FINE MULTIDEGREES OF MATRIX SCHUBERT VARIETIES

DAOJI HUANG AND MATT LARSON

ABSTRACT. We introduce fine Schubert polynomials, which record the multidegrees of the closures of matrix Schubert varieties in $(\mathbb{P}^1)^{n^2}$. We compute the fine Schubert polynomials of permutations w where the coefficients of the Schubert polynomials of w and w^{-1} are all either 0 or 1. We give a criterion for a collection of polynomials to be a universal Gröbner basis for an ideal in terms of the multidegree of the closure of the corresponding affine variety in $(\mathbb{P}^1)^n$. This criterion gives simple proofs of several existing results on universal Gröbner bases. We use this criterion to give a universal Gröbner basis for the ideal of a matrix Schubert variety of a permutation w where the coefficients of the Schubert polynomial of w and w^{-1} are all either 0 or 1.

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1. INTRODUCTION

For a permutation $w \in S_n$, let X_w be the associated matrix Schubert variety, i.e., the closure in the space of $n \times n$ complex matrices of the preimage of the Schubert cell labeled by w under the map from GL_n to the flag variety GL_n/B. These varieties, which were introduced by Fulton [Ful92], can be used to give algebraic proofs of geometric facts about Schubert varieties and to give an algebro-geometric interpretation

of Schubert polynomial [KM05]. We identify the space of $n \times n$ complex matrices with $\mathbb{A}^{n^2}$ and consider the natural inclusion of $\mathbb{A}^{n^2}$ into $(\mathbb{P}^1)^{n^2}$. Let $\overline{X}_w$ be the closure of X_w in $(\mathbb{P}^1)^{n^2}$.

It is known that X_w is a variety of dimension $n^2 - \ell(w)$, where $\ell(w)$ is the length of w. Therefore $\overline{X}_w$ defines a class $[\overline{X}_w] \in H_{2(n^2-\ell(w))}((\mathbb{P}^1)^{n^2})$ in the homology of $(\mathbb{P}^1)^{n^2}$. Let $[n] = \{1, \ldots, n\}$. For each $(i,j) \in [n]^2$, let $[H_{i,j}] \in H^2((\mathbb{P}^1)^{n^2})$ be the class of the hyperplane $\{(a_{k,\ell})_{(k,\ell)\in[n]^2} : a_{i,j} = 0\}$. There is a unique squarefree polynomial $\mathfrak{F}_w(z_{i,j}) \in \mathbb{Z}[z_{i,j} : (i,j) \in [n]^2]$ such that

$$[\overline{X}_w] = \mathfrak{F}_w([H_{i,j}]) \cap [(\mathbb{P}^1)^{n^2}].$$

We call this polynomial the **fine Schubert polynomial** of w because it records information about a very fine grading: it can be described in terms of the Hilbert function of the $\mathbb{Z}^{n^2}$-graded homogenization of the ideal of X_w, see Section 2. Note that $\mathfrak{F}_w(z_{i,j})$ is homogeneous of degree $\ell(w)$.

There is a canonical basis for $H_*((\mathbb{P}^1)^{n^2})$, given by the classes of coordinate subspaces. The basis for $H_{2k}((\mathbb{P}^1)^{n^2})$ is labeled by subsets S of $[n]^2$ of size $n^2 - k$, with S corresponding to the class of coordinate subspace $\{(a_{k,\ell})_{(k,\ell)\in[n]^2} : a_{i,j} = 0 \text{ if } (i,j) \in S\}$. The *multidegree* of $\overline{X}_w$ is the tuple of coefficients which are used to expand the class $[\overline{X}_w]$ in terms of this canonical basis. These coefficients are recorded by the coefficients of the fine Schubert polynomial: the coefficient of $z^S := \prod_{(i,j)\in S} z_{i,j}$ is the coefficient of the basis element corresponding to S.

If $|S| = \ell(w)$, the coefficient of z^S is also the degree of the composition $X_w \hookrightarrow (\mathbb{P}^1)^{n^2} \to (\mathbb{P}^1)^{S^c}$, where the second map is the projection onto the factors labeled by elements not in S. Concretely, if we fill in the entries of an $n \times n$ matrix which are labeled by elements of S^c with generic complex numbers, then the coefficient of z^S is the number of ways that the entries in S can be filled in so that the resulting matrix lies in X_w. In particular, the coefficients of fine Schubert polynomials are nonnegative.

Gröbner degenerations are a powerful tool to study the algebra, combinatorics, and geometry of matrix Schubert varieties [KM05, KMY09, HPW22, KW21]. Let $\varphi \colon \mathbb{G}_m \to \mathbb{G}_m^{n^2}$ be the inclusion of a 1-parameter subgroup into the torus which acts on $\mathbb{A}^{n^2}$. A Gröbner degeneration of X_w can be understood as the limit $\lim_{t\to 0} \varphi(t) \cdot X_w$, where $t \in \mathbb{G}_m$. If φ is sufficiently generic, then the limit will be a union of coordinate subspaces (each of dimension $n^2 - \ell(w)$), and so the degeneration is an essentially combinatorial object. Properties of X_w (such as its Hilbert function or Cohen–Macaulayness) can be analyzed in this degeneration. See [KM05, Section 1.7].

The polynomial $\mathfrak{F}_w(z_{i,j})$ "controls" the possible Gröbner degenerations of X_w. Indeed, $\lim_{t\to 0} \varphi(t) \cdot \overline{X}_w$ contains $\lim_{t\to 0} \varphi(t) \cdot X_w$. The multidegree of $\lim_{t\to 0} \varphi(t) \cdot \overline{X}_w$ is the same as the multidegree of $\overline{X}_w$, so if the coordinate subspace corresponding to $S \subset [n]^2$ is an irreducible component of the limit, then the coefficient of z^S in $\mathfrak{F}_w(z_{i,j})$ must be nonzero. One can show that the coefficient of z^S in the fine Schubert polynomial is equal to the maximal multiplicity of $\{(a_{k,\ell})_{(k,\ell)\in[n]^2} : a_{i,j} = 0 \text{ if } (i,j) \in S\}$ in any Gröbner degeneration of X_w. See Proposition 2.10.

Example 1.1. In [KM05], the authors show that, for certain φ (corresponding to an antidiagonal term order) $\lim_{t\to 0} \varphi(t) \cdot X_w$ is the union of the coordinate subspaces corresponding to the *reduced pipe dreams* of [BJS93]. In particular, if S is the set of crosses in a reduced pipe dream for w, then the coefficient of z^S in $\mathfrak{F}_w(z_{i,j})$ is positive.

Example 1.2. In [KW21], the authors show that there are certain φ for which $\lim_{t\to 0} \varphi(t) \cdot X_w$ is a union of coordinate subspaces corresponding to the blank tiles in the *bumpless pipe dreams* of [LLS21]. In particular, the coefficient of z^S in $\mathfrak{F}_w(z_{i,j})$ is at least the number of bumpless pipe dreams with S as the set of blank tiles. Unlike for pipe dreams, there can be multiple bumpless pipe dreams with the same set of blank tiles.

There are other combinatorial models for Schubert polynomials, such as hybrid pipe dreams [KU23], which conjecturally have Gröbner interpretations [Knu24], and so similarly should give lower bounds on the coefficients of fine Schubert polynomials. Fine Schubert polynomials could obstruct the existence of Gröbner interpretations for other combinatorial models, such as Bruhat chains [Yu24].

The **support** of a fine Schubert polynomial is the collection of subsets S of $[n]^2$ where the coefficient of z^S in the fine Schubert polynomial is nonzero. The support of a fine Schubert polynomial is the bases of a *matroid* of rank $\ell(w)$ on $[n]^2$. This matroid is dual to the *algebraic matroid* of the matrix Schubert variety: if $|S| = \ell(w)$, the coefficient of z^S is nonzero if and only if the coordinate functions $\{a_{i,j} : (i,j) \in S^c\}$ on X_w are a transcendence basis for the function field of X_w.

When $w = 1, 2, \ldots, r, q+1, q+2, \ldots, p+q-r, r+1, \ldots, q \in S_{p+q-r}$, the matrix Schubert variety X_w is isomorphic to the product of the locus of $p \times q$ matrices of rank at most r with an affine space. In this case, fine Schubert polynomials have been extensively studied in connection with rigidity theory [SC09, Ber17, Tsa24, BDG$^+$24]. Understanding the support of the fine Schubert polynomials of these permutations is a special case of understanding *bipartite rigidity* [KNN16] of graphs.

When $r \in \{0, 1, p-1, p\}$, these fine Schubert polynomials are easy to understand. When $r = 2$, the support of the fine Schubert polynomial was computed in [Ber17]. When $r \geq p - 3$, the support was computed in [BDG$^+$24]. In both of these cases, the coefficients of $\mathfrak{F}_w(z_{i,j})$ can be more than 1, and the coefficients are not understood.

It would be surprising if there is a simple description of the support of this family of fine Schubert polynomials, as the problem of understanding bipartite rigidity of graphs is considered to be of comparable difficulty to the longstanding problem of understanding graph rigidity. In [BDG$^+$24, Theorem 1.1], it is shown that understanding the support of $\mathfrak{F}_w(z_{i,j})$ in this case is equivalent to a difficult problem in information theory. For this reason, we do not expect there to be a simple recursion for $\mathfrak{F}_w(z_{i,j})$. Nevertheless, we will show that many techniques which are used to compute Schubert polynomials give partial information about fine Schubert polynomials.

The *Schubert polynomial* $\mathfrak{S}_w(x_i) \in \mathbb{Z}[x_1, \ldots, x_n]$ is a polynomial which represents the class of the Schubert variety corresponding to w in the cohomology of the flag variety. By [KM05], Schubert polynomials also encode the classes of matrix Schubert varieties in the $(\mathbb{C}^*)^n$-equivariant cohomology of $\mathbb{A}^{n^2}$. Using this interpretation, we can bound the coefficients of fine Schubert polynomials by the coefficients of Schubert polynomials. For a polynomial f and a monomial m, let $[m]f$ denote the coefficient of m in f.

Proposition 1.3. *Let $w \in S_n$ be a permutation, and let S be a subset of $[n]^2$ of size $\ell(w)$. Then*

$$[z^S]\mathfrak{F}_w(z_{i,j}) \leq [\prod_{(i,j)\in S} x_i]\mathfrak{S}_w(x_i).$$

In particular, if the coefficients of $\mathfrak{S}_w(x_i)$ are all either 0 or 1, then the same is true for $\mathfrak{F}_w(z_{i,j})$. As X_w is isomorphic to $X_{w^{-1}}$ via the map that sends a matrix to its transpose, we have $\mathfrak{F}_w(z_{i,j}) = \mathfrak{F}_{w^{-1}}(z_{j,i})$. Therefore, we can also bound the coefficients of $\mathfrak{F}_w(z_{i,j})$ by the coefficients of $\mathfrak{S}_{w^{-1}}(x_j)$. This gives the following corollary.

Corollary 1.4. *Let $w \in S_n$ be a permutation such that the coefficients of either $\mathfrak{S}_w(x_i)$ or $\mathfrak{S}_{w^{-1}}(x_j)$ are all either 0 or 1. Then the coefficients of $\mathfrak{F}_w(z_{i,j})$ are all either 0 or 1.*

Permutations w such that the coefficients of $\mathfrak{S}_w(x_i)$ are all either 0 or 1 were classified in [FMSD21] in terms of a pattern avoidance condition, and this family of permutations has attracted significant attention, see, e.g, [PS22, Kle23, CFY24, CDNG22].

If all coefficients of a fine Schubert polynomial are known to be either 0 or 1, then computing $\mathfrak{F}_w(z_{i,j})$ is equivalent to understanding its support. Let I_w be the ideal of X_w in $\mathbb{A}^{n^2}$. The **spread** of a polynomial $f \in I_w$ is the collection of $(k, \ell) \in [n]^2$ such that $a_{k,\ell}$ appears in some monomial with nonzero coefficient in f. For a subset S of $[n]^2$ of size $\ell(w)$, the coefficient of z^S is 0 if and only if there is some nonzero $f \in I_w$ whose spread is disjoint from S, see Proposition 2.3. So to compute $\mathfrak{F}_w(z_{i,j})$ in this case, it suffices to describe the spreads of nonzero elements of I_w which are minimal with respect to containment. This is equivalent to describing the matroid given by the support of $\mathfrak{F}_w(z_{i,j})$ in terms of its cocircuits.

If all coefficients of a fine Schubert polynomial are either 0 or 1, then a result of Brion [Bri03] implies that $\overline{X}_w$ has remarkable properties: $\overline{X}_w$ is projectively normal, arithmetically Cohen–Macaulay, and has rational singularities.

A collection of polynomials in an ideal is said to be a *universal Gröbner basis* for that ideal if they are a Gröbner basis for any term order. We follow [Stu96] for Gröbner basis conventions. While general results guarantee the existence of a universal Gröbner basis, they are only a few families of ideals where explicit universal Gröbner bases are known. The maximal minors of an $a \times b$ matrix form a universal Gröbner basis [BZ93, SZ93], see also [CDNG15]. There is also a known universal Gröbner basis for the ideal of 2×2 minors, which is a special case of [Stu96, Proposition 4.11 and 8.11] or [Vil15, Proposition 10.1.11]. See [BCRV22, Chapter 5] for a summary. In some cases, ideals associated to linear spaces [AB16, PS06] have explicit universal Gröbner bases.

We give a criterion for a collection of polynomials to be a universal Gröbner basis of an ideal. This can be used to give a simple proof of all of the above results. Let $X \subset \mathbb{A}^N$ be a subscheme whose irreducible components all have codimension d. Let $\overline{X}$ denote the closure of X in $(\mathbb{P}^1)^N$, and let $\mathfrak{F}_{\overline{X}}(z_i) \in \mathbb{Z}[z_1, \ldots, z_N]$ be the unique squarefree polynomial such that $[\overline{X}] = \mathfrak{F}_{\overline{X}}([H_i]) \cap [(\mathbb{P}^1)^N]$, where $[H_i]$ is the class of a hyperplane in the ith factor. Let I_X be the ideal of X.

Theorem 1.5. *Let X be a subscheme of $\mathbb{A}^N$ whose irreducible components all have codimension d. Let $f_1, \ldots, f_r \in I_X$ be nonzero squarefree polynomials. Suppose that, for all $S \subset [N]$, we have*

$$(1) \qquad S \cap \mathrm{spr}(f_i) = \emptyset \text{ for some } i \text{ if and only if } [z^T]\mathfrak{F}_{\overline{X}}(z_i) = 0 \text{ for all } T \subset S \text{ with } |T| = d.$$

Then $\{f_1, \ldots, f_r\}$ is a universal Gröbner basis for I_X, and all coefficients of $\mathfrak{F}_{\overline{X}}(z_i)$ are equal to either 0 or 1.

See Theorem 2.5 for a more general criterion, which is valid for any subscheme of $\mathbb{A}^N$. It is automatic that if $S \cap \mathrm{spr}(f_i) = \emptyset$ for some f_i, then $[z^T]\mathfrak{F}_{\overline{X}}(z_i) = 0$ for all $T \subset S$. See Lemma 2.1. For such a collection $\{f_1, \ldots, f_r\}$, the support of $\mathfrak{F}_{\overline{X}}(z_i)$ will be the sets $S \subset [n]^2$ of size d that intersect the spread of each f_i. In particular, giving such a collection computes $\mathfrak{F}_{\overline{X}}(z_i)$.

By considering lexicographic term orders, we see that any universal Gröbner basis for I_X must satisfy (1). If X is irreducible, then the support of $\mathfrak{F}_{\overline{X}}(z_i)$ is a matroid, and the spreads of the f_i are the cocircuits of the matroid given by the support of $\mathfrak{F}_{\overline{X}}(z_i)$. The f_i are sometimes called the *circuit polynomials* of X, see, e.g., [RST20].

1.1. **Universal Gröbner bases for a family of matrix Schubert varieties.**
We apply Theorem 1.5 to the matrix Schubert varieties X_w of a certain family of permutations. This gives a universal Gröbner basis for I_w and computes $\mathfrak{F}_w(z_{i,j})$ when w is in this family.

For $1 \le p, q \le n - 1$, let

$$r_{p \times q}(w) := \#\{(i, j) \le (p, q) : w(i) = j\}.$$

Let A denote the $n \times n$ matrix whose entry at (i, j) is the indeterminate $a_{i,j}$, and $A_{p \times q}$ the submatrix of A with rows in $1, \ldots, p$ and columns in $1, \ldots, q$. The ideal I_w is generated by all minors of size $1 + r_{p \times q}(w)$ in

$A_{p \times q}$ for all

$$(p, q) \in \text{ess}(w) := \{(i, j) \in [n-1]^2 : w(i) > j, w^{-1}(j) > i, w(i+1) \le j, w^{-1}(j+1) \le i\}.$$

See [Ful92]. These minors are called the *Fulton generators*. Following the terminology in [HPW22], we recall also the definition of the *CDG generators*. These are obtained from the Fulton generators by setting the monomials that contain variables already in I_w to 0, whenever the Fulton generator is a minor of size at least 3. If a generator factors after this specialization, it is redundant and we exclude it from $\text{CDG}(w)$. Denote the set of CDG generators for I_w as $\text{CDG}(w)$. It is immediate that the CDG generators also generate I_w. We now define an iterative procedure that produces a set of generators $\mathcal{R}_w$ from the CDG generators of I_w.

Let $f_1, f_2 \in \mathbb{Z}[a_{i,j} : (i,j) \in [n]^2]$, and suppose $f_1 = a_{\alpha,\beta}g_1 + r_1$ and $f_2 = a_{\alpha,\beta}g_2 + r_2$, where none of g_1, g_2, r_1, r_2 are divisible by $a_{\alpha,\beta}$. Define

$$\text{merge}_{(\alpha,\beta)}(f_1, f_2) := f_1 g_2 - f_2 g_1.$$

The set $\mathcal{R}_w$ is constructed as follows. For $S \subset [n]^2$, let $\text{row}(S)$ and $\text{col}(S)$ denote the set of row and column indices of S.

(1) Initialize $\mathcal{R}_w := \text{CDG}(w)$;
(2) Let

$$R := \{\text{merge}_{(\alpha,\beta)}(f_1, f_2) : f_1 \in \mathcal{R}_w, f_2 \in \text{CDG}(w), \text{spr}(f_1) \cap \text{spr}(f_2) = \{(\alpha, \beta)\}$$
$$\text{row}(\text{spr}(f_1)) \cap \text{row}(\text{spr}(f_2)) = \{\alpha\}, \text{col}(\text{spr}(f_1)) \cap \text{col}(\text{spr}(f_2)) = \{\beta\}\}$$

Enlarge $\mathcal{R}_w$ by R.
(3) Repeat Step (2), until no new relations are added in this step.

By construction, each element in $\mathcal{R}_w$ is square-free.

Example 1.6. Let $w = 426153$. There are two relations in $\mathcal{R}_w$ that are not in $\text{CDG}(w)$. Let $f_1 = \begin{vmatrix} 0 & a_{1,4} & a_{1,5} \\ a_{2,2} & a_{2,4} & a_{2,5} \\ a_{3,2} & a_{3,4} & a_{3,5} \end{vmatrix}$, $f_2 = \begin{vmatrix} 0 & a_{1,4} & a_{1,5} \\ a_{2,3} & a_{2,4} & a_{2,5} \\ a_{3,3} & a_{3,4} & a_{3,5} \end{vmatrix}$, $f_3 = \begin{vmatrix} 0 & a_{2,2} & a_{2,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \\ a_{5,1} & a_{5,2} & a_{5,3} \end{vmatrix}$, and $f_4 = \begin{vmatrix} 0 & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \\ a_{5,1} & a_{5,2} & a_{5,3} \end{vmatrix}$. Then $g_1 := \text{merge}_{(1,4)}(f_1, f_4) = \text{merge}_{(2,3)}(f_2, f_3)$ and $g_2 := \text{merge}_{(2,2)}(f_1, f_3) = \text{merge}_{(3,3)}(f_2, f_4)$ are the two extra relations. The sets $\text{spr}(g_1)$ and $\text{spr}(g_2)$ are shown in Figure 1. We have

$$g_1 = \det \begin{pmatrix} 0 & 0 & 0 & a_{1,4} & a_{1,5} \\ 0 & a_{2,2} & 0 & a_{2,4} & a_{2,5} \\ 0 & 0 & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{4,1} & a_{4,2} & a_{4,3} & 0 & 0 \\ a_{5,1} & a_{5,2} & a_{5,3} & 0 & 0 \end{pmatrix} \quad \text{and} \quad g_2 = \det \begin{pmatrix} 0 & 0 & 0 & a_{1,4} & a_{1,5} \\ 0 & 0 & a_{2,3} & a_{2,4} & a_{2,5} \\ 0 & a_{3,2} & 0 & a_{3,4} & a_{3,5} \\ a_{4,1} & a_{4,2} & a_{4,3} & 0 & 0 \\ a_{5,1} & a_{5,2} & a_{5,3} & 0 & 0 \end{pmatrix}.$$
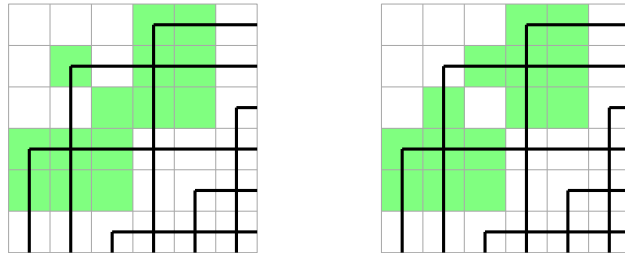

FIGURE 1. Spreads of generators in $\mathcal{R}_w$ not in $\text{CDG}(w)$ for $w = 426153$.

One can show that, in general, the elements $f \in \mathcal{R}_w$ can be described as determinants of matrices whose entries are 0 except on the spread of f.

Theorem 1.7. *Let w be a permutation such that all coefficients of both $\mathfrak{S}_w(x_i)$ and $\mathfrak{S}_{w^{-1}}(x_j)$ are either 0 or 1. Then*

$$S \cap \mathrm{spr}(f) = \emptyset \text{ for some } f \in \mathcal{R}_w \text{ if and only if } [z^T]\mathfrak{F}_w(z_{i,j}) = 0 \text{ for all } T \subset S \text{ with } |T| = \ell(w).$$

In particular, $\mathcal{R}_w$ is a universal Gröbner basis for I_w.

The second part of Theorem 1.7 follows from the first part and Theorem 1.5. As the relations in $\mathcal{R}_w$ are squarefree, Theorem 1.5 implies that all coefficients of $\mathfrak{F}_w(z_{i,j})$ are 0 or 1 in this case, so Theorem 1.7 computes $\mathfrak{F}_w(z_{i,j})$. Explicitly, we have

$$\mathfrak{F}_w(z_{i,j}) = \sum_{\substack{|S|=\ell(w), \\ S \cap \mathrm{spr}(f) \neq \emptyset \text{ for all } f \in \mathcal{R}_w}} \prod_{(i,j) \in S} z_{i,j}.$$

A permutation w for which the matrix Schubert variety corresponds to the variety of $p \times q$ matrices of rank at most r, with $p \leq q$, satisfies the condition in Theorem 1.7 if and only if $r \in \{0, 1, p-1, p\}$. Theorem 1.7 generalizes the known universal Gröbner bases for the ideals of maximal minors and the ideals of 2×2 minors. In Section 2.3, we illustrate the power of Theorem 1.5 by giving a short proof of the known universal Gröbner bases for the ideal of 2×2 minors.

We give an explicit description in terms of pattern avoidance of the permutations w for which all coefficients of both $\mathfrak{S}_w(x_i)$ and $\mathfrak{S}_{w^{-1}}(x_j)$ are either 0 or 1, see Proposition 4.1. We also give a positive description of these permutations in terms of their Rothe diagrams, see Theorem 4.5. See Figure 4 for an example of a large permutation in this class.

Remark 1.8. Matrix Schubert varieties can be defined over any field, and one can formulate the definition of the fine Schubert polynomial over any field (using Chow groups instead of homology). One can check that the multidegree of the closure of a matrix Schubert variety in $(\mathbb{P}^1)^{n^2}$ depends only on the characteristic of the field, but it is not obvious that it is independent of the field. See [BDG$^+$24, Section 2.3] for a discussion in the case of determinantal varieties. The proofs of all statements in this paper work over any field. In particular, for any w such that all coefficients of both $\mathfrak{S}_w(x_i)$ and $\mathfrak{S}_{w^{-1}}(x_j)$ are equal to 0 or 1, the multidegree of the closure of X_w in $(\mathbb{P}^1)^{n^2}$ is independent of the characteristic, and $\mathcal{R}_w$ is a universal Gröbner basis.

For arbitrary w, the relations in $\mathcal{R}_w$ are not a universal Gröbner basis, and they are not enough to compute the support of $\mathfrak{F}_w(z_{i,j})$. See Example 5.1.

In the course of proving Theorem 1.7, we develop several tools to relate $\mathfrak{F}_w(z_{i,j})$ to fine Schubert polynomials of "smaller"[1] permutations. These tools include a version of Lascoux's transition formula [Las01] (Proposition 3.17) and Knutson's cotransition formula [Knu22] (Proposition 3.6). We highlight one tool, which shows that fine Schubert polynomials behave well with respect to pattern containment. For $w \in S_n$, let

$$D(w) := \{(i,j) \in [n]^2 : w(i) > j, w^{-1}(j) > i\}$$

be the **Rothe diagram** of w. Define a **region** of $D(w)$ to be a maximal subset of $D(w)$ connected via adjacency. Note that for all (p,q) in a region, $r_{p \times q}(w)$ is the same. We call this the **rank** of this region.

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[1]This notion of "smaller" does not always mean smaller in Bruhat order.

Proposition 1.9. *Let $w \in S_n$ be a permutation. For some $k \in [n]$, let $v \in S_{n-1}$ be the permutation given by the relative ordering of $w(1), \ldots, w(k-1), w(k+1), \ldots, w(n)$. Set $M = \prod z_{i,j}$, where the product is over pairs (i,j) where $(i,j) \in D(w)$ and $i = k$ or $j = w(k)$. Then*

$$\mathfrak{F}_w(z_{i,j}) = M \cdot \mathfrak{F}_v(z_{i,j})_{i \in [n] \setminus k, j \in [n] \setminus w(k)} + E,$$

where E has non-negative coefficients.

In particular, the family of $w \in S_n$ where $\mathfrak{F}_w(z_{i,j})$ has all coefficients either 0 or 1 is closed under pattern containment. Proposition 1.9 is inspired by [FMSD21, Theorem 1.2], which gives a version of Proposition 1.9 for Schubert polynomials. Our proof of Proposition 1.9 generalizes to give a simple geometric proof of [FMSD21, Theorem 1.2], see Remark 3.4.

Our paper is organized as follows. In Section 2, we study the relationship between Gröbner bases and closures of varieties in a product of projective lines, and we prove Theorem 1.5. In Section 3, we develop several inductive tools to study fine Schubert polynomial. In Section 4, we analyze the permutations w with all coefficients of both $\mathfrak{S}_w(x_i)$ and $\mathfrak{S}_{w^{-1}}(x_j)$ equal to 0 or 1.

2. Fine multidegrees and Gröbner bases

In this section, we prove some results on how the ideal of an affine variety relates to the homology classes of the closure of that variety in a product of projective lines. We then prove Theorem 1.5.

2.1. **Fine multidegrees.** For a subscheme Y of $(\mathbb{P}^1)^N$ such that all irreducible components have codimension d, let $\mathfrak{F}_Y(z_i) \in \mathbb{Z}[z_1, \ldots, z_N]$ be the unique squarefree polynomial such that

$$[Y] = \mathfrak{F}_Y([H_i]) \cap [(\mathbb{P}^1)^N].$$

Then $\mathfrak{F}_Y(z_i)$ is homogeneous of degree d and has nonnegative coefficients. We call $\mathfrak{F}_Y(z_i)$ the **fine multidegree polynomial** of Y. If Y is irreducible, then this is the covolume polynomial of Y in the sense of [Alu24], and it is a typical example of a dually Lorentzian polynomial in the sense of [RSW23].

Let $\mathbb{C}[a_1, \ldots, a_N, b_1, \ldots, b_N]$ be the coordinate ring of $(\mathbb{P}^1)^N$, which has a $\mathbb{Z}^N$ grading, with the degree of a_i and b_i equal to the ith standard basis vector $\mathbf{e}_i$. Then the ideal $\mathcal{I}_Y$ of Y in $(\mathbb{P}^1)^N$ is also $\mathbb{Z}^N$-graded. When Y is pure dimensional, the fine multidegree polynomial can be computed in term of the fine Hilbert series of $\mathcal{I}_Y$, i.e., the $\mathbb{Z}^N$-graded Hilbert series of $\mathcal{I}_Y$, see, e.g., [MS05, Section 8.5].

Let $X \subset \mathbb{A}^N$ be a subscheme. Let $\overline{X}$ be the closure of X in $(\mathbb{P}^1)^N$. Let $\mathbb{C}[a_1, \ldots, a_N]$ be the coordinate ring of $\mathbb{A}^N$, and let $I_X \subset \mathbb{C}[a_1, \ldots, a_N]$ be the ideal of X. For $f \in \mathbb{C}[a_1, \ldots, a_N]$, let $f^h \in \mathbb{C}[a_1, \ldots, a_N, b_1, \ldots, b_N]$ be the multihomogenization of f, i.e., f^h is the unique multihomogeneous polynomial such that $f^h(a_1, \ldots, a_N, 1, \ldots, 1) = f(a_1, \ldots, a_N)$. Let

$$\mathcal{I}_X^h := (f^h : f \in I_X).$$

Note that $\mathcal{I}_X^h = \mathcal{I}_{\overline{X}}$, the ideal of $\overline{X}$ in $\mathbb{C}[a_1, \ldots, a_N, b_1, \ldots, b_N]$.

For a polynomial $f \in \mathbb{C}[a_1, \ldots, a_N]$, the **spread** of f is the subset

$$\mathrm{spr}(f) = \{i : a_i \text{ appears in a monomial with nonzero coefficient in } f\} \subset [N].$$

I.e., the spread of f is the variables which it uses. Similarly, for $f \in \mathbb{C}[a_1, \ldots, a_N, b_1, \ldots, b_N]$, we set

$$\mathrm{spr}(f) = \{i : a_i \text{ or } b_i \text{ appears in a monomial with nonzero coefficient in } f\} \subset [N].$$

Note that $\mathrm{spr}(f) = \mathrm{spr}(f^h)$, and if $g \in \mathbb{C}[a_1, \ldots, a_N, b_1, \ldots, b_N]$ is multihomogeneous of degree $(d_1, \ldots, d_N)$, then $\mathrm{spr}(g) = \{i : d_i \neq 0\}$.

We now prove a few results which relate the fine multidegree of $\overline{X}$ to the spreads of elements of I_X. For $U \subset [N]$, let $\pi_U \colon (\mathbb{P}^1)^N \to (\mathbb{P}^1)^U$ be the coordinate projection.

Lemma 2.1. *Let Y be a subscheme of $(\mathbb{P}^1)^N$. For $U \subset [N]$, we have $\pi_U(Y) \neq (\mathbb{P}^1)^U$ if and only if there is $f \in \mathcal{I}_Y$ with $\mathrm{spr}(f) \subset U$.*

Proof. Because π_U is a closed map, $\pi_U(Y)$ is a Zariski closed subset of $(\mathbb{P}^1)^U$, so it is a proper subset if and only if it is contained in some divisor, i.e., if there is a nonzero multihomogeneous polynomial $g \in \mathbb{C}[a_i, b_i : i \in U]$ such that $\pi_U(Y) \subset V(g)$. Then g defines an element of $\mathcal{I}_Y$ with spread contained in U, via the inclusion of $\mathbb{C}[a_i, b_i : i \in U]$ into $\mathbb{C}[a_1, \ldots, a_N, b_1, \ldots, b_N]$.

For the reverse direction, if $\mathrm{spr}(f) \subset U$ for some nonzero $f \in \mathcal{I}_Y$, then f, viewed as an element of $\mathbb{C}[a_i, b_i : i \in U]$, will vanish on $\pi_U(Y)$, so $\pi_U(Y)$ must be properly contained in $(\mathbb{P}^1)^U$. $\qquad\square$

The following lemma is closely related to the fact that, when Y is irreducible, the support of the fine multidegree polynomial of Y is the dual of the algebraic matroid of Y.

Lemma 2.2. *Suppose all irreducible components of Y have codimension d. If $\pi_U(Y) = (\mathbb{P}^1)^U$, then there is $V \supset U$ with $|V| = N - d$ and $\pi_V(Y) = (\mathbb{P}^1)^V$.*

Proof. By replacing Y with the reduction of a component which dominates $(\mathbb{P}^1)^U$, we can assume that Y is integral. It suffices to show that if $|U| < N - d$, then we can find $W \supset U$ with $|W| = |U| + 1$ and $\pi_W(Y) = (\mathbb{P}^1)^W$.

For each $k \in [N] \setminus U$, if $\pi_{U \cup k}(Y) \neq (\mathbb{P}^1)^{U \cup k}$, then by Lemma 2.1 there is $f_k \in \mathcal{I}_Y$ whose spread is contained in $U \cup k$. As $\pi_U(Y) = (\mathbb{P}^1)^U$, the spread of f_k must contain k. Inside of the function field of Y, f_k gives an algebraic relation satisfied by a_k over the field generated by $\{a_\ell : \ell \in U\}$. If this holds for all $k \in [N] \setminus U$, this implies that the function field of Y (which is generated by $\{a_1, \ldots, a_N\}$) is algebraic over the subfield generated by $\{a_\ell : \ell \in U\}$. But this contradicts that $|U| < N - d$. $\qquad\square$

Proposition 2.3. *If all irreducible components of Y have codimension d, then we can describe the spreads of nonzero elements of $\mathcal{I}_Y$, as follows. For $S \subset [N]$,*

there is $0 \neq f \in \mathcal{I}_Y$ with $\mathrm{spr}(f) \cap S = \emptyset$ if and only if $[z^T]\mathfrak{F}_Y(z_i) = 0$ for all $T \subset S$ with $|T| = d$.

Proof. Lemma 2.1 implies that if $\mathrm{spr}(f) \cap S = \emptyset$ for some nonzero $f \in \mathcal{I}_Y$, then $[z^T]\mathfrak{F}_Y(z_i) = 0$ for all $T \subset S$ because $\pi_{T^c}(Y)$ is a proper subset of $(\mathbb{P}^1)^{T^c}$. The other direction follows from Lemma 2.2. $\qquad\square$

Lemma 2.4. *There is $g \in \mathcal{I}_X^h$ with $\mathrm{spr}(g) = C$ if and only if there is $f \in I_X$ with $\mathrm{spr}(f) = C$.*

Proof. One direction follows from the fact that $\mathrm{spr}(f) = \mathrm{spr}(f^h)$.

For the other direction, note that if S is the spread of some $0 \neq g \in I_X$ and $T \supset S$, then $T = \mathrm{spr}(a^{T \setminus S} \cdot g)$. It therefore suffices to show that the minimal elements of $\{\mathrm{spr}(g) : 0 \neq g \in \mathcal{I}_X^h\}$ are the spreads of elements in I_X.

As the spread of $g \in \mathcal{I}_X^h$ is the union of the spreads of its multihomogeneous parts, we may assume that g is multihomogeneous. Then $\mathrm{spr}(g) = \mathrm{spr}(g(a_1, \ldots, a_N, 1, \ldots, 1))$. $\qquad\square$

2.2. **A Gröbner basis criterion.** We now prove a generalization of Theorem 1.5 which does not assume that all irreducible components of X have the same dimension.

Theorem 2.5. *Let X be a subscheme of $\mathbb{A}^N$, and let $f_1, \ldots, f_r \in I_X$ be nonzero squarefree polynomials. Suppose that, for all $S \subset [N]$, we have*

$$S \cap \mathrm{spr}(f_i) = \emptyset \text{ for some } i \text{ if and only if } \pi_{S^c}(\overline{X}) \neq (\mathbb{P}^1)^{S^c}.$$

Then $\{f_1, \ldots, f_r\}$ is a universal Gröbner basis for I_X. If X is pure dimensional, then all coefficients of $\mathfrak{F}_{\overline{X}}(z_i)$ are 0 or 1.

Proof of Theorem 1.5. By Theorem 2.5, it suffices to check that if all irreducible components of X have codimension d, then $\pi_{S^c}(\overline{X}) \neq (\mathbb{P}^1)^{S^c}$ if and only if $[z^T]\mathfrak{F}_{\overline{X}}(z_i) = 0$ for all $T \subset S$ with $|T| = d$. This follow by combining Proposition 2.3, Lemma 2.1, and Lemma 2.4. $\qquad\square$

The proof of Theorem 2.5 shows that $\{f_1^h, \ldots, f_r^h\}$ is a universal Gröbner basis for $\mathcal{I}_X^h$, and, if X is pure dimensional, $\mathcal{I}_X^h$ is a *Cartwright–Sturmfels** ideal in the sense of [CDNG18, CDNG20]. This family of ideals has remarkable homological properties. For example, any minimal generating set of a Cartwright–Sturmfels* ideal is a universal Gröbner basis [CDNG20, Proposition 1.9].

In the proof of Theorem 2.5, we will need a combinatorial result. Let Δ be a simplicial complex on $[N]$, and let $[N, \bar{N}] = [1, \ldots, N, \bar{1}, \ldots, \bar{N}]$ be a set of size $2N$ equipped with the obvious involution $\bar{\cdot}$. Let $q \colon [N, \bar{N}] \to [N]$ be the quotient by $\bar{\cdot}$. For a minimal nonface G of Δ, a **lift** $\widetilde{G}$ to $[N, \bar{N}]$ is a subset of $[N, \bar{N}]$ that contains exactly one of $\{i, \bar{i}\}$ for each $i \in G$ and has $q(\widetilde{G}) = G$.

Suppose we have chosen a lift $\widetilde{G}$ for each minimal nonface G of Δ. Let $\widetilde{\Delta}$ be the complex on $[N, \bar{N}]$ whose minimal nonfaces are given by the $\widetilde{G}$. For each face F of $\widetilde{\Delta}$, $\{i : \{i, \bar{i}\} \subset F\}$ is a face of Δ. We say that the choice of lifts is **full** if, for each facet of Δ, there is a face $\widetilde{F}$ of $\widetilde{\Delta}$ such that $\widetilde{F} \cap \{i, \bar{i}\} \neq \emptyset$ for all i, and $F = \{i : \{i, \bar{i}\} \subset \widetilde{F}\}$. We say that $\widetilde{F}$ **extends** F.

Lemma 2.6. *Suppose that $\widetilde{\Delta}$ arises from a full choice of lifts. Then every facet of Δ has a unique facet of $\widetilde{\Delta}$ extending it, and all facets of $\widetilde{\Delta}$ extend a facet of Δ.*

Proof. We induct on N; the statement has no content for $N = 0$. Let $D := \{E \in \Delta : N \notin E\}$ be the deletion of N and $L := \{E \in D : E \cup \{N\} \in \Delta\}$ be the link of N.

Each minimal nonface of Δ gives rise to a nonface of L, with a minimal nonface G which contains N giving rise to the nonface $G \setminus N$ of L, and a minimal nonface H which does not contain N giving rise to the nonface H of L. Every minimal nonface of L arises in this way, and these nonfaces all have induced choices of lift. Let F be a facet of Δ which contains N. Because Δ is full, there is a facet $\widetilde{F}$ of $\widetilde{\Delta}$ extending F. Then $\widetilde{F} \setminus \{N, \bar{N}\}$ extends the facet $F \setminus N$ of L, so we deduce that this induced choice of lifts for L is full.

The faces of L are of the form $F \setminus \{N\}$, where F is a face of Δ containing N. If we take a face of $\widetilde{L}$, we can obtain faces of $\widetilde{\Delta}$ by adding any (possibly empty) subset of $\{N, \bar{N}\}$.

Each minimal nonface G of Δ which does not contain N gives rise to a minimal nonface G of D, and every minimal nonface of D arises in this way. We have an induced choice of lifts for D. Because the choice of lifts for Δ are full, the choice of lifts for D are full.

There are two types of faces of D: faces F such that $F \cup \{N\}$ is a face of Δ, and faces K such that $K \cup \{N\}$ is not a face of Δ. Such faces F also correspond to faces of L. We can obtain a face of $\widetilde{\Delta}$ by taking a face of the first kind in $\widetilde{D}$ and adding a subset of $\{N, \bar{N}\}$; we get the same faces from $\widetilde{L}$.

Let $\widetilde{K}$ be a face of $\widetilde{D}$ which corresponds to the second kind of face of D. We know that $\widetilde{K}$ is a face of $\widetilde{\Delta}$ and that $\widetilde{K} \cup \{N, \bar{N}\}$ is not a face of $\widetilde{\Delta}$. Also, at most one of $\widetilde{K} \cup \{N\}$ and $\widetilde{K} \cup \{\bar{N}\}$ are faces of $\widetilde{\Delta}$ (because $K \cup \{N\}$ contains at least 1 minimal nonface of Δ). However, as the lift of $\widetilde{\Delta}$ is full, for each facet of the

second kind in D, we can extend it to a facet of $\widetilde{\Delta}$ that contains N or $\bar{N}$. So we can in fact extend $\widetilde{K}$ in exactly one way.

This describes all of the faces of $\tilde{\Delta}$. The claim follows, as this description of the facets of $\widetilde{\Delta}$ is in bijection with the facet of Δ. $\qquad\square$

Example 2.7. Suppose Δ has facets $\{1,4\}, \{1,5\}, \{2,5\}, \{3,5\}$. If we choose the lifts
$$\{\bar{4}, \bar{5}\}, \{\bar{3}, \bar{4}\}, \{\bar{2}, \bar{3}\}, \{\bar{1}, 2\}, \{\bar{1}, \bar{3}\}, \{2, \bar{4}\}$$
of the minimal nonfaces of Δ, then the facets of $\widetilde{\Delta}$ are
$$\{1, \bar{1}, \bar{2}, 3, 4, \bar{4}, 5\}, \{1, \bar{1}, \bar{2}, 3, 4, 5, \bar{5}\}, \{1, 2, \bar{2}, 3, 4, 5, \bar{5}\}, \{1, 2, 3, \bar{3}, 4, 5, \bar{5}\},$$
and so this lift is full.

Example 2.8. The external activity complex of a matroid [AB16] is an example of a full lift $\widetilde{\Delta}$, where Δ is the independence complex of the matroid.

By the Stanley–Reisner correspondence, the irreducible components of the vanishing locus in affine space of a squarefree monomial ideal are in bijection with the facets of the simplicial complex whose nonfaces are the squarefree monomials in the ideal, see, e.g., [MS05, Theorem 1.7]. We will need a variation of this for vanishing loci of squarefree monomial ideals in $(\mathbb{P}^1)^N$.

Let $\widetilde{\Delta}$ be a simplicial complex on $[N, \bar{N}]$. For each minimal nonface F of $\widetilde{\Delta}$, let $m_F = (\prod_{i \in F} a_i)(\prod_{\bar{j} \in F} b_j)$, and let $\mathcal{I} = (m_F : F \text{ minimal nonface of } \widetilde{\Delta})$. Then the irreducible components of the vanishing locus of $\mathcal{I}$ in $\mathbb{A}^{[N, \bar{N}]}$ are in bijection with facets of $\widetilde{\Delta}$, with the locus $\bigcap_{\ell \in [N] \setminus T} V(a_\ell) \cap \bigcap_{p \in [\bar{N}] \setminus T} V(b_p)$ corresponding to the facet T.

We obtain the vanishing locus of $\mathcal{I}$ in $(\mathbb{P}^1)^N$ by taking the vanishing locus in $\mathbb{A}^{[N, \bar{N}]}$, intersecting with the open set $U = \{a_i \neq 0 \text{ or } b_i \neq 0 \text{ for each } i\}$, and then taking the image under the map $U \to (\mathbb{P}^1)^N$ given by quotienting by the natural $\mathbb{G}_m^N$ action on U.

We see that the components of $V(\mathcal{I})$ in $(\mathbb{P}^1)^N$ are in bijection with facets T of Δ such that $\{i, \bar{i}\} \cap T \neq \emptyset$ for all i; with such T corresponding to the locus V_T in $(\mathbb{P}^1)^N$ where the ith coordinate is allowed to be arbitrary if $\{i, \bar{i}\} \subset T$, the ith coordinate is fixed to be ∞ if $i \in T$ but $\bar{i} \notin T$, and the ith coordinate is fixed to be 0 if $\bar{i} \in T$ but $i \notin T$.

The facets T of $\widetilde{\Delta}$ which have $T \cap \{i, \bar{i}\} = \emptyset$ for some i do not affect $V(\mathcal{I})$: they correspond to associated primes of $\mathcal{I}$ which are contained in the irrelevant ideal.

Proof of Theorem 2.5. Let $<$ be a term order on $\mathbb{C}[a_1, \ldots, a_N]$. We can extend $<$ to a term order on $\mathbb{C}[a_1, \ldots, a_N, b_1, \ldots, b_N]$ such that $a^S b^T < a^{S'} b^{T'}$ if $a^S < a^{S'}$; the initial term of a multihomogeneous polynomial in $\mathbb{C}[a_1, \ldots, a_N, b_1, \ldots, b_N]$ does not depend on the choice of extension. We have an inclusion
$$V(\mathrm{in}_<(\mathcal{I}_X^h)) \subset V(\mathrm{in}_<(f_1^h), \ldots, \mathrm{in}_<(f_r^h)) \subset (\mathbb{P}^1)^N.$$
Let Δ be the simplicial complex on $[N]$ whose faces are given by subsets S of $[N]$ such that $\pi_S(\overline{X}) = (\mathbb{P}^1)^S$. Every minimal nonface of Δ is the spread of some f_i. Indeed, if F is a minimal nonface of Δ, by assumption, there exists some f_i such that $\mathrm{spr}(f_i) \subset F$. By Lemma 2.1 together with Lemma 2.4, $\mathrm{spr}(f_i)$ is a nonface, so $\mathrm{spr}(f_i) = F$.

Because each f_i is squarefree, $\mathrm{in}_<(f_i^h)$ is a squarefree monomial where, for each j, at most one of $\{a_j, b_j\}$ appears. Additionally, we have $\mathrm{spr}(\mathrm{in}_<(f_i^h)) = \mathrm{spr}(f_i)$. We identify this squarefree monomial with a subset of $[N, \bar{N}]$, where the b-variables correspond to $[\bar{N}]$. Then this is a lift of the nonface $\mathrm{spr}(f_i)$ of Δ. Let $\widetilde{\Delta}$ be the complex on $[N, \bar{N}]$ with these as nonfaces. Note that $\widetilde{\Delta}$ is contained in a lift of Δ (in the sense of Lemma 2.6), if we use just the f_i corresponding to minimal nonfaces of Δ.

As described above, each facet T which intersect $\{i, \bar{i}\}$ for each i gives rise to a locus V_T in $(\mathbb{P}^1)^N$. We have

$$V(\operatorname{in}_<(f_1^h), \ldots, \operatorname{in}_<(f_r^h)) = \bigcup_{T \in \widetilde{\Delta}, \, T \cap \{i, \bar{i}\} \neq \emptyset \text{ for all } i} V_T.$$

As $\{\operatorname{spr}(f) : 0 \neq f \in I_X\} = \{\operatorname{spr}(\operatorname{in}_<(f^h)) : 0 \neq f \in I_X\}$, Lemma 2.1 implies that $\pi_S(V(\operatorname{in}_<(\mathcal{I}_X^h))) = (\mathbb{P}^1)^S$ for each $S \in \Delta$. This implies that, for each facet F of Δ, there is a face $\widetilde{F}$ of $\widetilde{\Delta}$ such that $F = \{i : \{i, \bar{i}\} \subset \widetilde{F}\}$. Then Lemma 2.6 gives that this assignment is a bijection between facets of $\widetilde{\Delta}$ and facets of Δ. In particular, there are no associated primes of $(\operatorname{in}_<(f_1^h), \ldots, \operatorname{in}_<(f_r^h))$ which are contained in the irrelevant ideal. Because $V(\operatorname{in}_<(f_1^h), \ldots, \operatorname{in}_<(f_r^h))$ is reduced, we deduce that $V(\operatorname{in}_<(\mathcal{I}_X^h)) = V(\operatorname{in}_<(f_1^h), \ldots, \operatorname{in}_<(f_r^h))$. Indeed, for every proper subscheme Y of $V(\operatorname{in}_<(f_1^h), \ldots, \operatorname{in}_<(f_r^h))$, there is some facet S of Δ such that $\pi_S(Y) \neq (\mathbb{P}^1)^S$: for each facet S of Δ, $V_{\widetilde{S}}$ is the unique component of $V(\operatorname{in}_<(f_1^h), \ldots, \operatorname{in}_<(f_r^h))$ which dominates $(\mathbb{P}^1)^S$. Therefore $\{f_1^h, \ldots, f_r^h\}$ is a Gröbner basis for $\mathcal{I}_X^h$. Dehomogenizing gives that $\{f_1, \ldots, f_r\}$ is a Gröbner basis with respect to $<$ (see, e.g., [Eis95, Exercise 15.39]), and so it is a universal Gröbner basis.

Suppose that X is pure dimensional, so $\mathfrak{F}_{\overline{X}}(z_i)$ is defined. The multidegree of $\overline{X}$ is the same as the multidegree of $\operatorname{in}_<(\mathcal{I}_X^h)$ [KM05, Theorem 1.7.1]. For $T \subset [N, \bar{N}]$ with $T \cap \{i, \bar{i}\} \neq \emptyset$ for each i, we have

$$\mathfrak{F}_{V_T}(z_i) = \prod_{\{i, \bar{i}\} \not\subset T} z_i, \quad \text{and} \quad \mathfrak{F}_{\overline{X}}(z_i) = \sum_{T \text{ facet of } \widetilde{\Delta}} \mathfrak{F}_{V_T}(z_i).$$

In particular, because the facets of $\widetilde{\Delta}$ are in bijection with the facet of Δ, the multidegrees of the V_T appearing are distinct. Therefore the coefficients in $\mathfrak{F}_{\overline{X}}(z_i)$ are all either 0 or 1. $\qquad \square$

Remark 2.9. Under the hypotheses of Theorem 2.5, $\operatorname{in}_\omega(I_X)$ is generated by $\{\operatorname{in}_\omega(f_1), \ldots, \operatorname{in}_\omega(f_r)\}$ for any weight ω. Indeed, if we choose some ω' corresponding to a monomial order, then $\operatorname{in}_{\omega'}(\operatorname{in}_\omega(I_X)) = \operatorname{in}_{\omega + \epsilon \omega'}(I_X)$ for some ϵ sufficiently small, and $\operatorname{in}_{\omega + \epsilon \omega'}(I_X)$ is a monomial ideal [Stu96, Proposition 1.11, Proposition 1.13]. Because ω' corresponds to a monomial order, the fact that $\{\operatorname{in}_{\omega'}(\operatorname{in}_\omega(f_1)), \ldots, \operatorname{in}_{\omega'}(\operatorname{in}_\omega(f_r))\}$ generates $\operatorname{in}_{\omega + \epsilon \omega'}(I_X)$ implies the claim.

A similar argument works in the setting of valued Gröbner basis in the sense of [CM19]: if $\operatorname{in}_\omega(I_X)$ is a monomial ideal, then the proof of Theorem 2.5 shows that the f_i are a Gröbner basis. At least if I is homogeneous, then the $\operatorname{in}_\omega(I_X)$ are a Gröbner basis even if $\operatorname{in}_\omega(I_X)$ is not monomial, using [MS15, Corollary 2.4.10] and [CM19, Remark 2.11].

Finally, we give a description of the fine multidegree of X in terms of Gröbner degenerations. This description is not needed in the sequel.

Proposition 2.10. *Let X be a subscheme of $\mathbb{A}^N$ whose irreducible components all have codimension d. For each set $S \subset [N]$ of size d, the coefficient of z^S in $\mathfrak{F}_{\overline{X}}(z_i)$ is the largest multiplicity with which the ideal $(a_i : i \in S)$ appears as an associated prime in a Gröbner degeneration of I_X.*

We first outline the geometry of Proposition 2.10. For a term order $<$, the coefficient of z^S in $\mathfrak{F}_{\overline{X}}(z_i)$ records the number of components (counted with multiplicity) of $V(\operatorname{in}_<(\mathcal{I}_X))$ which have multidegree S. We have

$$V(\operatorname{in}_<(I_X)) = V(\operatorname{in}_<(\mathcal{I}_X)) \cap \mathbb{A}^N,$$

so this coefficient gives an upper bound on the multiplicity of $V(a_i : i \in S)$ in $V(\operatorname{in}_<(I_X))$. If we choose a lexicographic order that starts with S, then all components of $V(\operatorname{in}_<(\mathcal{I}_X))$ with multidegree S intersect $\mathbb{A}^N$, so this bound is attained.

Proof of Proposition 2.10. Let $U = S^c$. Let $<$ be a term order on $\mathbb{C}[a_1, \ldots, a_N]$. As in the proof of Theorem 1.7, we extend $<$ to $\mathbb{C}[a_1, \ldots, a_N, b_1, \ldots, b_N]$. Then the multidegree of $\operatorname{in}_<(\mathcal{I}_X^h)$ is equal to the multidegree

of $\mathcal{I}_X^h$ [KM05, Theorem 1.7.1]. The multidegree of $\mathrm{in}_<(\mathcal{I}_X^h)$ is equal to the sum of the multiplicities of the components of $V(\mathrm{in}_<(\mathcal{I}_X^h))$ which are of the form V_T for some T with $U = \{i : \{i, \bar{i}\} \subset T\}$. The multiplicity that $(a_i : i \in S)$ appears as an associated prime in $\mathrm{in}_<(I_X)$ is equal to the multiplicity that $V_{U \cup \overline{U} \cup \overline{S}}$ appears. This shows that $[z^S]\mathfrak{F}_{\overline{X}}(z_i)$ is an upper bound for the multiplicity of $(a_i : i \in S)$.

To show that this upper bound is attained, it suffices to construct a term order for which the multiplicity of $V_{U \cup \overline{U} \cup \widetilde{S}}$ is 0 if $\widetilde{S}$ is a lift of S which is not $\overline{S}$. We may assume that $S = \{1, \dots, d\}$, and let $<$ be the lexicographic term order induced by $1 > 2 > \cdots > N$.

Let $\{f_1^h, \dots, f_r^h\}$ be a Gröbner basis for $<$ consisting of multihomogenizations of elements of I_X. For existence of such a Gröbner basis, see, e.g., [Eis95, Exercise 15.39]. Let $\widetilde{S}$ be a lift of S which is not $\overline{S}$, and let j be the smallest element of $S \cap \widetilde{S}$. The choice of term order guarantees that if $\mathrm{in}_<(f_k^h)$ contains b_j, then it must contain a_i for some $i < j$. In particular, as $i \notin \widetilde{S}$, if $\mathrm{in}_<(f_k^h)$ vanishes on $V_{U \cup \overline{U} \cup \widetilde{S}}$, then it also vanishes on $V_{U \cup \overline{U} \cup \widetilde{S} \cup j}$. This means $V_{U \cup \overline{U} \cup \widetilde{S}}$ is not an irreducible component of $V(\mathrm{in}_<(f_1^h), \dots, \mathrm{in}_<(f_r^h))$. $\square$

In particular, Proposition 2.10 shows that all coefficients of $\mathfrak{F}_{\overline{X}}(z_i)$ are either 0 or 1 if and only if all Gröbner degenerations of X are reduced.

The proof of Proposition 2.10 shows that, for any S, $[z^S]\mathfrak{F}_w(z_{i,j})$ is the multiplicity of $\mathbb{A}^{S^c}$ in $V(\mathrm{in}_<(I_w))$, where $<$ is a lexicographic term order in which S occurs first. In some cases, this case be used to compute $[z^S]\mathfrak{F}_w(z_{i,j})$. For example, if there is an antidiagonal term order which is a lexicographic term order in which S occurs first, then [KM05] shows that $[z^S]\mathfrak{F}_w(z_{i,j})$ is 1 if S is the set of crosses in a reduced pipe dream for w, and is 0 otherwise.

2.3. Gröbner bases for the varieties of rank 1 matrices.

We now use Theorem 1.5 to give a short proof of the known universal Gröbner basis for the ideal generated by the 2×2 minors of a generic matrix.

Let $X \subset \mathbb{A}^{pq}$ be the variety of $p \times q$ matrices of rank at most 1. It is well-known (and follows easily from the proof of Proposition 2.11) that X is irreducible of dimension $p + q - 1$. The ideal I_X of X is generated by 2×2 minors, but these are *not* a universal Gröbner basis. We will need to use the merge operation from the introduction in order to construct new relations.

For this, it is helpful to identify $[p] \times [q]$ with the edges of the complete bipartite graph $K_{p,q}$. Then the spread of a 2×2 minor corresponds to a 4-cycle in this graph. Suppose we have squarefree elements $f_1, f_2 \in I_X$ corresponding to subgraphs G_1 and G_2. Then they satisfy the hypothesis used to define $\mathcal{R}_w$ if and only if there is a unique edge (α, β) in both G_1 and G_2, and α and β are the only vertices used by both G_1 and G_2. In this case, the spread of $\mathrm{merge}_{(\alpha, \beta)}(f_1, f_2)$ corresponds to the subgraph $G_1 \cup G_2 \setminus \{(\alpha, \beta)\}$.

In particular, every cycle in $K_{p,q}$ can be obtained in this way: for $k \geq 2$, each $(2k + 2)$-cycle is obtained by merging a $2k$-cycle with a 4-cycle. Explicitly, given a cycle with edges $(i_1, j_1), (i_2, j_1), \dots, (i_k, j_k), (i_1, j_k)$, we have the relation

$$\text{(2)} \qquad \prod_{\ell=1}^k a_{i_\ell, j_\ell} - \prod_{\ell=1}^k a_{i_{\ell+1}, j_\ell} \in I_X, \qquad \text{where } i_{k+1} = i_1.$$

In particular, Proposition 2.3 shows that if $S \subset [p] \times [q]$ has $[z^S]\mathfrak{F}_{\overline{X}}(z_{i,j}) > 0$, then S^c correspond to a forest in $K_{p,q}$. As $\dim X = p + q - 1$, we must have $|S^c| = p + q - 1$, so S^c must be a spanning tree.

Proposition 2.11. [Stu96, Proposition 4.11 and 8.11] [Vil15, Proposition 10.1.11] *Let $X \subset \mathbb{A}^{pq}$ be the variety of $p \times q$ matrices of rank at most 1. Then the relations in (2) are a universal Gröbner basis for I_X.*

Proof. By Theorem 1.5, it suffices to show that if S^c is a spanning tree of $K_{p,q}$, then $[z^S]\mathfrak{F}_{\overline{X}}(z_{i,j}) > 0$. Let $U = S^c$. Every $p \times q$ matrix of rank at most 1 can be written as a product of a $p \times 1$ matrix and a $1 \times q$ matrix. This gives a surjective map

$$f \colon \mathbb{A}^p \times \mathbb{A}^q \to X \subset \mathbb{A}^{pq}, \qquad a_{i,j} \mapsto u_i \otimes v_j,$$

where $(u_1, \ldots, u_p)$ are the coordinates on $\mathbb{A}^p$ and $(v_1, \ldots, v_q)$ are the coordinates on $\mathbb{A}^q$.

By Lemma 2.1 and Proposition 2.3, $[z^S]\mathfrak{F}_{\overline{X}}(z_{i,j}) > 0$ if and only if the projection of X onto $\mathbb{A}^U$ is dominant. This happens if and only if the composition $\mathbb{A}^p \times \mathbb{A}^q \to \mathbb{A}^U$ is dominant. To prove that this is dominant when U is a spanning tree, it suffices to show that the Jacobian matrix has full rank. The Jacobian of the composition $\mathbb{A}^p \times \mathbb{A}^q \to \mathbb{A}^U$ is a $(p+q) \times (p+q-1)$ whose rows are labeled by vertices of $K_{p,q}$ and whose columns are labeled by edges in U. In the column labeled by an edge (i,j), as $\mathrm{d}(u_i v_j) = u_i \, \mathrm{d}\, v_j + v_j \, \mathrm{d}\, u_i$, we have an entry of v_j in the row labeled by the ith vertex, an entry of u_i in the row labeled by the jth vertex, and 0 for all other entries. This is the incidence matrix of the subgraph corresponding to U, and so it has full rank if (and only if) U is a spanning tree. □

Remark 2.12. The key to the simple proof of Proposition 2.11 is the explicit description of the subsets of $[p] \times [q]$ which avoid the spreads of the relations: they corresponds to forests in a bipartite graph. We do not know a simple description for the sets which avoid the spreads of elements of the sets $\mathcal{R}_w$ in Theorem 1.7.

Remark 2.13. The same argument can be used to give a short proof that the maximal minors are a universal Gröbner basis for the ideal that they generate [SZ93, BZ93]. Indeed, the Jacobian matrix of the map obtained from the fact that a matrix of rank at most $p-1$ can be written as a product of a $p \times (p-1)$ matrix and a $(p-1) \times q$ matrix is very simple, see [BDG$^+$24, Theorem 1.1(3), Corollary 3.2].

Remark 2.14. Let $X \subset \mathbb{A}^{16}$ be the variety of 4×4 matrices of rank at most 2. Then one can check that $[z_{1,1}z_{2,2}z_{3,3}z_{4,4}]\mathcal{F}_{\overline{X}}(z_{i,j}) = 2$ (in any characteristic), so Theorem 1.5 cannot be applied to X. Using Proposition 1.9, one can check that the fine multidegree polynomial of the locus of $p \times q$ matrices of rank at most r has all coefficients equal to 0 or 1 if and only if $r \in \{0, 1, p-1, q-1, p, q\}$.

3. Geometric tools

In this section, we develop a number of geometric tools for studying fine Schubert polynomials. As mentioned in the introduction, we do not know of an inductive formula for fine Schubert polynomials, so none of these tools can be used to compute fine Schubert polynomials in general. However, in the proof of Theorem 1.7 we show that, taken together, these tools are enough to compute the fine Schubert polynomial of certain permutations.

We set up some notation that we will use for the rest of the paper. For $w \in S_n$ and $(i,j) \in [n]^2$, define $w \cdot (i,j) := (w(i), j)$ and $(i,j) \cdot w := (i, w(j))$. For $U \subset [n]^2$, set $w \cdot U := \{w \cdot (i,j) : (i,j) \in U\}$, and define $U \cdot w$ similarly. Set $U_{A,B} = U \cap (A \times B)$. When A or B is $[n]$ we use $*$ as a shorthand, so $U_{i,*} := U \cap (\{i\} \times [n])$. For $v \in S_n$, let $v \cdot X_w$ be the variety in $\mathbb{A}^{n^2}$ obtained by applying the automorphism of $\mathbb{A}^{n^2}$ which applies the permutation v to the rows, and let $X_w \cdot v$ be the variety obtained by permuting the columns via v. For $1 \leq i < j \leq n$, let $t_{i,j}$ be the transposition (i,j), and set $s_i = t_{i,i+1}$.

We recall a consequence of Fulton's approach to intersection theory. Let X be a smooth projective variety, and let Y, Z be subvarieties whose irreducible components all have codimension k, ℓ, respectively. Let $[Y], [Z] \in H^*(X)$ be the fundamental classes of Y and Z in the cohomology of X. Let $C_1, \ldots, C_r$ be the irreducible components of $Y \cap Z$. By [Ful98, Section 6.1], we have an equality

$$[Y] \cup [Z] = \sum_{i=1}^r m_i \alpha_i,$$

where each m_i is positive and $\alpha_i \in H^{2(k+\ell)}(X)$ is pushed forward from C_i. In particular, if C_i has the expected dimension, i.e., C_i has codimension $k + \ell$, then $\alpha_i = [C_i]$. In this case, m_i is bounded above by the multiplicity of C_i [Ful98, Proposition 7.1], and so if C_i is generically reduced, then $m_i = 1$. If X

has globally generated tangent bundle, e.g., if X is a product of projective spaces, then each α_i is effective [Ful98, Theorem 12.2(a)].

We will frequently use the above properties in the following way: for some subvariety Z of $(\mathbb{P}^1)^{n^2}$, we will find an irreducible component C of $Z \cap \overline{X}_w$ of the expected dimension and use the above to obtain an equality $[Z] \cap [\overline{X}_w] = [C] + E$, where E is the class of an effective cycle, so E is a non-negative sum of coordinates subspaces. This gives a lower bound on the coefficients of $\mathfrak{F}_w(z_{i,j})$. We do not need to check that all other irreducible components of $Z \cap \overline{X}_w$ have the expected dimension.

We also recall one property of matrix Schubert varieties that will be used several times. For $w \in S_n$, let $p_w \in \mathbb{A}^{n^2}$ be the permutation matrix corresponding to w. Then $p_w \in X_w$.

Proposition 3.1. [Knu22, Lemma 4.2] *The tangent space to p_w in X_w is spanned by the matrix entries* not *in $D(w)$.*

In particular, the dimension of the tangent space of X_w at p_w is equal to $n^2 - \ell(w)$, which is the dimension of X_w. This means that if $Y \subset \mathbb{A}^{n^2}$ is a subscheme which contains X_w, and such that the tangent space of Y at p_w has dimension $n^2 - \ell(w)$, then X_w is an irreducible component of Y.

Finally, we state a general tool which will be used in this section and throughout Section 5, which states that swapping certain rows or columns induces an automorphism of the matrix Schubert variety.

Proposition 3.2. *Suppose that $w(i) < w(i+1)$. Then $s_i \cdot X_w = X_w$.*

Proof. The assumption that $w(i) < w(i+1)$ implies that $\mathrm{ess}(w)$ does not intersect the ith row. This means that multiplying on the left by s_i preserves the Fulton generators, and so it preserves X_w. □

Similarly, if $w^{-1}(i) < w^{-1}(i+1)$, then $X_w \cdot s_i = X_w$.

3.1. **Pattern avoidance.** We now prove Proposition 1.9. Choose some $k \in [n]$, and let $v := \mathrm{del}_k(w) \in S_{n-1}$ be the permutation given by the relative ordering of $w(1), \ldots, w(k-1), w(k+1), \ldots, w(n)$. View $\mathbb{A}^{(n-1)^2}$ as $\mathrm{Spec}\,\mathbb{C}[a_{i,j} : i \in [n] \setminus \{k\}, j \in [n] \setminus \{w(k)\}]$. We identify $\mathbb{A}^{(n-1)^2}$ with a subset of $\mathbb{A}^{n^2}$ via the map

$$\mathbb{C}[a_{i,j}] \to \mathbb{C}[a_{i,j} : i \in [n] \setminus \{k\}, j \in [n] \setminus \{w(k)\}] \quad a_{k,w(k)} \mapsto 1, \; a_{k,j}, \, a_{j,w(k)} \mapsto 0, \text{ and } a_{i,j} \mapsto a_{i,j} \text{ otherwise.}$$

In particular, we identify X_v with a subset of $\mathbb{A}^{n^2}$.

Lemma 3.3. *We have*

$$X_w \cap V(a_{k,w(k)} - 1) \cap V(a_{k,j} : j \neq w(k)) \cap V(a_{j,w(k)} : j \neq k) = X_v.$$

Proof. If we make the above substitutions, then each Fulton generator of I_w turns into a Fulton generator of I_v, and all Fulton generators of I_v are obtained in this way. □

Set $Z = V(a_{k,j} : (k,j) \notin D(w), j \neq w(k)) \cap V(a_{j,w(k)} : (j, w(k)) \notin D(w), j \neq k) \cap V(a_{k,w(k)} - 1)$.

Proof of Proposition 1.9. By Lemma 3.3, $\overline{X}_v$ is contained in $\overline{X}_w \cap \overline{Z}$. It follows from Proposition 3.1 that the tangent space at p_w of $\overline{X}_w \cap \overline{Z}$ is equal to the tangent space of $\overline{X}_v$ at p_w. This implies that $\overline{X}_v$ is an irreducible component of $\overline{X}_w \cap \overline{Z}$.

Recall that $M = \prod z_{i,j}$, where the product is over pairs (i,j) where $(i,j) \in D(w)$ and $i = k$ or $j = w(k)$. Let $M' = \prod_{i=k \text{ or } j=w(k)} z_{i,j}$. Note that $[\overline{Z}] = M'/M \in H^*((\mathbb{P}^1)^{n^2})$. As $\overline{X}_v$ is a component of $\overline{X}_w \cap \overline{Z}$, the discussion at the start of Section 3 gives that

$$[\overline{X}_w] \cup \overline{Z} = [\overline{X}_v] + E,$$

where E is an effective class. Because we are viewing $\overline{X}_v$ as a subspace of $(\mathbb{P}^1)^{n^2}$ via an embedding into a coordinate subspace, we have $[\overline{X}_v] = M' \cdot \mathfrak{F}_v(z_{i,j})_{i \in [n] \setminus k, j \in [n] \setminus w(k)}$. This gives that

$$M'/M \cdot \mathfrak{F}_w(z_{i,j}) = M' \cdot \mathfrak{F}_v(z_{i,j})_{i \in [n] \setminus k, j \in [n] \setminus w(k)} + E',$$

where E' has non-negative coefficients. This implies the result. $\qquad\square$

Remark 3.4. The same argument can be used to give a new proof of [FMSD21, Theorem 1.2], which gives a similar monotonicity result for Schubert polynomials, by taking the closure of X_w in $(\mathbb{P}^n)^n$ instead of $(\mathbb{P}^1)^{n^2}$, see Lemma 3.18.

3.2. **Cotransition.** In this section, we prove a version of Knutson's cotransition formula for Schubert polynomials [Knu22]. Let $w_0 = (n, n-1, \dots, 1)$ be the long element of S_n. We write $v \gtrdot w$ to mean that v covers w in Bruhat order.

Given $w \in S_n$, we say that we can *cotransition* at $k \in [n]$ if there is no $a < k$ such that $wt_{a,k} \gtrdot w$. If this is the case, then let $L_k(w) := \{wt_{k,b} : k < b, \ wt_{k,b} \gtrdot w\}$. For example, if $w \neq w_0$ and k is the minimal element such that $k + w(k) < n$, then we can cotransition at k, and $L_k(w) = \{w' \in S_n : w' \gtrdot w, \ w'(k) \neq w(k)\}$.

In [Knu22, Section 4], Knutson proves a special case of the following result, and his argument generalizes without modification.

Proposition 3.5. *Suppose we can cotransition at k. Then*

$$X_w \cap V(a_{k,w(k)}) = \bigcup_{w' \in L_k(w)} X_{w'}.$$

We use this to deduce that fine Schubert polynomials behave well with respect to cotransition.

Proposition 3.6. *Suppose we can cotransition at k. Then for any S not containing $(k, w(k))$, we have*

$$[z^S]\mathfrak{F}_w(z_{i,j}) \geq \sum_{w' \in L_k(w)} [z^{S \cup \{(k,w(k))\}}]\mathfrak{F}_{w'}(z_{i,j}).$$

Proof. In the decomposition

$$X_w \cap V(a_{k,w(k)}) = \bigcup_{w' \in L_k(w)} X_{w'},$$

each $X_{w'}$ is codimension 1 in X_w and is an irreducible component of the intersection. Therefore, for each $w' \in L_k(w)$, $\overline{X}_{w'}$ is an irreducible component of the expected dimension in $\overline{X}_w \cap V(a_{k,w(k)})$. The discussion at the start of Section 3 implies the desired bound. $\qquad\square$

3.3. **Deleting a special column.** In this section, we will develop tools that allow us to relate $\mathfrak{F}_w(z_{i,j})$ to the fine Schubert polynomial of a permutation w' that is obtained by "deleting a column" from the Rothe diagram of w. Define the *dominant part* of a permutation $w \in S_n$ to be

$$\mathrm{dom}(w) := \{(i,j) \in D(w) : (k, w(k)) \notin [i] \times [j] \text{ for all } k \in [n]\}.$$

Note that $\mathrm{dom}(w) = \emptyset$ if $w(1) = 1$. In other words, $\mathrm{dom}(w)$ is the region of $D(w)$ connected to $(1,1)$ when nonempty.

We say c is a **special column** if the following conditions are satisfied:

(1) $D(w)_{*,c} \setminus \mathrm{dom}(w) \neq \emptyset$;
(2) If r is maximal such that $(r, c) \in D(w)$, then $D(w)_{[r+1,n],[c,n]} = \emptyset$. Namely, there are no squares weakly southeast of $(r+1, c)$ in the Rothe diagram.

We will construct a permutation $\mathrm{del}_{*,c}(w)$ such that

$$(3) \qquad D(\mathrm{del}_{*,c}(w)) = \{(i,j) \in D(w) : j < c\} \cup \{(i, j-1) : j > c, (i,j) \in D(w)\}.$$

In other words, the diagram for $\mathrm{del}_{*,c}(w)$ is obtained from $D(w)$ by deleting the boxes in column c and shifting every box to the right of column c by one column to the left. See Figure 2 for an example.

When $(r, c+1) \notin D(w)$, namely (r,c) is a southeast corner, define

$$(4) \qquad \mathrm{del}_{*,c}(w)(i) := \begin{cases} w(i) & \text{if } w(i) < c \\ w(i) - 1 & \text{if } i \leq r, \ w(i) > c \\ \min([n] \setminus ([c] \cup \mathrm{del}_{*,c}(w)([i-1]))) & \text{if } i > r, \ w(i) \geq c \end{cases}$$

Otherwise, recursively define $\mathrm{del}_{*,c}(w) := \mathrm{del}_{*,c+1}(w)$.

Lemma 3.7. *We have that $\mathrm{del}_{*,c}(w)$ is a permutation that satisfies* (3).

Proof. It suffices to assume that $(r, c+1) \notin D(w)$. Let $w' := \mathrm{del}_{*,c}(w)$. To see that w' is a permutation, we observe from the definition that $w'(i) \leq n$ and $w'(i) \notin w'([i-1])$ for all $i \in [n]$.

The first case of the construction ensures that $D(w)$ and $D(w')$ agree in columns $< c$. The second case ensures that in rows $\leq r$, the cells in $D(w')$ are the cells in $D(w)$ shifted by one column to the left. The third case ensures that there are no blanks weakly southeast of $(r+1, c)$. $\qquad \square$

We now analyze the geometry of deleting a special column. For $S \subset [n]^2$, we view $\mathbb{A}^S$ as the coordinate subspace of $\mathbb{A}^{n^2}$ consisting of matrices whose entries are 0 except possibly for the entries in S. For a permutation w, let

$$\Phi(w) = \{(i,j) : \text{there is } (k, \ell) \in D(w) \text{ with } k \geq i, \ell \geq j\},$$

i.e., the subset of $[n] \times [n]$ which is northwest of some square in the Rothe diagram. We call this the **interesting part**, for the following reason. The spread of each Fulton generator of I_w is contained in $\Phi(w)$, and so $X_w \simeq (X_w \cap \mathbb{A}^{\Phi(w)}) \times \mathbb{A}^{[n]^2 \setminus \Phi(w)}$. We call $X_w \cap \mathbb{A}^{\Phi(w)}$ the interesting part of X_w. The support of every monomial with nonzero coefficient in $\mathfrak{F}_w(z_{i,j})$ is contained in $\Phi(w)$. Note that the dimension of $X_w \cap \mathbb{A}^{\Phi(w)}$ is $|\Phi(w)| - \ell(w)$.

Let c be a special column of w, and let $w' = \mathrm{del}_{*,c}(w)$. Then there is an injection $\iota_{*,c} \colon \Phi(w') \to \Phi(w)$ given by

$$\iota_{*,c}((i,j)) = \begin{cases} (i,j) & \text{if } j < c \\ (i, j+1) & \text{if } j \geq c. \end{cases}$$

Lemma 3.8. *The map $\mathbb{A}^{\Phi(w')} \to \mathbb{A}^{\Phi(w)}$ induced by $\iota_{*,c}$ restricts to an isomorphism from $X_{w'} \cap \mathbb{A}^{\Phi(w')}$ to $X_w \cap \mathbb{A}^{\Phi(w)} \cap V(a_{i,c} : (i,c) \in \Phi(w))$.*

Proof. The rank of every square (i,j) in $D(w')$ is the same as the rank of $\iota_{*,c}((i,j))$ in $D(w)$. Setting $a_{i,c} = 0$ for $(i,c) \in \Phi(w)$ kills the Fulton generators whose spreads intersect the cth column. Under the identification induced by $\iota_{*,c}$, the remaining Fulton generators for I_w are the same as the Fulton generators for $I_{w'}$. $\qquad \square$

We will now show that, in some situations, we can use Lemma 3.8 to bound the fine Schubert polynomial of w in terms of the fine Schubert polynomial of w'. See Example 3.13 for an example of the subtleties in this type of statement. We use the notation above.

Proposition 3.9. *Suppose $S \subset [n]^2$ satisfies $S_{*,c} = D(w)_{*,c}$. Then*

$$[z^S]\mathfrak{F}_w(z_{i,j}) \geq [z^{\iota_{*,c}^{-1}(S)}]\mathfrak{F}_{w'}(z_{i,j}).$$

Proof. By the discussion at the beginning of Section 3, it suffices to show that

$$X_{w'} \cap \mathbb{A}^{\Phi(w')} \text{ is an irreducible component of } X_w \cap \mathbb{A}^{\Phi(w)} \cap V(a_{i,c} : (i,c) \in \Phi(w) \setminus D(w)),$$

where we are using the identification in Lemma 3.8. For this, it suffices to check that there is a point of $X_w \cap \mathbb{A}^{\Phi(w)} \cap V(a_{i,c} : (i,c) \in \Phi(w) \setminus D(w))$ which is contained in $X_{w'} \cap \mathbb{A}^{\Phi(w')}$ where the dimension of the tangent space of $X_w \cap \mathbb{A}^{\Phi(w)} \cap V(a_{i,c} : (i,c) \in \Phi(w) \setminus D(w))$ is equal to the dimension of $X_{w'} \cap \mathbb{A}^{\Phi(w')}$.

Let p be the point of $X_w \cap \mathbb{A}^{\Phi(w)}$ which is 1 at each coordinate of the form $(i, w(i))$ in $\Phi(w)$ and is 0 at all other coordinates. Because X_w is the product of the interesting part with some affine factors, Proposition 3.1 implies that the tangent space of $X_w \cap \mathbb{A}^{\Phi(w)}$ at p is spanned by matrix entries in $\Phi(w) \setminus D(w)$. The "special column" condition guarantees that $(w^{-1}(c), c) \notin \Phi(w)$, so $p \in X_w \cap \mathbb{A}^{\Phi(w)} \cap V(a_{i,c} : (i,c) \in \Phi(w) \setminus D(w))$. From the definition of $\mathrm{del}_{*,c}(w)$, we see that $p \in X_{w'} \cap \mathbb{A}^{\Phi(w')}$, and the tangent space of $X_w \cap \mathbb{A}^{\Phi(w)} \cap V(a_{i,c} : (i,c) \in \Phi(w) \setminus D(w))$ at p is has dimension equal to the dimension of $X_{w'} \cap \mathbb{A}^{\Phi(w')}$, as desired. □

In order to better understand the geometry of deleting a special column, we will need another result on tangent spaces of matrix Schubert varieties.

Proposition 3.10. [Knu22, Lemma 4.4] *Let $v = wt_{i,j}$ be a cover of w. Let $p_v \in \mathbb{A}^{n^2}$ be the permutation matrix corresponding to v. Then the tangent space of X_w at p_v is spanned by the matrix entries not in $D(v)$ and the matrix entry $(i, w(i))$.*

We will use this to prove a version of Proposition 3.9 where the condition on how S meets $D(w)_{*,c}$ is different. Recall that $w' = \mathrm{del}_{*,c}(w)$.

Proposition 3.11. *Let c be a special column such that $D(w)_{*,c} \setminus \mathrm{dom}(w)$ is connected. Suppose $S \subset [n]^2$ has $|S_{*,c}| = |D(w)_{*,c}|$, $\mathrm{dom}(w)_{*,c} \subset S$, $S \subset \Phi(w)$, and $|S_{*,c} \setminus D(w)| = 1$. Then*

$$[z^S]\mathfrak{F}_w(z_{i,j}) \geq [z^{\iota_{*,c}^{-1}(S)}]\mathfrak{F}_{w'}(z_{i,j}).$$

In particular, if every $(i,c) \in D(w)_{*,c} \setminus \mathrm{dom}(w)$ has $r_{i,c}(w) = 1$, or if $|D(w)_{*,c} \setminus \mathrm{dom}(w)| = 1$, then either Proposition 3.9 or Proposition 3.11 gives that $[z^S]\mathfrak{F}_w(z_{i,j}) \geq [z^{\iota_{*,c}^{-1}(S)}]\mathfrak{F}_{w'}(z_{i,j})$.

Lemma 3.12. *Let c be a special column such that $D(w)_{*,c} \setminus \mathrm{dom}(w)$ is connected. Suppose $S \subset [n]^2$ has $|S_{*,c}| = |D(w)_{*,c}|$, $\mathrm{dom}(w)_{*,c} \subset S$, $S \subset \Phi(w)$, and $|S_{*,c} \setminus D(w)| = 1$. Then, under the identification in Lemma 3.8, $X_{w'}$ is an irreducible component of*

$$X_w \cap \mathbb{A}^{\Phi(w)} \cap \bigcap_{i \in \Phi(w),\, (i,c) \notin S} V(a_{i,c}).$$

Proof. Let (i,c) be the unique element of $S_{*,c} \setminus D(w)$, and let (j,c) be the unique element of $D(w)_{*,c} \setminus S$. The assumptions guarantee that $i < j$ and that $w(i) < w(j)$.

First suppose $v := t_{i,j}w$ is a cover of w. Note that $|S_{*,c} \cap D(v)| = 0$. Let $p \in \mathbb{A}^{\Phi(w)}$ be the point which is 1 in each coordinate of the form $(i, v(i))$ and 0 in all other coordinates. Proposition 3.10 implies that the tangent space to $X_w \cap \mathbb{A}^{\Phi(w)}$ is spanned by $(i, w(i))$ and the matrix entries not in $D(v)$. In particular, the dimension of the tangent space to $X_w \cap \mathbb{A}^{\Phi(w)} \cap V(a_{i,c} : (i,c) \in \Phi(w)_{*,c} \setminus S)$ has dimension equal to $|\Phi(w)| - \ell(w) - |D(w)_{*,c}|$. Note that, under the identification in Lemma 3.8, $X_{w'} \cap \mathbb{A}^{\Phi(w')}$ contains p, and the dimension of $X_{w'} \cap \mathbb{A}^{\Phi(w')}$ is $|\Phi(w')| - \ell(w') = |\Phi(w)| - \ell(w) - |D(w)_{*,c}|$. This implies the result in this case.

If $t_{i,j}w$ is not a cover of w, then the assumptions imply that $w(i) < w(i+1)$. By Proposition 3.2, we can replace S by $s_i \cdot S$. All of the assumptions still hold, and by repeatedly doing this we can reduce to the previous case. □
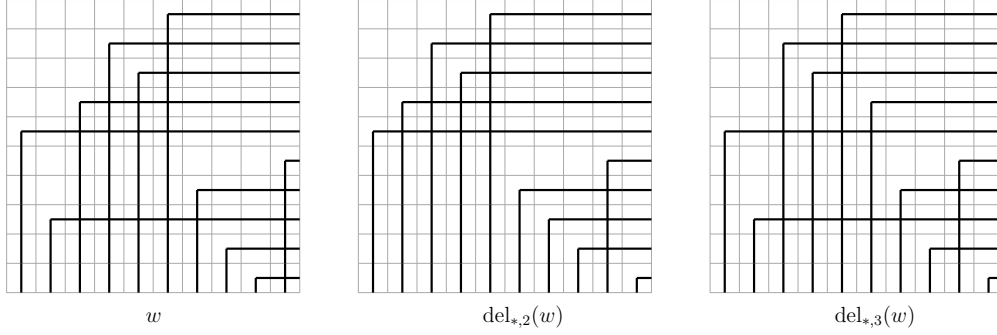
$$w \qquad\qquad \mathrm{del}_{*,2}(w) \qquad\qquad \mathrm{del}_{*,3}(w)$$

FIGURE 2. For the given w, $\mathrm{del}_{*,2}(w)$ is the deletion of a special column, and $\mathrm{del}_{*,3}$ is the deletion of a solid column.

Proof of Proposition 3.11. Lemma 3.12 implies that $\overline{X}_{w'}$ is an irreducible component of the expected dimension of

$$\overline{X}_w \cap (\mathbb{P}^1)^{\Phi(w)} \cap \bigcap_{\substack{i,\, (i,c)\notin S \\ (i,c)\in \Phi(w)}} V(a_{i,c}).$$

The discussion at the beginning of Section 3 implies the result.   □

We say that r is a **special row** for w if r is a special column for w^{-1}. If r is a special row, define $\mathrm{del}_{r,*}(w) = \mathrm{del}_{*,r}(w^{-1})^{-1}$, and define $\iota_{r,*}$ in a similar fashion to $\iota_{*,c}$. As X_w is isomorphic to $X_{w^{-1}}$ via the transpose map, all the results above can be immediately adapted to deleting a special row.

Example 3.13. Let $w = 145263$, and let $S = \{(1,1),(2,1),(3,3),(4,3),(5,3)\}$. Then the third column is special, and $\mathrm{del}_{*,3}(w) = 134256$. We have $[z^S]\mathfrak{F}_w(z_{i,j}) = 0$ because S^c contains the spread of the 2×2 minor in rows 1 and 2 and columns 2 and 3, but $[z^{\iota_{*,c}^{-1}(S)}]\mathfrak{F}_{\mathrm{del}_{*,3}(w)} = 1$. This shows that the hypothesis that $D(w)_{*,c} \setminus \mathrm{dom}(w)$ is connected cannot be omitted in Proposition 3.11.

3.4. **Deleting a solid column.** In the proof of Theorem 1.7, we will need to delete another class of columns whose intersection with the Rothe diagram is contained in the dominant part. We will analyze this case by reducing to the case of special columns.

We recursively say that a column c is **solid** if $D(w)_{*,c} \subset \mathrm{dom}(w)$, and either $c+1$ is solid, or $c+1$ is a special column with $|D(w)_{*,c+1} \setminus \mathrm{dom}(w)| = 1$.

Suppose $c, c+1, \ldots, c+k-1$ are solid columns, but $c+k$ is not solid. Then the $(c+k)$th column must be a special column with $|D(w)_{*,c+k} \setminus \mathrm{dom}(w)| = 1$. We will define a permutation $\mathrm{del}_{*,c}(w) \in S_n$ which "deletes" the solid column c. The definition uses that we have already defined how to delete a special column.

If $w^{-1}(c) < w^{-1}(c+1)$, then we recursively define $\mathrm{del}_{*,c}(w) := \mathrm{del}_{*,c+1}(w)$. Note that if $k = 1$, $\mathrm{del}_{*,c+1}(w)$ is deleting a special column, so it is defined.

If $w^{-1}(c) > w^{-1}(c+1)$, let $c+l_1, \ldots, c+l_j$ be the set of all columns such that $0 < l_1 < \cdots < l_j < k$ and $w^{-1}(c) < w^{-1}(c+l_1) < \cdots < w^{-1}(c+l_j)$. If this set is nonempty, let $\widetilde{w} = t_{c,c+l_1} t_{c+l_1,c+l_2} \ldots t_{c+l_{j-1},c+l_j} t_{c+l_j,c+k} w$; otherwise set $\widetilde{w} = t_{c,c+k} w$. Define $\mathrm{del}_{*,c}(w) := \mathrm{del}_{*,c}(\widetilde{w})$, as c is a special column of $\widetilde{w}$.

Set $w' = \mathrm{del}_{*,c}(w)$. The Rothe diagram of w' is obtained from the Rothe diagram of w informally by deleting the cth column, removing the unique square in $D(w)_{*,c+k} \setminus \mathrm{dom}(w)$, and sliding columns $\{c+1, \ldots, n\}$ to the left. See Figure 2 for a comparison between deleting a special column and deleting a solid column.

Lemma 3.14. *The map $\mathbb{A}^{\Phi(w')} \to \mathbb{A}^{\Phi(w)}$ induced by $\iota_{*,c}$ restricts to an isomorphism from $X_{w'} \cap \mathbb{A}^{\Phi(w')}$ to $X_w \cap \mathbb{A}^{\Phi(w)} \cap V(a_{i,c} : (i,c) \in \Phi(w))$.*

Proof. After setting $a_{i,c} = 0$ for $(i,c) \in \Phi(w)$, the Fulton generators for w either vanish or become Fulton generators for w', and all Fulton generators for w' are obtained in this way. $\square$

Proposition 3.15. *Suppose that $S \subset [n]^2$ satisfies $|S \cap \Phi(w)_{*,c}| = 1$. Then*

$$[z^S]\mathfrak{F}_w(z_{i,j}) \geq [z^{\iota_{*,c}^{-1}(s)}]\mathfrak{F}_{w'}(z_{i,j}).$$

As before, Proposition 3.15 follows from the following lemma.

Lemma 3.16. *Suppose that $S \subset [n]^2$ satisfies $|S \cap \Phi(w)_{*,c}| = 1$. Then, under the identification induced by $\iota_{*,c}$, $X_{w'} \cap \Phi(w')$ is an irreducible component of*

$$X_w \cap \mathbb{A}^{\Phi(w)} \cap \bigcap_{i,\, (i,c)\notin S} V(a_{i,c}).$$

Proof. For a column d and a permutation v, let $f_d(v)$ be the first row appearing in $\Phi(v)_{*,d} \setminus \mathrm{dom}(v)$. We prove the lemma by induction first on k, and then on $f_c(w) - f_{c+1}(w)$.

First suppose that $f_c(w) = f_{c+1}(w)$. The special column condition implies that this happens if and only if $w^{-1}(c) < w^{-1}(c+1)$. Then, by Proposition 3.2, $X_w \cdot s_c = X_w$. If $k = 1$, then the result follows from Lemma 3.12. If $k > 1$, then the result follows from induction on k.

Now suppose that $f_c(w) > f_{c+1}(w)$. Because $D(w)_{*,c} \subset \mathrm{dom}(w)$, $w(i) < c$ for all i with $w^{-1}(c+1) < i < w^{-1}(c)$. Let $\widetilde{w} = wt_{f_c(w)-1,w^{-1}(c)}$. Then c is still a solid column for $\widetilde{w}$, with $f_c(\widetilde{w}) - f_{c+1}(\widetilde{w}) = f_c(w) - f_{c+1}(w) - 1$. If we delete column c from $\widetilde{w}$, then we still get w'.

By induction on $f_c(w) - f_{c+1}(w)$, we know that $X_{w'} \cap \Phi(w')$ is an irreducible component of

$$X_{\widetilde{w}} \cap V(a_{f_c(w)-1,c}) \cap \mathbb{A}^{\Phi(w')} \cap \bigcap_{i,\, (i,c)\notin S} V(a_{i,c}).$$

By Proposition 3.6, $X_{\widetilde{w}} \cap V(a_{f_c(w)-1,c}) = X_w$ because $L_{f_c(w)-1}(\widetilde{w}) = \{w\}$. As $\mathbb{A}^{\Phi(\widetilde{w})} = \mathbb{A}^{\Phi(w)}$, this gives the result. $\square$

We say that a row r is a **solid row** of w if column r is a solid column of w^{-1}. In this way, we can use Proposition 3.15 to delete solid rows.

3.5. **Transition.**
In this section, we prove a version of Lascoux's transition formula [Las01].

Proposition 3.17. *Let $w \in S_n$, and let $(r,c) \in \mathrm{ess}(w)$ such that for all $(r',c') \in D(w)$, $r' < r$ or $c' < c$. Let $w' := wt_{r,w^{-1}(c)}$. Then for any S which contains (r,c), we have*

$$[z^S]\mathfrak{F}_w(z_{i,j}) = [z^{S\setminus\{(r,c)\}}]\mathfrak{F}_{w'}(z_{i,j}).$$

The permutation w' satisfies the following:

(1) $\ell(w') = \ell(w) - 1$,
(2) $(r,c) \in \Phi(w)$, but $(r,c) \notin \Phi(w')$, and
(3) For every $(i,j) \in D(w')$, we have $r_{i \times j}(w') = r_{i \times j}(w)$.

Note that, unlike all other results in this section, Proposition 3.17 exactly computes some coefficients of the fine Schubert polynomial of w in terms of the fine Schubert polynomial of w'.

Proof of Proposition 3.17. Let $\pi\colon (\mathbb{P}^1)^{n^2} \to (\mathbb{P}^1)^{n^2-1}$ be the projection away from the factor labeled by (r,c). We claim that $\pi(\overline{X}_w) = \pi(\overline{X}_{w'})$. Note that, because $(r,c) \notin \Phi(w')$, $\overline{X}_{w'} = \pi(\overline{X}_{w'}) \times \mathbb{P}^1$, and so $\pi(\overline{X}_{w'})$ is an irreducible variety of dimension $n^2 - \ell(w') - 1 = n^2 - \ell(w)$.

Also, $\pi(\overline{X}_w)$ is an irreducible variety of dimension $n^2 - \ell(w)$. That it is irreducible and has dimension at most $n^2 - \ell(w)$ is automatic; that it has dimension $n^2 - \ell(w)$ follows from the fact that $(r,c) \in \Phi(w)$.

As $\pi(\overline{X}_w)$ and $\pi(\overline{X}_{w'})$ are irreducible varieties of the same dimension, it suffices to show that $\pi(X_{w'}) \supset \pi(X_w)$. Because $(r,c) \notin \Phi(w')$, it is clear that the variety $\pi(X_{w'})$ is cut out by the Fulton generators for $I_{w'}$. Each Fulton generator for $I_{w'}$ is a Fulton generator for I_w, which implies the containment.

Note that the map $\overline{X}_w \to \pi(\overline{X}_w)$ is birational: over some dense open in X_w, any CDG generator whose spread contains (r,c) allows us to solve for $a_{r,c}$ in terms of the other variables.

We have
$$[\pi(\overline{X}_{w'})] = \mathfrak{F}_{w'}([H_{i,j}]) \cap [(\mathbb{P}^1)^{n^2-1}] \in H_*((\mathbb{P}^1)^{n^2-1}).$$
Note that this makes sense, because $\mathfrak{F}_{w'}(z_{i,j})$ does not involve the variable $z_{r,c}$ as $(r,c) \notin \Phi(w')$. On the other hand,
$$[\pi(\overline{X}_w)] = \pi_*[\overline{X}_w] = \sum_{S \ni (r,c)} [z^S]\mathfrak{F}_w(z_{i,j}) \cdot [H]^{S \setminus \{(r,c)\}} \cap [(\mathbb{P}^1)^{n^2-1}].$$
Here $[H]^T := \prod_{(i,j) \in T}[H_{i,j}]$, and the first equality holds because the degree of the map $\overline{X}_w \to \pi(\overline{X}_w)$ is 1. This gives the result. $\qquad\square$

3.6. **Relation to Schubert polynomials.** We now prove Proposition 1.3. Let $(\mathbb{P}^n)^n$ be the projective completion of the rows of the space of $n \times n$ matrices, so there an inclusion of $\mathbb{A}^{n^2}$ into $(\mathbb{P}^n)^n$. For $X \subset \mathbb{A}^{n^2}$, let $\mathrm{cl}(X)$ denote the closure in $(\mathbb{P}^n)^n$, and let $\overline{X}$ be the closure in $(\mathbb{P}^1)^{n^2}$. Let $[H_i] \in H^2((\mathbb{P}^n)^n)$ be the class of a hyperplane in the ith row.

Lemma 3.18. *We have* $[\mathrm{cl}(X_w)] = \mathfrak{S}_w([H_i]) \cap [(\mathbb{P}^n)^n]$ *in the homology of* $(\mathbb{P}^n)^n$.

Proof. Let $(\mathbb{P}^{n-1})^n$ be the projectivization of the rows of the space of $n \times n$ matrices. By [KM05] (see [MS05, Notes to Chapter 8]), the class of the projectivization of the rows of X_w in $(\mathbb{P}^{n-1})^n$ is given by $\mathfrak{S}_w$ evaluated at the hyperplane classes. As $\mathrm{cl}(X_w)$ is the cone over the projectivization of the rows of X_w, this implies the result. $\qquad\square$

Proof of Proposition 1.3. Let $U = S^c$. For each $(i,j) \in U$, let $\lambda_{i,j}$ be a generic complex number, and let $V_{i,j} = \{a_{i,j} = \lambda_{i,j}\} \subset \mathbb{A}^{n^2}$. The fundamental class of $\mathrm{cl}(V_{i,j})$ is $[H_i]$.

As the $\lambda_{i,j}$ are generic, a version of Bertini's theorem [Ful98, Lemma B.9.1] implies that $\overline{X}_w \cap \bigcap_{(i,j) \in U} \overline{V}_{i,j}$ consists of $[z^S]\mathfrak{F}_w(z_{i,j})$-many reduced points which are contained in $\mathbb{A}^{n^2}$. By Lemma 3.18, we have
$$(\prod_{(i,j) \in U}[\mathrm{cl}(V_{i,j})]) \cap [\mathrm{cl}(X_w)] = (\mathfrak{S}_w([H_i]) \cup \prod_{(i,j) \in U}[H_i]) \cap [(\mathbb{P}^n)^n] = [\prod_{(i,j) \in S}x_i]\mathfrak{S}_w(x_i) \cdot [\mathrm{pt}],$$
where $[\mathrm{pt}]$ is the class of a point.

Each of the $[z^S]\mathfrak{F}_w(z_{i,j})$-many reduced points is an irreducible component of $\mathrm{cl}(X_w) \cap \bigcap_{(i,j) \in U} \mathrm{cl}(V_{i,j})$, and these points contribute $[z^S]\mathfrak{F}_w(z_{i,j}) \cdot [\mathrm{pt}]$ to the intersection number. Because the tangent bundle of $(\mathbb{P}^n)^n$ is globally generated, the other irreducible components of $\mathrm{cl}(X_w) \cap \bigcap_{(i,j) \in U} \mathrm{cl}(V_{i,j})$ contribute non-negatively, which gives the result. $\qquad\square$

Remark 3.19. The above argument can be used to give a positive answer to a special case of [CDNG22, Question 7.4]. However, the question has a negative answer in general: a positive answer would imply that if all coefficients of $\mathfrak{S}_w(x_i)$ are 0 or 1, then all coefficients of the double Schubert polynomial $\mathfrak{S}_w(x_i, y_j)$ are 0 or 1. This fails for $w = 2143$.
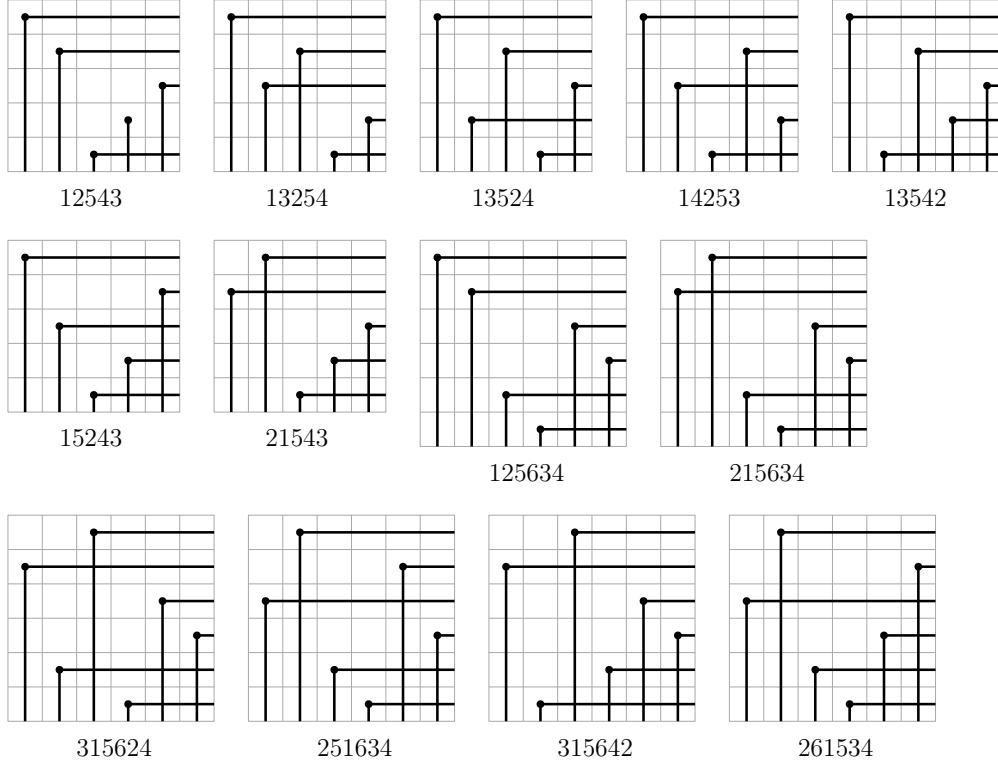
FIGURE 3. Patterns in $\mathcal{P}$

4. DESCRIPTION OF w WHEN $\mathfrak{S}_w$ AND $\mathfrak{S}_{w^{-1}}$ HAVE ALL COEFFICIENTS 0 OR 1

4.1. **Rothe diagrams.** In this section, we provide a characterization of the permutations w when $\mathfrak{S}_w(x_i)$ and $\mathfrak{S}_{w^{-1}}(x_j)$ have all coefficients equal to 0 or 1 in terms of Rothe diagrams. Our proof for Theorem 1.7 will heavily rely on the structure of these permutations.

Proposition 4.1. *For $w \in S_n$, $\mathfrak{S}_w(x_i)$ and $\mathfrak{S}_{w^{-1}}(x_j)$ have all coefficients equal to 0 or 1 if and only if w avoids 12543, 13254, 13524, 14253, 13542, 15243, 21543, 125634, 215634, 315624, 251634, 315642, and 261534.*

We denote this set of patterns by $\mathcal{P}$. See Figure 3.

Proof. By [FMSD21, Theorem 4.8], $\mathfrak{S}_w(x_i)$ has all coefficients equal to 0 or 1 if and only if w avoids 12543, 13254, 13524, 13542, 21543, 125364, 125634, 215364, 215634, 315264, 315624, and 315642. The set $\mathcal{P}$ is obtained by adding the transposes of these patterns and removing the redundant ones. $\square$

We say that $(p, w(p))$ is an **outer corner** of $\mathrm{dom}(w)$ if $(p-1, w(p)) \in \mathrm{dom}(w)$ and $(p, w(p)-1) \in \mathrm{dom}(w)$, whenever they are defined. We say that $(i, j) \in D(w)$ is a **northwest corner** if $(i-1, j) \notin D(w)$ and $(i, j-1) \notin D(w)$.

Lemma 4.2. *Suppose w avoids $\mathcal{P}$, $(p, w(p))$ is an outer corner of $\mathrm{dom}(w)$, and $(p+1, w(p)+1) \in D(w)$. Let $D_p := \{(i, j) \in D(w) : i > p, j > w(p)\}$, and let $D'_p \subset D_p$ be the region connected to $(p+1, w(p)+1)$. Then if $(i, j) \in D_p \setminus D'_p$, then either $i = p+1$ or $j = w(p)+1$.*
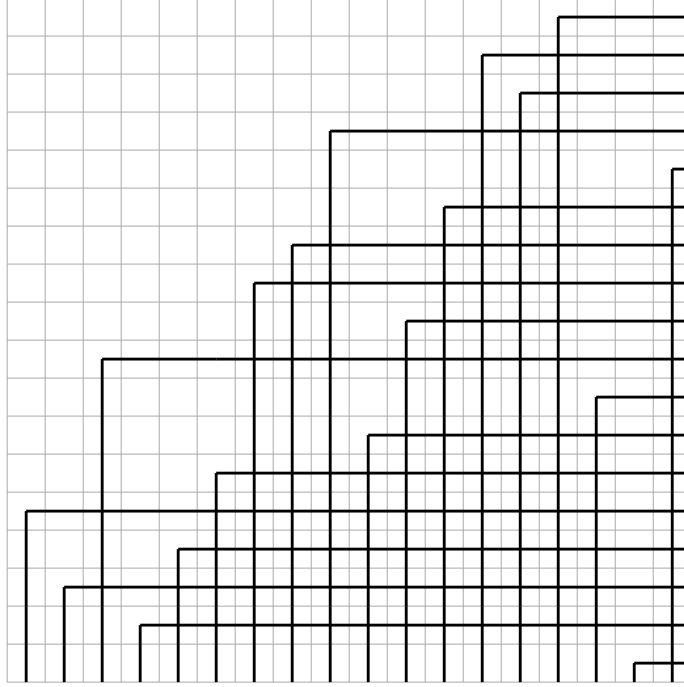
FIGURE 4. A large permutation which avoids $\mathcal{P}$

Proof. Let a be the largest row index such that $(a, w(p) + 1) \in D'_p$, and let b be the largest column index such that $(p + 1, b) \in D'$.

First suppose that there is a region of $D_p \setminus D'_p$ which has a northwest corner that is contained in the $(p + 1)$st row, but the region is not contained in row $p + 1$. Let $(p + 1, j)$ be such a northwest corner with j minimal. Then $(p + 2, j) \in D_p$, so $w(p + 2) > j$, and necessarily $(p + 2, w(p) + 1) \in D_p$. Notice also that, by the minimality of j, for all $b < k < j$ we have $(p + 1, k) \notin D(w)$. Therefore, for all $b < j' < j$, we have $w^{-1}(j') < p$.

- If $w(p + 1) > w(p + 2)$, we have $w^{-1}(j - 1) < p < p + 1 < p + 2 < w^{-1}(j)$ and $w(p) < j - 1 < j < w(p + 2) < w(p + 1)$, realizing the pattern 21543.
- If $w(p + 1) < w(p + 2)$:
  - If $w^{-1}(w(p) + 1) < w^{-1}(j)$, we have $w^{-1}(j - 1) < p < p + 1 < p + 2 < w^{-1}(w(p) + 1) < w^{-1}(j)$ and $w(p) < w(p) + 1 < j - 1 < j < w(p + 1) < w(p + 2)$, so we realize the pattern 315624.
  - If $w^{-1}(w(p) + 1) > w^{-1}(j)$, we have $w^{-1}(j - 1) < p < p + 1 < p + 2 < w^{-1}(j) < w^{-1}(w(p) + 1)$ and $w(p) < w(p) + 1 < j - 1 < j < w(p + 1) < w(p + 2)$, so we realize the pattern 315642.

If there is a region of $D_p \setminus D'_p$ which has a northwest corner that is contained in the $(w(p) + 1)$st column, but the region is not contained in column $w(p) + 1$, the we are in the situation transpose to the previous case. We can therefore realize the patterns 21543, 251634 or 261534.

Finally, suppose (i, j) is a northwest-most northwest corner of a region in $D_p \setminus D'_p$ (meaning that there is no $(i_0, j_0) \in D_p \setminus D'_p$ such that $i_0 < i$, $j_0 < j$ that is a northwest corner), and $i > p + 1$, $j > w(p + 1)$. Then $w(i - 1) < j$, $w(i) > j$, $w^{-1}(j - 1) < i$, and $w^{-1}(j) > i$. There are the following cases:

Case $i \leq a$. By assumption, for all $p+1 \leq i' \leq a$ and $j' < w(p)$, we have $(i', j') \in D(w)$ and $(i', w(p)+1) \in D(w)$. Therefore, since $p + 1 \leq i - 1 \leq a$, we must have $w(i - 1) > w(p) + 1$. Furthermore, we have $w^{-1}(w(p) + 1) > a$.

Then if $w^{-1}(j) < w^{-1}(w(p)+1)$, we have $p < i-1 < i < w^{-1}(j) < w^{-1}(w(p)+1)$ and $w(p) < w(p)+1 < w(i - 1) < j < w(i)$, realizing the pattern 13542. Otherwise $p < i - 1 < i < w^{-1}(w(p) + 1) < w^{-1}(j)$ and $w(p) < w(p) + 1 < w(i - 1) < j < w(i)$, realizing the pattern 13524.

Case $j \leq b$. This is transpose to the case $i \leq a$, and we can realize patterns 15243 and 14253.

Case $i > a + 1$ and $j > b + 1$. (Notice that it is impossible that $i = a + 1$ or $j = b + 1$). There are a few subcases.

Subcase 1: $w(i) < w(p + 1)$ and for all $b < j' < j$, we have $w^{-1}(j') < p$. It must be the case that $(p + 1, j) \in D(w)$, so $w(p + 1) > j$. We have $w^{-1}(j - 1) < p < p + 1 < i < w^{-1}(j)$ and $w(p) < j - 1 < j < w(i) < w(p + 1)$, realizing the pattern 21543.

Subcase 2: $w^{-1}(w(p) + 1) > w^{-1}(j)$, and for all $a < i' < i$, we have $w(i') < w(p)$. We are in the situation transpose to Subcase 1, and we can again realize 21543 with row indices $p < i - 1 < i < w^{-1}(j) < w^{-1}(w(p) + 1)$.

Subcase 3: $w(i) > w(p + 1)$; $w^{-1}(w(p) + 1) < w^{-1}(j)$; for all $b < j' < j$, we have $w^{-1}(j') < p$; and for all $a < i' < i$ we have $w(i') < w(p)$. Then it must be the case that $(i, w(p)+1) \in D(w)$, so $w^{-1}(w(p)+1) > i$. We have $p < p+1 < i-1 < i < w^{-1}(w(p)+1) < w^{-1}(j)$ and $w(i-1) < w(p) < w(p)+1 < j < w(p+1) < w(i)$, realizing the pattern 251634.

Subcase 4: $w(i) > w(p + 1)$; for all $b < j' < j$, we have $w^{-1}(j') < p$; and there exists $a < i' < i$ such that $w(i') > w(p)$. Since $(i, j) \in D(w)$ is a northwest corner, we must have $w(i') < j$. Again we have $(p + 1, j) \in D(w)$. We have $p < p + 1 < i' < i < w^{-1}(j)$ and $w(p) < w(i') < j < w(p + 1) < w(i)$, realizing the pattern 14253.

Notice that Subcases 1–4 cover all the situations when, for all $b < j' < j$, we have $w^{-1}(j') < p$. By considering the transpose, we cover all the situations when, for all $a < i' < i$, we have $w(i') < w(p)$. We are left with one last case.

Subcase 5: There exists $b < j' < j$ such that $w^{-1}(j') > p$ and there exists $a < i' < i$ such that $w(i') > w(p)$. Since $(i, j) \in D(w)$ is a northwest-most northwest corner, we must have $w(i') < j$ and $w^{-1}(j') < i$.

First we assume that i', j' also satisfy $w^{-1}(j') \leq a$ and $w(i') \leq b$. We then have $p < w^{-1}(j') < i' < i < w^{-1}(j)$ and $w(p) < w(i') < j' < j < w(i)$, realizing the pattern 13245.

We then consider the case when for all $a < \tilde{i} < i$ such that $w(\tilde{i}) > w(p)$, we also have $w(\tilde{i}) > b$. It follows that $w^{-1}(w(p) + 1) > i$, since $w^{-1}(w(p) + 1) > a$. If $w^{-1}(w(p) + 1) < w^{-1}(j)$, we have $p < i' < i < w^{-1}(w(p) + 1) < w^{-1}(j)$ and $w(p) < w(p + 1) < w(i') < j < w(i)$, realizing the pattern 13524. If $w^{-1}(w(p) + 1) > w^{-1}(j)$, we realize the pattern 13542.

The case when, for all $b < \tilde{j} < j$ such that $w^{-1}(\tilde{j}) > p$, we also have $w^{-1}(\tilde{j}) > a$, is transpose to the previous case. $\qquad\square$

Lemma 4.3. *Suppose w avoids $\mathcal{P}$, $(p, q) \in D(w)$ is a northwest corner of a non-dominant region, and either (a) $w(p-1) \neq q-1$, or (b) $w(p-1) = q-1$ and $w(p-2) = q-2$. Let $D_{p,q} := \{(i, j) \in D(w) : i \geq p, j \geq q\}$. Then $D_{p,q}$ is either entirely contained in row p or entirely contained column q.*

Proof. Since $(p, q) \in D(w)$ is a northwest corner of a non-dominant region, when (a) holds we have $w(p-1) < q - 1$ and $w^{-1}(q - 1) < p - 1$.

First suppose $(i, j) \in D_{p,q}$ such that $i > p$, $j > q$.

Case 1: Suppose $(p, j), (i, q) \in D_{p,q}$. Then we have $w(p) > j$, $w(i) > j$, $w^{-1}(q) > i$, and $w^{-1}(j) > i$.

If $w(p) < w(i)$ and $w^{-1}(q) < w^{-1}(j)$, then when (a) holds, we have $w^{-1}(q - 1) < p - 1 < p < i < w^{-1}(q) < w^{-1}(j)$ and $w(p - 1) < q - 1 < q < j < w(p) < w(i)$, realizing the pattern 215634. When (b) holds, we can realize the pattern 125634.

If $w(p) > w(i)$, then when (a) holds, we have $w^{-1}(q-1) < p-1 < p < i < w^{-1}(j)$ and $w(p-1) < q-1 < j < w(i) < w(p)$, realizing the pattern 21543. When (b) holds, we can realize 12543. The case $w^{-1}(q) > w^{-1}(j)$ is similar.

<u>Case 2</u>: Suppose $(i,q) \notin D_{p,q}$ and $(p,j) \notin D_{p,q}$. We must have some $p < i' < i$ such that $w(i') = q$, and some $q < j' < j$ such that $w(p) = j'$. We have $p-1 < p < i' < i < w^{-1}(j)$ and $w(p-1) < q < w(p) < j < w(i)$, realizing the pattern 13254.

<u>Case 3</u>: Suppose $(i,q) \notin D_{p,q}$ and $(p,j) \in D_{p,q}$. Then we must have some $p < i' < i$ such that $w(i') = q$, as well as $w(p) > j$. If $w(p) < w(i)$, we have $p-1 < p < i' < i < w^{-1}(j)$ and $w(p-1) < q < j < w(p) < w(i)$, realizing the pattern 14253. If $w(p) > w(i)$, we realize the pattern 15243.

<u>Case 4</u>: If $(i,q) \in D_{p,q}$ and $(p,j) \notin D_{p,q}$, we are in the situation transpose to Case 3, realizing the pattern 13524 or 13542.

The above argument shows that we cannot have $(i,j) \in D_{p,q}$ such that $i > p$, $j > q$. It remains to show that there cannot simultaneously exist (p,j) and (i,q) in $D_{p,q}$ with $j > q$ and $i > p$.

Suppose otherwise. We have $w(p) > j$, $w^{-1}(q) > i$. Since there are no blanks strictly to the southeast of (p,q), we have $(i,j) \notin D_{p,q}$. Therefore, either there exists some $p < i' < i$ such that $w(i') = j$, or there exists some $q < j' < j$ such that $w(i) = j'$. Suppose the former is true; the latter case is similar. When (a) holds, we have $w^{-1}(q-1) < p-1 < p < i' < w^{-1}(q)$ and $w(p-1) < q-1 < q < j < w(p)$, realizing the pattern 21543. When (b) holds, we realize the pattern 12543. $\qquad\square$

Lemma 4.4. *If w contains a pattern $v \in \mathcal{P}$, then w does not satisfy the properties in Lemma 4.2 and Lemma 4.3.*

Proof. By inspection, if $w \in \mathcal{P}$, then $D(w)$ violates the properties in Lemma 4.2 and 4.3.

Let $v \in \mathcal{P}$, so $v \in S_m$ ($m = 5$ or 6), and let $w \in S_n$. Suppose we have $k_1 < \cdots < k_m$, and the relative order of $w(k_1), \ldots, w(k_m)$ given by v. Suppose $\{l_1, \ldots l_m\} = \{w(k_1), \ldots w(k_m)\}$ and $l_1 < \cdots < l_m$. Let $\iota \colon [m]^2 \to [n]^2$ given by $\iota((i,j)) := (k_i, l_j)$. Then (i,j) lies in $D(v)$ if and only if $\iota((i,j))$ lies in $D(w)$, and $r_{k_i \times l_j}(w) \geq r_{i \times j}(v)$. If $D(v)$ violates the conditions in Lemma 4.2, then $D(w)$ violates the conditions in Lemma 4.2 if the rank 1 region stays rank 1, and $D(w)$ violates the conditions in Lemma 4.3 if the rank 1 region increases rank. If $D(v)$ violates the conditions in Lemma 4.3, then $D(w)$ also does. $\qquad\square$

In summary, Lemmas 4.2, 4.3, and 4.4 give the following.

Theorem 4.5. *A permutation w avoids $\mathcal{P}$ if and only if its diagram $D(w)$ satisfies the conditions in Lemmas 4.2 and 4.3.*

By considering how the various operations on a permutation in Section 3 change the Rothe diagram, we have the following corollary.

Corollary 4.6. *Suppose w is a permutation that avoids $\mathcal{P}$ and w' is a permutation obtained from w by applying one of del_k, $\mathrm{del}_{r,*}$, $\mathrm{del}_{*,c}$, or transition as in Proposition 3.17. Then w' avoids $\mathcal{P}$.*

Proposition 4.7. *Suppose w avoids $\mathcal{P}$, and let $w' \in L_k(w)$ be a term obtained by cotransitioning at k. Then w' also avoids $\mathcal{P}$.*

Proof. By Proposition 4.1, $\mathfrak{S}_w(x_i)$ and $\mathfrak{S}_{w^{-1}}(x_j)$ have all coefficients equal to 0 or 1. Since $x_k \mathfrak{S}_w(x_i) = \sum_{w' \in L_k(w)} \mathfrak{S}_{w'}(x_i)$, each $\mathfrak{S}_{w'}(x_i)$ must have all coefficients equal to 0 or 1. From [Knu22], we also have $(x_k - y_{w(k)}) \mathfrak{S}_w(x_i, y_j) = \sum_{w' \in L_k(w)} \mathfrak{S}_{w'}(x_i, y_j)$ for double Schubert polynomials. Since $\mathfrak{S}_w(x_i) = \mathfrak{S}_w(x_i, 0)$ and $\mathfrak{S}_w(x_i, y_j) = \mathfrak{S}_{w^{-1}}(-y_j, -x_i)$ (see e.g., [AF23, Section 11.4]), it follows that $\mathfrak{S}_{w'^{-1}}(x_j)$ also has all coefficients 0 or 1 for $w' \in L_k(w)$. $\qquad\square$

4.2. **Relations in $\mathcal{R}_w$.** Equipped with the characterization of the Rothe diagrams of w, we state some (mostly straightforward) properties of relations in $\mathcal{R}_w$ as technical preparation for the next section.

Let $f \in \mathcal{R}_w$, and let $\mathrm{cdg}(f)$ be the set of CDG generators merged to get f. (This may not be unique in general; we may take the ones picked by any specific implementation of the algorithm.) Let $\mathfrak{m}(f) := \{(i,j) : \exists g_1 \neq g_2 \in \mathrm{cdg}(f), \text{ such that } (i,j) \in \mathrm{spr}(g_1) \cap \mathrm{spr}(g_2)\}$, the set of coordinates at which merging occurred. By construction, we have $\mathrm{spr}(f) = (\bigcup_{g \in \mathrm{cdg}(f)} \mathrm{spr}(g)) \setminus \mathfrak{m}(f)$.

Lemma 4.8. *Let $f \in \mathcal{R}_w$. If $\mathrm{spr}(f)_{*,c} \neq \emptyset$, then $|\mathrm{spr}(f)_{*,c}| > 1$.*

Proof. We induct on the number of CDG generators merged to construct f. If $f \in \mathrm{CDG}(w)$, this is clear by definition of the CDG generators (Section 1). Suppose $f = \mathrm{merge}_{(\alpha,\beta)}(g,h)$. If $\beta \neq c$, then $\mathrm{spr}(f)_{*,c} = \mathrm{spr}(g)_{*,c}$ or $\mathrm{spr}(f)_{*,c} = \mathrm{spr}(h)_{*,c}$ and the claim follows by induction. If $\beta = c$, then $\mathrm{spr}(f)_{*,c} = \mathrm{spr}(g)_{*,c} \cup \mathrm{spr}(h)_{*,c} \setminus \{(\alpha,c)\}$. Since $\mathrm{spr}(g)_{*,c} \cap \mathrm{spr}(h)_{*,c} = \{(\alpha,c)\}$, the claim follows. $\square$

Lemma 4.9. *Suppose $(r,c) \in \mathrm{ess}(w)$ is rank m, and there is $r' < r$ such that $w(r') = c-1$ and $(r'+1,c) \in D(w)$, and $c' < c$ such that $w^{-1}(c') = r-1$ and $(r,c'+1) \in D(w)$. Suppose also that for $i \in [r-(m-1), r-1]$, $w(i) < w(r')$. If $f \in \mathcal{R}_w$ has some $g \in \mathrm{cdg}(f)$ such that g is obtained from an $(m+1)$-minor and $\mathrm{spr}(g) \subset [r] \times [c]$, then $\mathrm{spr}(f)_{[r-m-1,r],[c-1,c]} = ([r-m-1,r] \times [c-1,c])$.*

Proof. By the definition of the CDG generators, g cannot use more than one row above row $r-(m-1)$, and it also must use both column c and $c-1$. Since it is an $(m+1)$-minor, $[r-m-1,r] \times [c-1,c] \subset \mathrm{spr}(g)$. Since none of the coordinates in $[r-m-1,r] \times [c-1,c]$ can be in $\mathfrak{m}(f)$, the result follows. $\square$

5. Proof of Theorem 1.7

We first outline the strategy that we use to prove Theorem 1.7. Given $w \in S_n$ and a subset U of $[n]^2$, we say that U is X_w-*completable* if $\pi_U(\overline{X}_w) = (\mathbb{P}^1)^U$, i.e., if we fill in the entries corresponding to U in an $n \times n$ matrix with generic complex numbers, then we can fill in the entries in $[n]^2 \setminus U$ in some way so that the resulting matrix lies in X_w. Combining Lemma 2.1 and Proposition 2.3 gives that U is X_w-completable if and only if there is $T \subset U^c$ with $|T| = \ell(w)$ such that $[z^T]\mathfrak{F}_w(z_{i,j}) \neq 0$. If V is X_w-completable and $U \subset V$, then U is X_w-completable.

We say that U is $\mathcal{R}_w$-*avoiding* if U does not contain the spread of any $f \in \mathcal{R}_w$. If V is $\mathcal{R}_w$-avoiding and $U \subset V$, then U is $\mathcal{R}_w$-avoiding. By Lemma 2.1, if U is X_w-completable, then it is $\mathcal{R}_w$-avoiding. In general, the converse does not hold.

Example 5.1. Let $w = 21543$. A Macaulay2 computation [GS] shows that
$$f = a_{1,2}a_{1,4}a_{2,1}a_{3,1}a_{4,3} - a_{1,3}a_{1,4}a_{2,1}a_{3,1}a_{4,2} + a_{1,2}a_{1,3}a_{2,1}a_{3,4}a_{4,1}$$
$$-a_{1,2}a_{1,4}a_{2,1}a_{3,3}a_{4,1} - a_{1,2}a_{1,3}a_{2,4}a_{3,1}a_{4,1} + a_{1,3}a_{1,4}a_{2,2}a_{3,1}a_{4,1} \in I_w.$$

The spread of f does not contain the spread of any relation which can be obtained by merging CDG generators, so $\mathrm{spr}(f)$ is $\mathcal{R}_w$-avoiding but not X_w-completable.

Theorem 5.2. *Let w be a permutation which avoids $\mathcal{P}$. Then U is $\mathcal{R}_w$-avoiding if and only if it is X_w-completable.*

Proof of Theorem 1.7. By Proposition 1.9, w avoids $\mathcal{P}$ if and only if $\mathfrak{S}_w(x_i)$ and $\mathfrak{S}_{w^{-1}}(x_j)$ have all coefficients equal to 0 or 1. Theorem 5.2 implies that $\mathcal{R}_w$ satisfies the criterion in Theorem 1.5, as the relations in $\mathcal{R}_w$ are squarefree. $\square$

We now explain the strategy of the proof of Theorem 5.2. The results of Section 3 describe a number of "smaller" permutations $w' \in S_n$ that can be constructed out of w. We will show that if U is $\mathcal{R}_w$-avoiding, then some related set U' is $\mathcal{R}_{w'}$-avoiding. Corollary 4.6 and Proposition 4.7 show that w' also avoids $\mathcal{P}$. As

w' is "smaller" than w, we deduce by induction that U' is $X_{w'}$-completable. By Lemma 2.2, there is some $\widetilde{U}'$ containing U' such that $|\widetilde{U}'| = n^2 - \ell(w')$ and $\widetilde{U}'$ is still $X_{w'}$-completable. The results in Section 3 construct some $\widetilde{U}$ with $|\widetilde{U}| = n^2 - \ell(w)$ that is X_w-completable. The set $\widetilde{U}$ will contain U, so U is X_w-completable.

We start with a simple but important lemma.

Lemma 5.3. *Let $U \subset [n]^2$ and $w \in S_n$. Suppose $[r, r'] \subset [n]$ satisfies that, for all $r \leq i < r'$, $w(i) < w(i+1)$. If $v \in S_n$ fixes $[n] \setminus [r, r']$, then U is $\mathcal{R}_w$-avoiding (resp., X_w-completable) if and only if $v \cdot U$ is $\mathcal{R}_w$-avoiding (resp., X_w completable).*

We will often use the transposed version of this lemma, where we replace w with w^{-1} and $v \cdot U$ with $U \cdot v$ in the statement.

Proof. This follows from Proposition 3.2 and the construction of $\mathcal{R}_w$. $\square$

Proof of Theorem 5.2. It suffices to show the forward direction. We proceed by induction first on the size of the smallest rectangle containing $\Phi(w)$, then on the size of $\Phi(w) \setminus \mathrm{dom}(w)$. Our base case is when w is the identity permutation. Suppose $U \subset [n]^2$ is $\mathcal{R}_w$-avoiding. We may assume U contains all entries outside of $\Phi(w)$, since these are not contained in the spread of any element of $\mathcal{R}_w$.

If $D(w)_{*,[w(1),n]} = \emptyset$, let $w' := \mathrm{del}_{1,*}(w)$ and $U' := \{(i-1, j) : (i,j) \in U, i \geq 2\}$. Since U is $\mathcal{R}_w$-avoiding, U' must be $\mathcal{R}_{w'}$-avoiding, and by the induction hypothesis it is $X_{w'}$-completable. Therefore, by the row version of Proposition 3.9 and the discussion in the beginning of this section, U is X_w-completable.

Otherwise, let (r, c) be the essential box such that c is maximal. The uniqueness of such an essential box is a consequence of Lemma 4.2 and 4.3. We have two cases.

Case I: (r, c) is not rank 1. By assumption $c > w(1)$. By Lemma 4.3, we know $D(w)_{*,[w(1),n]} \subset \{r\} \times [w(1) + 1, c]$. It follows that w^{-1} is increasing in $[w(1), c]$.

Suppose there is $j \in [w(1), c]$ such that $|U_{*,j}| \leq n - 1$ and that $(i, j) \in \Phi(w) \setminus U$ for some i. If $|U_{*,j}| < n - 1$, we may enlarge U so that it contains everything except (i, j) in column j, since if $f \in \mathcal{R}_w$ satisfies $\mathrm{spr}(f)_{*,j} \neq \emptyset$, it must be the case that $\mathrm{spr}(f)_{*,j} = [r] \times \{j\}$: indeed, this is apparent when f is a CDG generator, and if (i, j) is contained in the spreads of two CDG generators, they both must use all of row 1 through r, so these generators cannot be merged. We may also without the loss of generality assume $j = c$, since by Lemma 5.3 we may replace U with $U \cdot t_{j,c}$ if $j \neq c$.

Let $w' := \mathrm{del}_{*,c}(w)$, and let $U' := U \setminus U_{*,c}$. Since $\mathcal{R}_{w'} \subset \mathcal{R}_w$, U' is $\mathcal{R}_{w'}$-avoiding. By the induction hypothesis, U' is $X_{w'}$-completable. By Proposition 3.9 or Proposition 3.11 (depending on whether $i = r$), U is X_w-completable.

Now, suppose that for all $j \in [w(1), c]$, we have $|U_{*,j}| = n$. Then let $w' := \mathrm{del}_1(w)$, and let $U' := U \setminus (U_{*,w(1)} \cup U_{1,*})$. Suppose to the contrary that there exists some $f \in \mathcal{R}_{w'}$ such that $\mathrm{spr}(f) \subset U'$. If $\mathrm{spr}(f) \cap ([n] \times [w(1), c]) = \emptyset$, the construction of w' implies that $f \in \mathcal{R}_w$. Otherwise, there must exist a unique $g \in \mathrm{CDG}(w')$ such that either $g = f$, or $f = \mathrm{merge}_{(r,\beta)}(g, h)$ for some $h \in \mathcal{R}_{w'}$ and $\beta < w(1)$. Let $\widetilde{g}$ be the unique generator in $\mathrm{CDG}(w)$ such that after setting $a_{1,w(1)} = 1$, $a_{1,j} = 0$ for all $w(1) < j \leq c$, and $a_{i,w(1)} = 0$ for all $1 < i \leq r$ in $\widetilde{g}$ we get g. Then set $\widetilde{f} := \mathrm{merge}_{(r,\beta)}(\widetilde{g}, h)$ if h exists, and set $\widetilde{f} := \widetilde{g}$ otherwise; we must have $\mathrm{spr}(\widetilde{f}) \subset U$ and $\widetilde{f} \in \mathcal{R}_w$. This contradicts that U is $\mathcal{R}_w$-avoiding. Therefore, U' is $\mathcal{R}_{w'}$-avoiding, and by the induction hypothesis, it is $X_{w'}$-completable. By Proposition 1.9, U is X_w-completable.

Case II: (r, c) is rank 1. We reduce to the case when (r, c) is the unique essential box in its region in $D(w)$ (i.e., this region is rectangular.) To see this, let r' be the largest row index such that $(r', w(1) + 1)$ is in $D(w)$ and in the same region as (r, c). If this region is not rectangular, there is some $r < \rho \leq r'$ such that $w(1) < w(\rho) \leq c$, $(\rho - 1, w(\rho))$ lies in $D(w)$, and $(\rho, w(\rho) - 1)$ lies in $D(w)$. Let $\rho' > \rho$ be the smallest such that $w(\rho') > w(\rho)$. Then set $\widetilde{w} := w t_{\rho,\rho'}$, which is a cover of w. Since there are no cells in $D(w)$ weakly

southeast of $(\rho, w(\rho))$, we have $\mathrm{CDG}(w) \subset \mathrm{CDG}(\widetilde{w})$. It follows that U is $\mathcal{R}_{\widetilde{w}}$-avoiding, so we may replace w with $\widetilde{w}$. By Proposition 3.17, if U is $X_{\widetilde{w}}$-completable, it is also X_w-completable.

Because (r, c) is an essential box in a rectangular region, w is increasing on $[r]$. Notice that $X_w \cap \mathbb{A}^{[1,r] \times [w(1),c]}$ is the variety of rank 1 matrices. By the discussion in Section 2.3, a subset $V \subset [1, r] \times [w(1), c]$ can be identified with a subgraph of a complete bipartite graph with vertex set $[1, r] \sqcup [w(1), c]$. If V is $\mathcal{R}_w$-avoiding, this subgraph must be a tree or a forest. To do induction, we first try to delete a "row leaf" or a "column leaf," i.e., a row or column which has only one entry in $U \cap \Phi(w)$.

By Lemma 4.2 and the assumption on (r, c), we see that for all $1 < i \le r$, all but one of the entries in $\Phi(w)_{i,*}$ are in $D(w)$, and for all $w(1) + 2 \le j \le c$, all but one of the entries $\Phi(w)_{*,j}$ are in $D(w)$.

Suppose there is $i \le r$ such that $|U_{i,*} \cap \Phi(w)| \le 1$. If $U_{i,*} \cap \Phi(w) = \emptyset$, we may pick some $(i, j) \in \Phi(w) \setminus \mathrm{dom}(w)$ to add to U; the result will still be $\mathcal{R}_w$-avoiding by Lemma 4.8. Suppose $(i, j) \in U$. If $i < r$, we replace U with $U \cdot t_{i,r}$ using Lemma 5.3, so we may assume that $i = r$. Let $w' := \mathrm{del}_{r,*}(w)$. Then, by Lemma 3.8, under the identification of $\iota_{r,*}$, $\mathcal{R}_{w'}$ is contained in $\mathcal{R}_w$. Therefore $\iota_{r,*}^{-1}(U \setminus \{(r, j)\})$ is $\mathcal{R}_{w'}$-avoiding. By the induction hypothesis it is $X_{w'}$-completable, which means that there exists some S' such that $(S')^c \supset \iota_{r,*}^{-1}(U \setminus \{(r, j)\})$, and $[z^{S'}]\mathfrak{F}_{w'} > 0$. Let $S := \iota_{r,*}(S') \cup ((\Phi(w))_{r,*} \setminus \{(r, j)\})$. By Proposition 3.9 or Proposition 3.11 (depending on whether $j = w(1)$), $[z^S]\mathfrak{F}_w > 0$. Since S^c contains U, U is X_w-completable.

By similar reasoning, if there is $j \in [w(1) + 2, c]$ such that $|U_{*,j}| = 1$, we induct using $w' := \mathrm{del}_{*,c}$.

We may now assume that for all $i \le r$, $|U_{i,*} \cap \Phi(w)| \ge 2$, and for all $j \in [w(1) + 2, c]$, $|U_{*,j} \cap \Phi(w)| \ge 2$.

Claim 5.4. Under these assumptions, if $U_{[r],[w(1),c]}$ is identified with a subgraph of the complete bipartite graph with vertex set $[r] \sqcup [w(1), c]$, U is a path graph, and $|U_{[r],w(1)}| = |U_{[r],w(1)+1}| = 1$.

Proof. Let $m := |U_{[r],[w(1),c]}|$ and $k = c - w(1) + 1$. Using the graph-theoretic interpretation, since U is $\mathcal{R}_w$-avoiding, $U_{[r],[w(1),c]}$ must be identified with a tree or a forest. Therefore $m \le r + k - 1$. By assumption, $m \ge 2r$ and $m \ge 2(k-2)$, so $m \ge r + k - 2$. Therefore $m = r + k - 1$ or $m = r + k - 2$. If $m = r + k - 1$, the only possibility is that $U_{[r],[w(1),c]}$ is a path graph, and $|U_{[r],w(1)}| = |U_{[r],w(1)+1}| = 1$. Otherwise, $U_{[r],[w(1),c]}$ is a forest, so it either has an isolated vertex (which must be one of column $w(1)$ and $w(1) + 1$), or at least four leaves. In either case, at least one row in $[r]$ or one column in $[w(1) + 2, c]$ must be a leaf, contradicting the assumption. $\square$

Since we may permute rows in $[r]$ using Lemma 5.3, we may assume that $(1, w(1)) \in U$ and that $(i, w(1) + 1) \in U$ for some $i \le r$.

Let $(r_2, c_2) \in \mathrm{ess}(w)$ such that $c_2 < c$ is maximal. Namely, (r_2, c_2) is the second essential box in $\Phi(w)$, scanning northeast to southwest. We have $r_2 > r$. If $c_2 < w(1)$, then X_w is isomorphic to the product of $X_w \cap \mathbb{A}^{[n] \times [w(1)-1]}$ and $X_w \cap \mathbb{A}^{[n] \times [w(1),n]}$, which are both smaller matrix Schubert varieties, so the claim follows by induction. Thus by Lemma 4.2 we assume $c_2 = w(1) + 1$.

Case II.1. There exists some $c' < c_2$ such that $(r_2, c') \in D(w)$ and (r_2, c') is not rank 0. Then we have $D(w)_{[r+1,r_2-1],[w(1),w(1)+1]} = \emptyset$. See Figure 5.

Claim 5.5. We may assume U contains $[r + 1, r_2] \times \{w(1) + 1\}$.

Proof. If $U \supset [r + 1, r_2] \times \{w(1)\}$, we may replace U with $U \cdot s_{w(1)}$ by Lemma 5.3. If U contains neither $[r+1, r_2] \times \{w(1)\}$ nor $[r+1, r_2] \times \{w(1)+1\}$, by Lemma 4.9, we may replace U with $U \cup ([r+1, r_2] \times \{w(1)+1\})$, since if $f \in \mathcal{R}_w$ has $\mathrm{spr}(f)_{[r+1,r_2],\{w(1)+1\}} \ne \emptyset$, there is $h \in \mathrm{cdg}(f)$ such that h is obtained from a Fulton generator of size $r_2 - r + 1$ and $\mathrm{spr}(h) \subset [r_2] \times [c_2]$. $\square$

We can cotransition at 1, and $w' := s_{w(1)}w$ is in $L_1(w)$, i.e., it is a term that appears on the right-hand side of the cotransition formula. We show that $U \setminus \{(1, w(1))\}$ is $\mathcal{R}_{w'}$-avoiding.
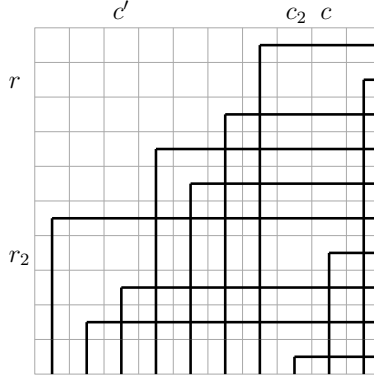
FIGURE 5. Case II.1

Suppose to the contrary that there is $f \in \mathcal{R}_{w'}$ which has $\mathrm{spr}(f) \subset U \setminus \{(1, w(1))\}$. We will construct a relation in $\mathcal{R}_w$ whose spread is contained in U. If f satisfies $\mathrm{spr}(f) \cap [r_1 + 1, r_2] \times \{w(1)\} = \emptyset$, then $f \in \mathcal{R}_w$. Since $(r_2, w(1)) \in \mathrm{ess}(w')$ has rank $r_2 - r - 1$, $\mathrm{spr}(f) \supset [r+1, r_2] \times \{w(1)\}$. Either $f \in \mathrm{CDG}(w')$, in which case we set $h := f$, or we have $f = \mathrm{merge}_{\alpha, \beta}(f_1, h)$, where $f_1 \in \mathcal{R}_{w'}$, $h \in \mathrm{CDG}(w')$ and $\mathrm{spr}(h) \supset [r+1, r_2] \times \{w(1)\}$. Note that since $\mathrm{spr}(f_1) \cap [r_1 + 1, r_2] \times \{w(1)\} = \emptyset$, we have $f_1 \in \mathcal{R}_w$. Let $\widetilde{h} \in \mathrm{CDG}(w)$ be the (unique) generator such that $\mathrm{spr}(\widetilde{h}) \supset \mathrm{spr}(h) \cup ([r+1, r_2] \times \{w(1)+1\}) \cup \{(1, w(1))\}$. If $f = h$, let $\widetilde{f} := \widetilde{h}$; otherwise let $\widetilde{f} := \mathrm{merge}_{\alpha, \beta}(f_1, \widetilde{h})$. Then if $(1, w(1)+1) \in U$, we have $\mathrm{spr}(\widetilde{f}) \subset U$. Otherwise, $\mathrm{spr}(\widetilde{f}) \setminus \{(1, w(1)+1)\} \subset U$. By Claim 5.4, there exists $f_2 \in \mathcal{R}_w$ with $\mathrm{spr}(f_2) \subset U_{[r], [w(1)+1, c]} \cup \{(1, w(1)+1)\}$. Then $\mathrm{merge}_{(1, w(1)+1)}(\widetilde{f}, f_2)$ lies in $\mathcal{R}_w$ and its spread is contained in U.

By induction, using Proposition 3.6, we conclude that U is X_w-completable.

Case II.2. For all $c' < c_2$, if $(r_2, c') \in D(w)$, then $(r_2, c') \in \mathrm{dom}(w)$. Let $r' > r$ denote the smallest row index such that $(r', c_2) \in D(w)$. See Figure 6.
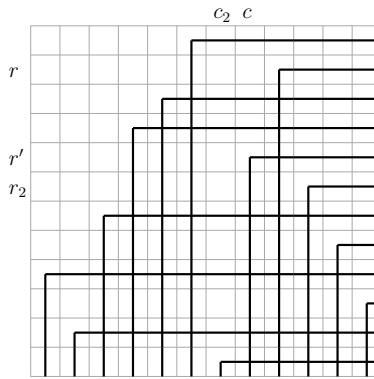

FIGURE 6. Case II.2

Let $m := r' - r - 1$. By Lemma 4.3, we have $w([r + 1, r' - 1]) = [w(1) - m, w(1) - 1]$. Let $V := ([r + 1, r'] \times [w(1) - m, w(1) + 1]) \setminus \mathrm{dom}(w)$.

We argue that we cannot have $V \subset U$. Suppose otherwise. Let $h \in \mathrm{CDG}(w)$ such that $\mathrm{spr}(h) = V \cup \{(1, w(1)), (1, w(1) + 1)\}$. Since U is $\mathcal{R}_w$-avoiding and $(1, w(1)) \in U$, it must be the case that $(1, w(1) + 1) \notin U$. However, by Claim 5.4, there is $f \in \mathcal{R}_w$, with $\mathrm{spr}(f) \subset U_{[r],[w(1)+1,c]} \cup \{(1, w(1) + 1)\}$. Then $\mathrm{merge}_{(1,w(1)+1)}(h, f)$ lies in $\mathcal{R}_w$, and its spread is contained in U.

Therefore, there exists $(i, j) \in V$ such that $(i, j) \notin U$. We may also assume that $|(U \cap \Phi(w))_{i,*}| = |V_{i,*}| - 1$, since if $\mathrm{spr}(f)$ for some $f \in \mathcal{R}_w$ intersects $V_{i,*}$ it must contain $V_{i,*}$. Let $w' := \mathrm{del}_{i,*}(w)$. If $i = r'$ and $j = w(1) + 1$, we apply Proposition 3.9; if $i = r'$ and $j < w(1) + 1$, we apply Proposition 3.11; if $i < r'$, we apply Proposition 3.15. The claim then follows by induction. $\qquad\square$

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