \dots, y_n) \in Y\}$ in \mathbb{A}^n . We denote this by $X * Y$. If we perform the change of variables $s_j = x_j + i y_j$ and $t_j = x_j - i y_j$, then

$$(32) \quad (s_i - s_j)(t_i - t_j) = (x_i - x_j)^2 + (y_i - y_j)^2.$$

Let L_n be the image of the map $\mathbb{C}^n \rightarrow \mathbb{C}^{\binom{n}{2}}$ given by $(t_1, \dots, t_n) \mapsto (t_i - t_j)_{i < j}$. Then L_n is a realization of the matroid of the complete graph $M(K_n)$, so $\dim L_n = n - 1$. From (32), we see that CM_n is equal to $L_n * L_n$. This description of CM_n , which dates back at least to [LY82], is extremely useful.

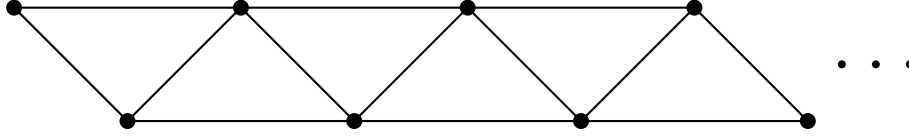
Given an integral subvariety X of \mathbb{A}^n which intersects the open torus, let X^{-1} be the closure in \mathbb{A}^n of the image of X under the inversion map on the torus. We previously analyzed the semi-inverted Hadamard product of two linear spaces, i.e., $L_2^{-1} * L_1$ for L_1, L_2 linear subspaces of \mathbb{A}^n .

Proposition 10.1. *Let X and Y be integral subvarieties of \mathbb{A}^n which intersect the open torus. Then the algebraic matroid of $X * Y$ is equal to the algebraic matroid of $X^{-1} * Y$.*

In particular, the dimension of $X * Y$ is equal to the dimension of $X^{-1} * Y$. To prove Proposition 10.1, we will use tools from tropical geometry. Given an integral subvariety X of \mathbb{A}^n which intersects the open torus in \mathbb{A}^n , its *tropicalization* $\text{trop}(X)$ is a subset of \mathbb{R}^n . This subset admits the structure of a rational fan of dimension equal to the dimension of X , and this fan is equipped with a function w from the set of maximal cones to the positive integers. See [MS15].

The tropicalization can be described in terms of Minkowski weights; see [Kat09, Proposition 9.4]. There is a fan structure on the tropicalization of X and a smooth projective toric variety Y such that the tropicalization of X is a subfan of the fan of Y . Furthermore, the fundamental class of the strict transform of X in Y defines a Minkowski weight [FS97], and this Minkowski weight is the tropicalization of X . We can obtain such a Y by repeatedly blowing up any toric compactification of \mathbb{A}^n . In other words, the tropicalization of X contains exactly the data of the homology class of the strict transform of X in toric varieties.

Lemma 10.2. [Yu17, Lemma 2] *Let X be a subvariety of \mathbb{A}^n which intersects the open torus. Then the coordinate projection $X \rightarrow \mathbb{A}^S$ is dominant if and only if there is a cone of $\text{trop}(X)$ whose image under the coordinate projection to \mathbb{R}^S is full-dimensional.*

FIGURE 4. A basis for $D(M(K_n), M(K_n))$

Proof. Let Y be a smooth projective toric variety which is obtained by repeatedly blowing up $(\mathbb{P}^1)^n$, and such that the fundamental class of the strict transform of X is its tropicalization. Then the projection $X \rightarrow \mathbb{A}^S$ is dominant if and only if the pushforward of the fundamental class of the strict transform of X to $(\mathbb{P}^1)^S$ is non-zero. That the pushforward is non-zero if and only if there is a cone of the tropicalization of X whose image under the coordinate projection to \mathbb{R}^S is full-dimensional follows from the description of the pushforward map in [FS97, (2.6)]. \square

Proof of Proposition 10.1. It follows from a different description of the tropicalization [MS15, Theorem 3.2.3(3)] that $\text{trop}(X * Y) = \text{trop}(X) + \text{trop}(Y)$, where the “+” denotes Minkowski sum. Similarly, $\text{trop}(X^{-1} * Y) = (-\text{trop}(X)) + \text{trop}(Y)$. Given cones C_1 of $\text{trop}(X)$ and C_2 of $\text{trop}(Y)$, the image of $C_1 + C_2$ in \mathbb{R}^S is full-dimensional if and only if the image of $(-C_1) + C_2$ in \mathbb{R}^S is full-dimensional, so the result follows from Lemma 10.2. \square

It has been long known in the graph rigidity literature that semi-inverted Hadamard products are simpler than Hadamard products, and that this can be helpful to understand 2-dimensional rigidity; see [Whi96, Section 4.1] and [Mar03]. We use this to give a description of graph rigidity in the plane.

For a graph G on n vertices, let $M(G)$ denote the corresponding graphic matroid. This is the matroid whose ground set is the set of edges of G , and whose bases are the spanning forests in G .

Theorem 10.3. [PG27, Lam70] *The algebraic matroid of CM_n is $D(M(K_n), M(K_n))$.*

Proof. As described in (32), CM_n is the Hadamard product $L_n * L_n$, where L_n is a realization of $M(K_n)$. By Proposition 10.1, its algebraic matroid is the same as the algebraic matroid of the semi-inverted Hadamard product $L_n^{-1} * L_n$, which is computed to be $D(M(K_n), M(K_n))$ in Proposition 9.24. \square

Remark 10.4. The proof of Theorem 10.3 shows that the algebraic matroid of a Hadamard product of two linear spaces with matroids M_1 and M_2 is $D(M_1, M_2)$. This result was previously proven in [Ber22, Theorem 3.1] using a different technique.

As a consequence, a graph G is generically rigid in the plane if and only if the edges of G contain a basis for $D(M(K_n), M(K_n))$. We claim that for $n \geq 3$, the rank of $D(M(K_n), M(K_n))$ is $2n - 3$, i.e., it has the expected rank. To see this, it suffices to exhibit a single independent set in $D(M(K_n), M(K_n))$ of size $2n - 3$. It is easy to check that the version of the graph G depicted in Figure 4 with n vertices has $2n - 3$ edges, and if e is any edge of K_n , then $G \cup e$ can be written as a disjoint union of two forests. This implies that G is a basis for $D(M(K_n), M(K_n))$, and so $D(M(K_n), M(K_n))$ has rank $2n - 3$.

A graph on n vertices with $2n - 3$ edges which corresponds to a basis of $D(M(K_n), M(K_n))$ is called *minimally rigid*, because it is rigid, but no spanning subgraph is rigid. Such graphs have been extensively studied; see [Whi96, Section 2.2]. For example, there is a recursive construction of any such graph using two operations called “Henneberg moves.” Using an alternative description of the diagonal Dilworth truncation [Edm70, Theorem 8], one can show that a graph on n vertices with $2n - 3$ edges is minimally rigid if and only if every induced subgraph with at least $m \geq 2$ vertices has at most $2m - 3$ edges.

Given a collection of n generic points in \mathbb{R}^2 and a graph on n vertices, one can ask how many configurations of n points in the plane there are such that for every edge (u, v) , the distance from u to v is the same as the distance from u to v in the given configuration. If G is generically rigid and one counts up to isometry, there will be finitely many. However, the number of such configurations will depend on the configuration; the answer is not constant on a dense subset of $(\mathbb{R}^2)^n$. See [JO19, Figures 1 and 2].

However, if one counts *complex* realizations, then this problem goes away. This can be described as computing the degree of the projection map from CM_n onto its image in \mathbb{C}^E , where E is the set of edges in the graph. Counting the number of complex realizations of all minimally rigid graphs on n vertices is equivalent to describing the fundamental class of the closure of CM_n in $(\mathbb{P}^1)^{\binom{n}{2}}$. The following conjecture has attracted a lot of attention; see, e.g., [CGG⁺18, GGS20, BEV22, CDT⁺25].

Conjecture 10.5. [JO19, Conjecture 10.3] *Let G be a minimally rigid graph on n vertices. Then G has at least 2^{n-3} complex realizations.*

For a rigid graph G on n vertices, let L_G be the image of L_n in k^E , where E is the set of edges of G . Because G is rigid, it is connected, and so $\dim L_G = n - 1$. Let $\mathbb{P}\text{CM}_n$ denote the projectivization of CM_n . We have a diagram

$$\begin{array}{ccc} \mathbb{P}L_n \times \mathbb{P}L_n & \dashrightarrow & \mathbb{P}\text{CM}_n \\ \downarrow & & \downarrow \\ \mathbb{P}L_G \times \mathbb{P}L_G & \dashrightarrow & \mathbb{P}^{|E|-1}. \end{array}$$

The number of complex realizations of G is the degree of the right vertical arrow onto its image, which is generically finite because G is rigid. Note that the left vertical map is an isomorphism. The top horizontal map has degree 2.

Counting complex realizations of a minimally rigid graph is a special case of the following problem: let L_1 and L_2 be linear subspaces of k^n realizing matroids M_1 and M_2 with ranks r_1 and r_2 . Suppose that $r_1 + r_2 = n + 1$, and that $D(M_1, M_2)$ has the expected rank, i.e., it is Boolean. What is the degree of the dominant generically finite map $\mathbb{P}L_1 \times \mathbb{P}L_2 \dashrightarrow \mathbb{P}^{n-1}$ induced by the Hadamard product? When $L_1 = L_2 = L_G$, this is twice the complex realization number of G .

Remark 10.6. In this framework, one might generalize Conjecture 10.5 to the statement that if L is a linear subspace of k^n of dimension r with $2r = n + 1$, then the degree of the Hadamard product map $\mathbb{P}L \times \mathbb{P}L \dashrightarrow \mathbb{P}^{n-1}$ is at least 2^{r-1} if the map is dominant. However, a counterexample is given in [CDG⁺25, Example 9.3], when L is a realization of the dual of the Fano matroid. The Hadamard product map is dominant with degree 6, which is less than 2^{4-1} .

If $r_1 + r_2 < n + 1$ and $D(M_1, M_2)$ has expected rank, then the map $\mathbb{P}L_1 \times \mathbb{P}L_2 \dashrightarrow \mathbb{P}^{n-1}$ is generically finite onto its image, and one can ask about the degree of this map. This generalizes the problem of counting the number of complex realizations of a rigid graph which is not minimally rigid: when $L_1 = L_2 = L_G$, it is twice the complex realization number of G . In general, this degree is not determined by the matroids of L_1 and L_2 .

Example 10.7. Let L be a subspace of k^n realizing a connected matroid M , with $n \geq 2$. Then $D(M, M^\perp)$ is $U_{n-1, n}$, and so it has the expected rank. By Proposition 5.26, the closure of the image of the Hadamard product map $\mathbb{P}L \times \mathbb{P}L^\perp \dashrightarrow \mathbb{P}^{n-1}$ is a hyperplane, and the map has degree $\beta(M)$ onto its image. In particular, if L is a realization of a uniform matroid $U_{r, n}$, then L^\perp is a realization of the uniform matroid $U_{n-r, n}$, and the degree of the Hadamard product map onto its image is $\beta(U_{r, n}) = \binom{n-2}{r-1}$.

Example 10.8. Let L_1 be a general realization of $U_{r,n}$ for some $0 < r < n$, and let L_2 be a general realization of $U_{n-r,n}$. In [BCK16, Theorem 6.8], the authors show that the closure of the image of the Hadamard product map $\mathbb{P}L_1 \times \mathbb{P}L_2 \dashrightarrow \mathbb{P}^{n-1}$ is a hypersurface of degree $\binom{n-2}{r-1}$, and the Hadamard product map is birational onto its image.

Note that in these examples, the product of the degree of the map with the degree of the image is $\binom{n-2}{r-1}$, independent of the choices of realization. We show that this is a general phenomenon.

Proposition 10.9. *Let L_1 and L_2 be linear subspaces of k^n which realize loopless matroids M_1 and M_2 of ranks r_1 and r_2 , respectively. Let Y be the closure of the image of the rational map $m: \mathbb{P}L_1 \times \mathbb{P}L_2 \dashrightarrow \mathbb{P}^{n-1}$. Let $\deg(Y)$ denote the degree of Y , viewed as a subvariety of \mathbb{P}^{n-1} , and let $\deg(m)$ denote the degree of m , which is defined to be 0 if m is not generically finite. Then*

$$\deg(Y) \cdot \deg(m) = \deg_{M_2}(c_{r_2-1}(\mathcal{S}_{M_1^\perp}^\vee)).$$

Proof. As discussed in Section 5.3, the projective bundle $\mathbb{P}_{W_{L_2}}(\mathcal{Q}_{L_1^\perp}^\vee)$ is a subvariety of $W_{L_2} \times \mathbb{P}^{n-1}$, it is birational to $\mathbb{P}L_1 \times \mathbb{P}L_2$, and the projection of $\mathbb{P}_{W_{L_2}}(\mathcal{Q}_{L_1^\perp}^\vee)$ to \mathbb{P}^{n-1} is identified with m . By [EH16, Proposition 9.13], we have

$$(33) \quad [\mathbb{P}_{W_{L_2}}(\mathcal{Q}_{L_1^\perp}^\vee)] = \sum_{i=0}^{r_2-1} s_i(\mathcal{Q}_{L_1^\perp}^\vee) \cdot c_1(\mathcal{O}(1))^{n-r_1-i} \text{ in } A^\bullet(W_{L_2} \times \mathbb{P}^{n-1}),$$

where $s_i(\mathcal{Q}_{L_1^\perp}^\vee)$ is the i th Segre class of $\mathcal{Q}_{L_1^\perp}^\vee$. The short exact sequence $0 \rightarrow \mathcal{Q}_{L_1^\perp}^\vee \rightarrow \mathcal{O}^{\oplus n} \rightarrow \mathcal{S}_{L_1^\perp}^\vee \rightarrow 0$ implies that $s_i(\mathcal{Q}_{L_1^\perp}^\vee) = c_i(\mathcal{S}_{L_1^\perp}^\vee)$, which is equal to $c_i(\mathcal{S}_{M_1^\perp}^\vee)$. When we pushforward $[\mathbb{P}_{W_{L_2}}(\mathcal{Q}_{L_1^\perp}^\vee)]$ to $A^\bullet(\mathbb{P}^{n-1})$, we get $\deg(m) \cdot [Y]$. On the other hand, using (33), we see that the pushforward of $[\mathbb{P}_{W_{L_2}}(\mathcal{Q}_{L_1^\perp}^\vee)]$ is $\deg_{M_2}(c_{r_2-1}(\mathcal{S}_{M_1^\perp}^\vee)) \cdot c_1(\mathcal{O}(1))^{n-r_1-r_2+1}$: the pushforward of a class $a \cdot c_1(\mathcal{O}(1))^j$, with $a \in A^\bullet(W_{L_2})$, is equal to $\deg_{M_2}(a) \cdot c_1(\mathcal{O}(1))^j$ in $A^\bullet(\mathbb{P}^{n-1})$. \square

When $r_1 + r_2 = n + 1$, this recovers [CDG⁺25, Theorem 1.3]. In this case, if the Hadamard product map is generically finite, then it is dominant, and so $\deg(Y) = 1$. The phenomenon shown in Examples 10.7 and 10.8 does not occur in this case: the degree of the Hadamard product map is determined by the matroids of L_1 and L_2 .

This symmetry of the Hadamard product and an argument using Theorem 4.10 to reduce to the case of realizable matroids imply that, for any loopless matroids M_1 and M_2 , we have

$$\deg_{M_2}(c_{r_2-1}(\mathcal{S}_{M_1^\perp}^\vee)) = \deg_{M_1}(c_{r_1-1}(\mathcal{S}_{M_2^\perp}^\vee)).$$

If M_1 and M_2 are realized by L_1 and L_2 , then the Hadamard product map is generically finite if and only if $D(M_1, M_2)$ has the expected rank. In particular, we have the following corollary.

Corollary 10.10. *Let M_1 and M_2 be loopless matroids on $[n]$ which are realizable over the same field. Then*

$$\deg_{M_2}(c_{r_2-1}(\mathcal{S}_{M_1^\perp}^\vee)) > 0 \text{ if and only if } D(M_1, M_2) \text{ has the expected rank.}$$

We conjecture that the conclusion of Corollary 10.10 holds for any loopless matroids M_1 and M_2 , even if they are not realizable. If $\text{rk}(M_1) + \text{rk}(M_2) = n + 1$, then [Ber22, Theorem 3.3] implies that the conclusion of Corollary 10.10 holds.

Remark 10.11. The behavior of Hadamard products of linear spaces is much more complicated than the behavior of semi-inverted Hadamard products of linear spaces. For example, if L_1 and L_2 are linear subspaces of k^n with $\dim L_1 + \dim L_2 = n + 1$, then the semi-inverted Hadamard product map $\mathbb{P}L_1 \times \mathbb{P}L_2 \dashrightarrow \mathbb{P}^{n-1}$

is birational if it is dominant; see Example 8.13. Example 8.13 also implies that semi-inverted Hadamard products of linear spaces are normal and Cohen–Macaulay, properties that can fail for Hadamard products of linear spaces. See Example 9.9.

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Email address: mattlars@princeton.edu