DIFFERENTIAL OPERATORS, ANISOTROPY, AND SIMPLICIAL SPHERES

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ABSTRACT. We find identities involving differential operators in the generic artinian reduction of the Stanley–Reisner ring of a simplicial sphere in any positive characteristic. These identities generalize the characteristic 2 identities used by Papadakis and Petrotou to give a proof of the algebraic g-conjecture. We use these identities to prove the anisotropy of certain forms on the generic artinian reduction of the Stanley–Reisner ring and to prove weak Lefschetz results.

1. INTRODUCTION

In [PP20], Papadakis and Petrotou found some identities involving differential operators in the generic artinian reduction of the Stanley–Reisner ring of a simplicial sphere over a field of characteristic 2, and they used those identities to give a proof of the algebraic g-conjecture for simplicial spheres. They conjectured a generalization of their identities, and this conjecture was proven in [KX23]. These identities have enabled much recent progress in the study of simplicial spheres [APP21, LSN24, Oba24].

We prove analogous identities in the generic artinian reduction of the Stanley–Reisner ring of a simplicial sphere over a field of characteristic p for any prime p. We use these identities to deduce algebraic properties of such generic artinian reductions, including proving conjectures of Adiprasito, Papadakis, and Petrotou. Our results hold for all connected oriented simplicial pseudomanifolds, and we work in that level of generality. Recall that a simplicial pseudomanifold of dimension $d-1$ is a pure simplicial complex Δ of dimension $d-1$ such that every $(d-2)$-dimensional face lies in exactly two facets, and each connected component of the geometric realization of Δ remains connected after we remove its $(d-3)$-skeleton.

Let Δ be a connected oriented simplicial pseudomanifold of dimension $d-1$ over a field k with vertex set $V = \{1, \ldots, n\}$. Set $K = k(a_{i,j})_{1 \leq i \leq d,\, 1 \leq j \leq n}$. Let $K[\Delta]$ be the Stanley–Reisner ring of Δ, and set $\mu_i = \sum_j a_{i,j} x_j$ for $i \in \{1, \ldots, d\}$, so $\mu_1, \ldots, \mu_d$ is a linear system of parameters for $K[\Delta]$. Let $H(\Delta) := K[\Delta]/(\mu_1, \ldots, \mu_d)$ be the generic artinian reduction of $K[\Delta]$. Then $H^d(\Delta)$ is 1-dimensional, and there is a degree map

$$\deg \colon K[x_1, \ldots, x_n]_d \to H^d(\Delta) \longrightarrow K$$

which induces a distinguished isomorphism from $H^d(\Delta)$ to K, that, with a slight abuse of notation, we also denote by deg. Here $K[x_1, \ldots, x_n]_d$ denotes the degree d part of the polynomial ring. The degree map has the following concrete description: $H^d(\Delta)$ is generated as a K-vector space by the monomial x^F associated to any facet $F = \{j_1 < \cdots < j_d\}$ of Δ. Fix an orientation of Δ. This gives an ordering of the vertices of F, up to even permutation. Let $\epsilon_F \in \{1, -1\}$ be the sign of the permutation that takes $(j_1, \ldots, j_d)$ to this ordering. Let $[F]$ denote the determinant of the $d \times d$ matrix whose (i,m)th entry is a_{i,j_m}. Then $\deg(x^F) = \epsilon_F/[F] \in K$.

If $L = (\nu_{i,j})$ is an $m \times n$ matrix of nonnegative integers for some positive integer m, then define

$$x^L := \prod_{i,j} x_j^{\nu_{i,j}} = \prod_j x_j^{\sum_i \nu_{i,j}}.$$

For example, if F is a facet and L is the $1 \times n$ matrix whose jth entry is 1 if $j \in F$ and 0 otherwise, then $x^L = x^F$. We use $(x^L)^{1/p}$ to denote the monomial whose pth power is x^L if such a monomial exists and zero

otherwise. For each $i \in \{1, \ldots, d\}$ and $j \in V$, the differential operator $\frac{\partial}{\partial a_{i,j}}$ acts on K. If $I = (I_{i,j})$ is a $d \times n$ matrix of nonnegative integers, define

$$\partial^I \coloneqq \prod_{i,j} \left(\frac{\partial}{\partial a_{i,j}} \right)^{I_{i,j}}.$$

Theorem 1.1. *Let Δ be a connected oriented simplicial pseudomanifold of dimension $d-1$ over a field k of characteristic p with vertex set $V = \{1, \ldots, n\}$. Let I be a $d \times n$ matrix of nonnegative integers such that all row sums are $p-1$. Let J be a $1 \times n$ matrix of nonnegative integers with row sum d. Then*

$$\partial^I \deg(x^J) = (-1)^d \deg((x^I \cdot x^J)^{1/p})^p,$$

where $(x^I \cdot x^J)^{1/p}$ is the pth root of $x^I \cdot x^J$ if $x^I \cdot x^J$ is a pth power, and is zero otherwise.

When $p = 2$, the above theorem was conjectured in [PP20, Conjecture 14.1] and proved in [KX23, Theorem 4.1], and it generalizes [PP20, Proposition 5.1 and 5.7]. Our argument imitates the proof of [KX23, Theorem 4.1], but it requires some new ideas, see Remark 2.10.

Example 1.2. Suppose that $d = 2$, $p = 3$ and $F = \{1 < 2\}$ is a facet. Let $I = \begin{bmatrix} 2 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \end{bmatrix}$ and $J = (1, 1, 0, \cdots)$. Then Theorem 1.1 says that $\partial^I \deg(x^F) = \deg(x^F)^3$. This corresponds to the fact that $\left(\frac{\partial}{\partial a_{1,1}} \right)^2 \left(\frac{\partial}{\partial a_{2,2}} \right)^2 \frac{1}{a_{1,1}a_{2,2} - a_{2,1}a_{1,2}} = \frac{1}{(a_{1,1}a_{2,2} - a_{2,1}a_{1,2})^3}$ over a field of characteristic 3.

We use Theorem 1.1 to deduce results about a quotient of $H(\Delta)$. Let $\overline{H}(\Delta)$ denote the *Gorenstein quotient* of $H(\Delta)$, i.e., the quotient by the kernel of the pairing $H^\bullet \times H^{d-\bullet} \to K$ given by $(g, h) \mapsto \deg(g \cdot h)$. For example, if Δ is a homology sphere over k, then $\overline{H}(\Delta) = H(\Delta)$. Set $\ell = x_1 + \cdots + x_n \in \overline{H}^1(\Delta)$.

Theorem 1.3. *Let Δ be a connected oriented simplicial pseudomanifold of dimension $d-1$ over a field k of characteristic p. Then for any nonzero $g \in \overline{H}^m(\Delta)$ with $m \leq d/p$, $\ell^{d-pm} \cdot g^p$ is nonzero.*

I.e., the form on $\overline{H}^m(\Delta)$ given by $g \mapsto \deg(\ell^{d-pm} \cdot g^p)$ is *anisotropic*. When $p = 2$, this result recovers [KX23, Theorem 4.4].

In particular, Theorem 1.3 shows that if $g \in \overline{H}^m(\Delta)$ is nonzero for some $m \leq d/p$, then g^p is nonzero. This proves [APP21, Conjecture 7.3]. Also, $\ell^{d-pm} \cdot g$ is nonzero, so the map from $\overline{H}^m(\Delta)$ to $\overline{H}^{d-(p-1)m}(\Delta)$ given by multiplication by ℓ^{d-pm} is injective. In particular, the map from $\overline{H}^m(\Delta)$ to $\overline{H}^{m+1}(\Delta)$ given by multiplication by ℓ is injective for $m \leq \frac{d-1}{p}$. This implies that the the weak Lefschetz property holds for $\overline{H}(\Delta)$ in degrees at most $\lfloor (d-1)/p \rfloor$, and $\dim \overline{H}^0(\Delta) \leq \cdots \leq \dim \overline{H}^{\lfloor (d-1)/p \rfloor}(\Delta)$. Even for $\mathbb{Q}$-homology spheres, these inequalities cannot in general be proven using the characteristic 2 results of [PP20] or [KX23], see Example 3.7. A proof of the strong Lefschetz property for $\overline{H}(\Delta)$ was announced in [APP21], see also [Adi18].

We can use Theorem 1.1 to deduce results when k has characteristic 0 as well. For this, we need to know that the dimension of $\overline{H}(\Delta)$ does not depend on the characteristic of k. A result of Novik and Swartz [NS09] guarantees that this holds when Δ is a connected oriented simplicial $\mathbb{Z}$-homology manifold whose homology has no torsion. Recall that a $(d-1)$-dimensional simplicial complex Δ is an A-homology manifold for some abelian group A if the link of every nonempty face G of Δ has the same homology as a sphere of dimension $d - |G| - 1$ over A. We can then prove the following result, which answers the question in [PP20, Remark 14.3] when k has characteristic 0.

Theorem 1.4. *Let Δ be a connected oriented simplicial $\mathbb{Z}$-homology manifold of dimension $d-1$ over a field k of characteristic 0. Assume that the integral homology of Δ is torsion-free. For any nonzero $g \in \overline{H}^m(\Delta)$ with $m \leq d/t$, $\ell^{d-tm} \cdot g^t$ is nonzero.*

This result, which is new even for simplicial spheres, is proved by using Theorem 1.3 for each prime p which divides t.

Our paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we prove Theorem 1.3 and Theorem 1.4. See [KX23, Section 2] for background on pseudomanifolds, orientations, and degree maps.

2. Proof of the identity

2.1. **Reductions.** Let k be a field of characteristic p. We begin the proof of Theorem 1.1 by reducing to the case when Δ is the boundary of a simplex. It was shown in [KX23, Theorem 3.2] that the degree map for a connected oriented simplicial pseudomanifold Δ over k with M facets can be expressed as a sum of degree maps

$$(1) \qquad \deg_\Delta = \deg_{\Pi_1} + \deg_{\Pi_2} + \ldots + \deg_{\Pi_M}.$$

Here the summands correspond to facets σ_i of Δ. Let 0 be a new vertex. Then the complex Π_i corresponding to $\sigma_i = \{j_1, \ldots, j_d\}$ is the boundary of the d-simplex $\{0, j_1, \ldots, j_d\}$. The orientation on Π_i is such that the orientations on σ_i agree in Δ and Π_i. The map $\deg_{\Pi_i}$ is viewed as a function on $K[x_1, \ldots, x_n]_d$ via the composition

$$K[x_1, \ldots, x_n]_d \to K[x_0, x_{j_1}, \ldots, x_{j_d}] \xrightarrow{\deg_{\Pi_i}} K,$$

where the left map sets all x_j that do not lie in $\{x_{j_1}, \ldots, x_{j_d}\}$ equal to zero.

Lemma 2.1. *Theorem 1.1 for all Π_i implies it for Δ.*

Proof. After substituting in (1) and using linearity of pth powers, the statement of Theorem 1.1 is

$$\sum_{i=1}^{M} \partial^I \deg_{\Pi_i}(x^J) = \sum_{i=1}^{M} (-1)^d \deg_{\Pi_i}((x^I \cdot x^J)^{1/p})^p.$$

We check that the corresponding summands on both sides are equal. If x^J involves a variable x_j such that j is not a vertex of Π_i, then $\deg_{\Pi_i}$ vanishes on both sides. Similarly, if x^I involves such a variable x_j, then the term on the right hand side vanishes; on the left hand side, $\deg_{\Pi_i}$ does not depend on variables $a_{i,j}$, so the derivative vanishes. In the case when $x^I x^J$ only involves variables x_j such that j is a vertex of Π_i, the two sides are equal by Theorem 1.1 applied to Π_i. $\qquad\square$

From now on, we will assume that $\Delta = \Pi$ is the boundary of a d-dimensional simplex with vertices $\{0, 1, \ldots, d\}$. Let $A = (a_{i,j})_{1 \leq i \leq d, 0 \leq j \leq d}$, and note that we index the columns by $0, 1, \ldots, d$. We use a similar convention for the matrices I and J in the statement of Theorem 1.1.

The degree map for $\Delta = \Pi$ has a simple form. For $0 \leq j \leq d$, let X_j be $(-1)^j$ times the determinant of A with the jth column removed. We write X^J for a monomial in the X_j. If x^J has degree d, then

$$(2) \qquad \deg_\Pi(x^J) = \frac{X^J}{X_0 X_1 \cdots X_d}.$$

This formula, together with (1), can be used to give an explicit expression for the degree map $\deg_\Delta$. The resulting formula is the same as the one given by Lee [Lee96]. It can also be deduced from the integration map of Brion [Bri97].

Remark 2.2. Note that the determinants X_j lie in $\mathbb{Z}[a_{i,j}]$, and the expression for $\deg_\Delta$ is independent of the field k. This can be used to specialize the degree map from characteristic 0 to characteristic p. For any $g \in \mathbb{Z}[a_{i,j}][x_j]$, we obtain the same result if we first reduce g mod p and then compute its degree in $\mathbb{F}_p(a_{i,j})$, or if we first compute its degree in $\mathbb{Q}(a_{i,j})$ and then reduce this mod p.

The next theorem is a stronger version of Theorem 1.1 in the case where $\Delta = \Pi$. When $p = 2$, it was proved in [KX23, Theorem 4.6].

Theorem 2.3. *Let I be a $d \times (d+1)$ matrix of nonnegative integers such that all row sums are $p-1$. Let J be a $1 \times (d+1)$ matrix of nonnegative integers whose row sum is $-1 \pmod p$. Then*

$$\partial^I X^J = (-1)^d X^{pL}$$

if there exists an integer vector L such that $X^I X^J = X^{pL}(X_0 \cdots X_d)^{p-1}$, and $\partial^I X^J = 0$ otherwise.

Lemma 2.4. *Theorem 2.3 implies Theorem 1.1 for Π.*

Proof. The statement of Theorem 1.1 is

$$\partial^I \frac{X^J}{X_0 \cdots X_d} = \begin{cases} (-1)^d X^I X^J/(X_0 \cdots X_d)^p & \text{if } X^I X^J = X^{pL}, \\ 0 & \text{otherwise.} \end{cases}$$

In characteristic p, partial derivatives commute with pth powers. Multiplying both sides by $(X_0 \cdots X_d)^p$ gives the equivalent statement

$$\partial^I X^J (X_0 \cdots X_d)^{p-1} = \begin{cases} (-1)^d X^{pL} & \text{if } X^I X^J = X^{pL}, \\ 0 & \text{otherwise.} \end{cases}$$

The above equation is the statement of Theorem 2.3 for I and J' such that $X^{J'} = X^J(X_0 \cdots X_d)^{p-1}$. Note that the row sum of J' is $d + (p-1)(d+1) = pd + (p-1)$, which is $-1 \pmod p$. $\square$

2.2. **SL(d,k)-invariance.** Let $B \in SL(d,k)$. Multiplication by B from the left $A \mapsto B \cdot A$ gives a linear change of variables on the $a_{i,j}$, so it gives an action of $SL(d,k)$ on the polynomial ring $k[a_{i,j}]$. If k is infinite, then the invariants of this action are the subring $k[X_j] \subseteq k[a_{i,j}]$. If k is finite, then this subring is the set of absolute invariants, i.e., the polynomials in $k[a_{i,j}]$ that are invariant under the action of $SL(d,\overline{k})$, where $\overline{k}$ is the algebraic closure of k. The following lemma generalizes [KX23, Lemma 4.8].

Lemma 2.5. *In the setup of Theorem 2.3, $\partial^I X^J \in k[a_{i,j}]$ is $\mathrm{SL}(d,k)$-invariant for any field k of characteristic p. In particular, $\partial^I X^J \in \mathbb{F}_p[a_{i,j}]$ is an absolute $\mathrm{SL}(d,\mathbb{F}_p)$-invariant.*

Proof. The group $SL(d,k)$ is generated by elementary matrices (transvections) that act on A by adding a constant c times a row r to a row s. It suffices to prove invariance under this action when $r = 2$ and $s = 1$. Consider new variables

$$a'_{i,j} = \begin{cases} a_{i,j} + ca_{2,j} & \text{if } i = 1, \\ a_{i,j} & \text{otherwise.} \end{cases}$$

Then the action maps the variable $a_{i,j}$ to $a'_{i,j}$. For a polynomial $f \in k[a_{i,j}]$, we denote by $f(a')$ the result of substituting $a'_{i,j}$ into $a_{i,j}$. We need to prove that

$$(\partial^I X^J)(a') = \partial^I X^J.$$

Let $f \in k[a_{i,j}]$. Then for any $0 \le m \le d$, the chain rule gives

$$\frac{\partial}{\partial a_{2,m}}\left(f(a')\right) = \frac{\partial}{\partial a_{2,m}} f(a_{1,j} + ca_{2,j}, a_{2,j}, \ldots)$$

$$= \left(\frac{\partial}{\partial a_{2,m}} f\right)(a') + \left(c\frac{\partial}{\partial a_{1,m}} f\right)(a') = \left(\left(\frac{\partial}{\partial a_{2,m}} + c\frac{\partial}{\partial a_{1,m}}\right) f\right)(a').$$

For $i \ne 2$,

$$\frac{\partial}{\partial a_{i,m}}\left(f(a')\right) = \left(\frac{\partial}{\partial a_{i,m}} f\right)(a').$$

Now let $f = X^J$. We apply the derivatives in ∂^I to $X^J(a')$ one at a time. The result is

$$\left(\partial^I(a')X^J\right)(a'),$$

where $\partial^I(a')$ is the derivative ∂^I with each $\frac{\partial}{\partial a_{2,m}}$ replaced by $\frac{\partial}{\partial a_{2,m}} + c\frac{\partial}{\partial a_{1,m}}$. The derivative $\partial^I(a')$ has one term equal to ∂^I. All other terms contain at least p derivatives of the form $\frac{\partial}{\partial a_{1,m}}$. Lemma 2.6 below shows that all these other terms vanish, so

$$\partial^I\left(X^J(a')\right) = (\partial^I X^J)(a').$$

Since X^J is invariant under the $SL(d,k)$ action, the left hand side is equal to $\partial^I X^J$. $\qquad\square$

Lemma 2.6. *If the row sum of J is -1 (mod p), then*

$$\frac{\partial}{\partial a_{r,j_1}} \frac{\partial}{\partial a_{r,j_2}} \cdots \frac{\partial}{\partial a_{r,j_p}} X^J = 0.$$

for any $1 \le r \le d$ and $0 \le j_1, ..., j_p \le d$.

Proof. Let r be fixed. We argue by induction on the row sum of J.

First assume that the row sum of J is $p-1$. Then since each X_j is homogeneous of degree 1 in the variables $\{a_{r,j} : 0 \le j \le d\}$, X^J is homogeneous of degree $p-1$. Applying $\frac{\partial}{\partial a_{r,j}}$ to a homogeneous polynomial either kills it or decreases its degree by 1, so $\frac{\partial}{\partial a_{r,j_1}} \frac{\partial}{\partial a_{r,j_1}} \cdots \frac{\partial}{\partial a_{r,j_p}} X^J = 0$.

Now assume that the row sum of J is strictly greater than $p-1$. We use relations between the X_j to decrease the row sum.

Let $\tilde{A}$ be the matrix $A = (a_{i,j})$ augmented with an extra column e_r (the standard basis vector). Denote by Y_{jm} the determinant of $\tilde{A}$ with two columns j and m removed. The determinants Y_{jm} satisfy the classical Plücker relations: for any $0 \le j_1 < j_2 < j_3 < j_4 \le d+1$, we have

$$Y_{j_1 j_2} Y_{j_3 j_4} - Y_{j_1 j_3} Y_{j_2 j_4} + Y_{j_1 j_4} Y_{j_2 j_3} = 0.$$

We specialize to the case where $(j_1, j_2, j_3, j_4) = (1, 2, j, d+1)$ for some $2 < j < d+1$. Using that $X_j = (-1)^j Y_{j,d+1}$, the relation becomes

$$(-1)^j Y_{12} X_j - Y_{1j} X_2 - X_1 Y_{2j} = 0.$$

This relation allows us to write

(3) $$X_j = b_1 X_1 + b_2 X_2,$$

where $b_1 = (-1)^j Y_{2j}/Y_{12}$ and $b_2 = (-1)^j Y_{1j}/Y_{12}$ are elements of K that are independent of the variables $\{a_{r,j} : 0 \le j \le d\}$. Substituting the expression from (3) into X^J for all $j > 2$, we reduce to the case when $X^J = X_1^{m_1} X_2^{m_2}$ for some nonnegative integers m_1, m_2 such that $m_1 + m_2$ is the row sum of J. Since the row sum of J is -1 (mod p) and is strictly greater than $p-1$, it must be at least $2p-1$. Hence either $m_1 \ge p$ or $m_2 \ge p$. Without loss of generality, assume that $m_1 \ge p$. Then we can write $X^J = X_1^p X^{J'}$ with the row

sum of J' equal to $-1 \pmod{p}$ and strictly less than the row sum of J. By induction, and since derivatives commute with pth powers, we have

$$\frac{\partial}{\partial a_{r,j_1}} \frac{\partial}{\partial a_{r,j_2}} \cdots \frac{\partial}{\partial a_{r,j_p}} X^J = X_1^p \frac{\partial}{\partial a_{r,j_1}} \frac{\partial}{\partial a_{r,j_2}} \cdots \frac{\partial}{\partial a_{r,j_p}} X^{J'} = 0. \qquad \square$$

2.3. **Proof of Theorem 2.3.** We will now prove Theorem 2.3, which also finishes the proof of Theorem 1.1.

Recall the setup. Let $I = (I_{i,j})$ be a $d \times (d+1)$ matrix of nonnegative integers such that all row sums are $p - 1$. Let J be a $1 \times (d+1)$ matrix of nonnegative integers whose row sum is $-1 \pmod{p}$. We write

$$X^I X^J = X^M (X_0 \cdots X_d)^{p-1},$$

for some $1 \times (d+1)$ matrix of integers M. Observe that if $(X^M)^{1/p}$ exists, then, since $X^M (X_0 \cdots X_d)^{p-1}$ has nonnegative exponents, the coefficients of M are nonnegative integers. Theorem 2.3 states that $\partial^I X^J$ is equal to $(-1)^d X^M$ if the pth root $(X^M)^{1/p}$ exists, and it is zero otherwise.

We give $k[a_{i,j}]$ a multigrading by $\mathbb{Z}^{d+1}$, where $a_{i,j}$ has degree e_j (the standard basis vector). Then X_j lies in degree $(1, 1, \ldots, 1) - e_j$. Since the degrees of the X_j for $j = 0, \ldots, d$ are linearly independent, there is at most one monomial in $k[X_0, \ldots, X_d]$ in each degree.

Lemma 2.5 implies that $\partial^I X^J$ lies in the ring $k[X_0, \ldots, X_d]$. Since X^J is homogeneous with respect to the multigrading and each partial derivative with respect to $a_{i,j}$ reduces the degree by e_j, it follows that $\partial^I X^J$ is also homogeneous. Its multidegree is

$$\mathrm{mdeg}\, \partial^I X^J = \mathrm{mdeg}\, X^J - \sum_{i,j} I_{i,j} e_j = \mathrm{mdeg}\, X^J + \mathrm{mdeg}\, X^I - d(p-1)(1, 1, \ldots, 1)$$

$$= \mathrm{mdeg}\, X^I X^J - \mathrm{mdeg}(X_0 \cdots X_d)^{p-1} = \mathrm{mdeg}\, X^M.$$

It follows that $\partial^I X^J$ is a constant multiple of the monomial X_M. We need to prove that this constant is equal to $(-1)^d$ if $(X^M)^{1/p}$ exists, and it is zero otherwise.

Lemma 2.7. *If $(X^M)^{1/p}$ does not exist, then $\partial^I X^J = 0$.*

Proof. Lemma 2.6 gives that the partial derivative of $\partial^I X^J$ with respect to any $a_{i,j}$ is zero. This implies that $\partial^I X^J$ is a pth power in $k[a_{i,j}]$, and since the X_j are irreducible, it is also a pth power in $k[X_0, \ldots, X_d]$. $\square$

Lemma 2.8. *If $X^M = X^{pL}$ for some L, then $\partial^I X^J = (-1)^d X^{pL}$.*

We have seen that $\partial^I X^J$ is a constant multiple of the monomial $X_M = X^{pL}$. We consider I, J, L satisfying the equation

$$(4) \qquad\qquad X^I X^J = X^{pL}(X_0 \cdots X_d)^{p-1}.$$

For every I, there is a unique monomial X^J of smallest degree such that this equation has a solution for some L. Any other X^J is the product of this minimal degree monomial with the pth power of a monomial. This implies that if I, J, L satisfy (4), then

$$\partial^I X^J = c_I X^{pL}$$

for some constant c_I that only depends on I. To prove that $c_I = (-1)^d$ as stated in the lemma, we compare the constants c_I for different I.

Lemma 2.9. *Assume that the matrix I has last column equal to zero. Let $\overline{I}_{\hat{r}}$ be the matrix obtained from I by removing the last column and row r for some $1 \le r \le d$. Then*

$$c_{\overline{I}_{\hat{r}}} = -c_I.$$

Let us prove Lemma 2.8 using Lemma 2.9.

Proof of Lemma 2.8. We prove the lemma in dimension $d - 1$. Start with an arbitrary matrix $\bar{I}$ of size $(d-1) \times d$, and create a matrix I of size $d \times (d+1)$ by adding to $\bar{I}$ an extra row with row sum $p - 1$ and a zero column. From the matrix I we can construct the matrix $\bar{I}_{\hat{r}}$ by removing the last column and row r. One choice of r takes us back to the original $\bar{I}$. Other choices of r correspond to removing from $\bar{I}$ a row and adding another row. Lemma 2.9 states that the constants $c_{\bar{I}_{\hat{r}}}$ are the same for all choices of r. We can change rows of $\bar{I}$ one by one, without changing the constant $c_{\bar{I}}$, until the matrix $\bar{I}$ is given by

$$\bar{I}_{i,j} = \begin{cases} p - 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that we index columns of $\bar{I}$ by $0, 1, \ldots, d - 1$, so column 0 of $\bar{I}$ is zero. For this $\bar{I}$ we may take $X^J = X_0^{p-1}$ and $X^L = 1$. Then

$$\partial^{\bar{I}} X^J = \left((p-1)!\right)^{d-1} X^{pL}.$$

As $(p-1)! \equiv -1 \pmod{p}$, we get that $c_{\bar{I}} = (-1)^{d-1}$. $\qquad\square$

Proof of Lemma 2.9. Recall that X_j is $(-1)^j$ times the determinant of the matrix A with column j removed. We write $\bar{X}_j$, $j = 0, \ldots, d - 1$, for similar signed determinants in dimension $d - 1$. Let $\bar{J}$ be the vector J with its last entry removed. Let I_i be the ith row of I, and let $\bar{I}_i$ be the row I_i with its last entry removed. The vectors $\bar{I}_i$ for $i \neq r$ form the rows of $\bar{I}_{\hat{r}}$. Similarly, let $\bar{L}$ be L with its last entry removed.

Since the last column of I is zero, we may assume that the last entry of J is $p - 1$ and the last entry of L is zero. By definition,

$$(5) \qquad\qquad \partial^I X^J = c_I X^{pL}.$$

Consider the k-algebra map $\pi \colon k[a_{i,j}]_{1 \le i \le d, \, 0 \le j \le d} \to k[a_{i,j}]_{1 \le i \le d, \, 0 \le j \le d-1}$ defined by

$$\pi(a_{i,j}) = \begin{cases} a_{i,j} & \text{if } j < d, \\ (-1)^{r+d} & \text{if } (i,j) = (r,d), \\ 0 & \text{otherwise.} \end{cases}$$

We claim that if we apply π to both sides of equation (5), we get

$$(6) \qquad\qquad \partial^{\bar{I}_{\hat{r}}} \bar{X}^{\bar{J} + \bar{I}_r} = -c_I \bar{X}^{p\bar{L}}.$$

Here $\bar{I}_{\hat{r}}, \bar{J} + \bar{I}_r, \bar{L}$ satisfy equation (4) in dimension $d - 1$, hence $c_{\bar{I}_{\hat{r}}} = -c_I$.

It remains to establish (6). By the cofactor expansion of the determinant, we compute $\pi(X_j) = \bar{X}_j$ for $0 \le j \le d - 1$, so applying π to the right hand side of (5) gives $c_I \bar{X}^{p\bar{L}}$.

Since ∂^I does not involve the variables $\{a_{i,d} : 1 \le i \le d\}$, applying ∂^I commutes with applying π. Write ∂^{I_r} for the derivative corresponding to row r of I. Applying π to the left hand side of (5) gives

$$\partial^I \pi(X^J) = \partial^{\bar{I}_{\hat{r}}} \bar{X}^{\bar{J}} \left(\partial^{I_r} \pi(X_d)^{p-1} \right).$$

We use that $(p-1)! \equiv -1 \pmod{p}$ and $(-1)^{p-1} \equiv 1 \pmod{p}$ to compute

$$\partial^{I_r} \pi(X_d^{p-1}) = (-1)^{(p-1)(d+r+1)} \partial^{I_r} \left(\sum_{j=0}^{d-1} a_{r,j} \bar{X}_j \right)^{p-1} = (p-1)! \bar{X}^{\bar{I}_r} = -\bar{X}^{\bar{I}_r}.$$

We conclude that $\partial^I \pi(X^J) = -\partial^{\bar{I}_{\hat{r}}} \bar{X}^{\bar{J} + \bar{I}_r}$, which establishes (6). $\qquad\square$

Remark 2.10. The proof of Theorem 1.1 is similar to the argument in [KX23, Section 4], but there are a few places where new arguments are needed, especially Lemma 2.6 and Lemma 2.8. For example, the proof of [KX23, Lemma 4.11], which is the special case of Lemma 2.8 when $p = 2$, uses that $\mathbb{F}_2$ has size 2 and so all nonzero elements of $\mathbb{F}_2$ are equal to 1.

3. APPLICATIONS

3.1. **Applications in characteristic p.** Fix a prime p, and assume that k is a field of characteristic p. Let Δ be a connected oriented simplicial pseudomanifold of dimension $d-1$ over k with vertex set $V = \{1, \ldots, n\}$. We apply Theorem 1.1 to deduce results about the Gorenstein quotient $\overline{H}(\Delta)$. Firstly, we have the following generalization of [KX23, Corollary 4.2].

Corollary 3.1. *Let $h \in \overline{H}^m(\Delta)$ for some $m \le d/p$. Let I be a $d \times n$ matrix of nonnegative integers such that all rows sums are $p-1$. Let J be a $1 \times n$ matrix of nonnegative integers such that the row sum is $d - pm$. Then*

$$\partial^I \deg(h^p \cdot x^J) = (-1)^d \deg(h \cdot (x^I \cdot x^J)^{1/p})^p,$$

where $(x^I \cdot x^J)^{1/p}$ is the pth root of $x^I \cdot x^J$ if $x^I \cdot x^J$ is a pth power, and is zero otherwise.

Proof. Write $h = \sum_L \lambda_L x^L$ for some $1 \times n$ matrices of nonnegative integers L and some $\lambda_L \in K$. Using Theorem 1.1 and that partial derivative are linear over pth powers, we compute

$$\begin{aligned}
\partial^I \deg(h^p \cdot x^J) &= \sum_L \partial^I \lambda_L^p \deg(x^{pL} \cdot x^J) \\
&= \sum_L \lambda_L^p \partial^I \deg(x^{pL} \cdot x^J) \\
&= (-1)^d \sum_L \lambda_L^p \deg(x^L \cdot (x^I \cdot x^J)^{1/p})^p \\
&= (-1)^d \deg(h \cdot (x^I \cdot x^J)^{1/p})^p. \qquad \square
\end{aligned}$$

Secondly, we have the following generalization of [KX23, Corollary 4.3].

Corollary 3.2. *Suppose $g \in \overline{H}^m(\Delta)$ has $\ell^r \cdot g^p = 0$ in $\overline{H}^{r+pm}(\Delta)$ for some $0 \le r \le d - pm$. Let $t = \lfloor r/p \rfloor$. Then $\ell^t \cdot g = 0$ in $\overline{H}^{t+m}(\Delta)$.*

Proof. Write $r = tp + s$, where $t = \lfloor r/p \rfloor$ and $0 \le s < p$. For any $d \times n$ matrix I of nonnegative integers with all row sums equal to $p - 1$ and any $1 \times n$ matrix J of nonnegative integers with row sum $d - pm - r$, we have

$$0 = \deg(\ell^r \cdot g^p \cdot x^J) = \deg((\ell^t \cdot g)^p \cdot x^J \cdot \ell^s) = \sum_{(v_1, \ldots, v_s)} \deg((\ell^t \cdot g)^p \cdot x^J \cdot x_{v_1} \cdots x_{v_s}),$$

where $(v_1, \ldots, v_s)$ varies over all sequences of s vertices in V. By Corollary 3.1, we have

$$(7) \qquad 0 = \sum_{(v_1, \ldots, v_s)} \partial^I \deg((\ell^t \cdot g)^p \cdot x^J \cdot x_{v_1} \cdots x_{v_s}) = \sum_{(v_1, \ldots, v_s)} \deg(\ell^t \cdot g \cdot (x^I \cdot x^J \cdot x_{v_1} \cdots x_{v_s})^{1/p})^p.$$

Suppose that $\ell^t \cdot g$ is nonzero. By the nondegeneracy of the pairing between $\overline{H}^{m+t}(\Delta)$ and $\overline{H}^{d-m-t}(\Delta)$, there is a $1 \times n$ matrix L of nonnegative integers with row sum $d - m - t$ such that $\deg(\ell^t \cdot g \cdot x^L)$ is nonzero. Let $L = (\nu_1, \ldots, \nu_n)$, so $x^L = x_1^{\nu_1} \cdots x_n^{\nu_n}$. Let $M = (m_1, \ldots, m_n)$ be a sequence of nonnegative integers

that sum to s with the property that x^M divides x^{pL}. For example, if ν_i is nonzero, then one can take $M = (0, \ldots, 0, s, 0, \ldots, 0)$, where the s is in the ith coordinate. Choose I and J so that

$$(8) \qquad x^{pL}/x^M = x^I \cdot x^J$$

Note that both sides have degree $p(d - m) - r$, so this is always possible.

Now, $x^I \cdot x^J \cdot x_{v_1} \cdots x_{v_s} = (x^{pL}/x^M) \cdot x_{v_1} \cdots x_{v_s}$ is a pth power if and only if $x^{(p-1)M} \cdot x_{v_1} \cdots x_{v_s}$ is a pth power. The latter holds if and only if $x_{v_1} \cdots x_{v_s} = x^M$. Indeed, if $x_{v_1} \cdots x_{v_s} = x_1^{b_1} \cdots x_n^{b_n}$ for some nonnegative integers satisfying $b_1 + \cdots + b_n = s$, then $x^{(p-1)M} \cdot x_{v_1} \cdots x_{v_s}$ is a pth power if and only if $b_i = m_i$ mod p for all i. Since $b_i \leq s < p$, this holds if and only if $b_i = m_i$ for all i.

The number of sequences $(v_1, \ldots, v_s)$ such that $x_{v_1} \cdots x_{v_s}$ is equal to x^M is the multinomial coefficient $s!/((m_1)! \cdots (m_n)!)$, which is nonzero in k. Therefore (7) implies that

$$0 = \frac{s!}{(m_1)! \cdots (m_n)!} \deg(\ell^t \cdot g \cdot (x^I \cdot x^J \cdot x^M)^{1/p})^p = \frac{s!}{(m_1)! \cdots (m_n)!} \deg(\ell^t \cdot g \cdot x^L)^p.$$

This contradicts the assumption that $\deg(\ell^t \cdot g \cdot x^L)$ is nonzero. $\qquad\square$

Proof of Theorem 1.3. Suppose that $g \in \overline{H}^m(\Delta)$ is nonzero. We claim that $\ell^r \cdot g^p \in \overline{H}^{r+pm}(\Delta)$ is nonzero for any $r \leq d - pm$. The case when $r = 0$ follows from Corollary 3.2. Assume that $r > 0$ and let $t = \lfloor r/p \rfloor < r$. By induction, $\ell^t \cdot g^p = (\ell^t \cdot g) \cdot g^{p-1}$ is nonzero, so Corollary 3.2 implies that $\ell^r \cdot g^p$ is nonzero. $\qquad\square$

3.2. Applications in characteristic 0.

We now give applications of our results to simplicial $\mathbb{Z}$-homology manifolds in characteristic 0. Our strategy is similar to [KX23, Section 5]. The following application of results of Novik and Swartz allows us to transfer information from characteristic p to characteristic 0.

Proposition 3.3. [NS09, Theorem 1.3 and 1.4] *Let Δ be a connected oriented simplicial k-homology manifold of dimension $d - 1$ over a field k with vertex set $V = \{1, \ldots, n\}$. Let $K = k(a_{i,j})_{1 \leq i \leq d, 1 \leq j \leq n}$. Let $\mu_1, \ldots, \mu_d$ be an l.s.o.p. for $K[\Delta]$. Then the dimension of the Gorenstein quotient of $K[\Delta]/(\mu_1, \ldots, \mu_d)$ depends only on the h-vector of Δ and the Betti numbers of Δ over k.*

If Δ is a simplicial complex whose integral homology has no p-torsion, then the universal coefficient theorem implies that its Betti numbers are the same in characteristic 0 and characteristic p. An orientation of Δ in characteristic 0 also gives an orientation of Δ in characteristic p. Proposition 3.3 then gives the following corollary.

Corollary 3.4. *Let Δ be a connected oriented simplicial $\mathbb{Z}$-homology manifold whose homology has no p-torsion. Then the dimensions of the graded pieces of the Gorenstein quotient of a generic artinian reduction of $K[\Delta]$ are the same when k has characteristic 0 or characteristic p.*

Proposition 3.5. *Let Δ be a connected oriented simplicial $\mathbb{Z}$-homology manifold of dimension $d - 1$ whose homology has no p-torsion. Let k be a field of characteristic 0. For any nonzero $g \in \overline{H}^m(\Delta)$ with $m \leq d/p$, $\deg(\ell^{d-pm} \cdot g^p)$ is nonzero.*

Proof. Because k has characteristic 0, it is an extension of $\mathbb{Q}$. By the Chevalley extension theorem (see, e.g., [EP05, Theorem 3.1.1]), we can extend the p-adic valuation on $\mathbb{Q}$ to a valuation on k. Let κ denote the residue field.

Let $\overline{H}(\Delta)$ be the Gorenstein quotient of the generic artinian reduction of $k(a_{i,j})[\Delta]$, and define $\overline{H}_\kappa(\Delta)$ similarly. Choose a basis for $\overline{H}_\kappa^m(\Delta)$ and for $\overline{H}_\kappa^{d-m}(\Delta)$ consisting of monomials. Denote these bases by $v_1, \ldots, v_r$ and $w_1, \ldots, w_r$, where $r = \dim \overline{H}_\kappa^m(\Delta)$. Let M_κ be the matrix whose (i, j)th entry is $\deg_\kappa(v_i \cdot w_j)$, i.e., using the degree map on $\overline{H}_\kappa(\Delta)$. By construction, M_κ is invertible. Let M be the matrix whose (i, j)th entry is $\deg(v_i \cdot w_j)$, i.e., using the degree map on $\overline{H}(\Delta)$. We obtain M_κ from M by reducing modulo p, see

Remark 2.2, so M is invertible. This implies that $\{v_1, \ldots, v_r\}$ is linearly independent in $\overline{H}^m(\Delta)$, and so it is a basis for $\overline{H}^m(\Delta)$ by Corollary 3.4.

Let $\lambda \in k[a_{i,j}]$ be a polynomial such that, when we express λg in this basis, all coefficients are polynomials with coefficients in the valuation ring of k, and at least one of these polynomials has a coefficient whose image in κ is nonzero. Let $\overline{\lambda g}$ denote the image of λg in $\overline{H}_\kappa(\Delta)$, which is nonzero. By Theorem 1.3, $\deg(\ell^{d-pm} \cdot (\overline{\lambda g})^p)$ is nonzero. This implies that $\deg(\ell^{d-pm} \cdot (\lambda g)^p) = \lambda^p \deg(\ell^{d-pm} \cdot g^p)$ is nonzero, and so $\deg(\ell^{d-pm} \cdot g^p)$ is nonzero. $\qquad\square$

Proof of Theorem 1.4. Let p be a prime dividing t. By Proposition 3.5, g^p is nonzero in $\overline{H}^{pm}(\Delta)$. Replacing g by g^p and continuing in this fashion gives the result. $\qquad\square$

Remark 3.6. The proof of Theorem 1.4 works for any connected oriented simplicial $\mathbb{Q}$-homology manifold such that the homology of the link of any (possibly empty) face has no p-torsion for each prime p dividing t.

Example 3.7. Let Δ be a triangulation of $\mathbb{RP}^{d-1}$, where d is even. Then Δ is a connected orientable simplicial $\mathbb{Z}$-homology manifold whose homology has no 3-torsion. If k has characteristic 0 or 3, then Proposition 3.5 or Theorem 1.3 implies that $\overline{H}(\Delta)$ has the weak Lefschetz property in degrees at most $\lfloor (d-1)/3 \rfloor$, and so $\dim \overline{H}^0(\Delta) \leq \cdots \leq \dim \overline{H}^{\lfloor (d-1)/3 \rfloor}(\Delta)$. The results of [PP20] or [KX23] cannot be applied to such Δ because the dimension of $\overline{H}(\Delta)$ is different in characteristic 2.

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