VANISHING THEOREMS FOR COMBINATORIAL GEOMETRIES

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ABSTRACT. We establish strong vanishing theorems for line bundles on wonderful varieties of hyperplane arrangements, and we show that the resulting positivity properties of Euler characteristics extend to all matroids. We achieve this by showing that every degeneration of a wonderful variety within the permutohedral toric variety is reduced and Cohen–Macaulay. The same holds for a larger class of subschemes in products of projective lines that we call "kindred," which are characterized by matroidal Hilbert polynomials. Our results give a new proof of the 20-year-old f-vector conjecture of Speyer and resolve the conjecture of Tohǎneanu that higher order Orlik–Terao algebras are Cohen–Macaulay.

1. INTRODUCTION

For a nonnegative integer n, let $[n] = \{1, \ldots, n\}$. For a subset $S \subseteq [n]$, let $\mathbf{e}_S := \sum_{i \in S} \mathbf{e}_i \in \mathbb{R}^n$ denote the sum of standard basis vectors indexed by S. A *matroid* M of *rank* r on $[n]$ is a nonempty collection $\mathcal{B}$ of subsets of $[n]$ of cardinality r, called the set of *bases* of M, satisfying the property:

> For every edge of the polytope $P(\mathrm{M}) :=$ the convex hull of $\{\mathbf{e}_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$,
> there is a pair $\{i \neq j\} \subseteq [n]$ such that the edge is parallel to $\mathbf{e}_i - \mathbf{e}_j$.

The polytope $P(\mathrm{M})$ is called the *base polytope* of M. We say that M is *loopless* if every $i \in [n]$ is contained in a basis of M. We point to [Wel76, Oxl11] as standard references on matroid theory and to [GGMS87] for the equivalence of the definition of matroids given here with theirs.

The prototypical example of a matroid is given by a linear subspace $L \subseteq \mathbb{k}^{[n]}$ over a field $\mathbb{k}$, which defines a matroid of rank $r = \dim L$ with set of bases

$$\{B \subseteq [n] : \text{the composition of } L \hookrightarrow \mathbb{k}^{[n]} \text{ with the projection } \mathbb{k}^{[n]} \twoheadrightarrow \mathbb{k}^B \text{ is an isomorphism}\}.$$

Matroids that arise in this way are called *realizable matroids*, and they provide an interface between matroid theory and geometry. A key medium of this interface occurs through the *wonderful variety* W_L of $L \subseteq \mathbb{k}^{[n]}$, introduced by de Concini and Procesi [dCP95]. To avoid trivialities, we assume that L is not contained in a coordinate hyperplane of $\mathbb{k}^{[n]}$, or, equivalently, that the matroid of L is loopless. Then W_L is constructed as follows. Write $\mathbb{P}V$ for the projective space of lines in a vector space V. Define a rational map

$$\phi \colon \mathbb{P}(\mathbb{k}^{[n]}) \dashrightarrow \prod_{i \neq j \in [n]} \mathbb{P}(\mathbb{k}^{\{i,j\}}) = (\mathbb{P}^1)^{\binom{n}{2}},$$

where each factor $\mathbb{P}(\mathbb{k}^{[n]}) \dashrightarrow \mathbb{P}(\mathbb{k}^{\{i,j\}})$ is the rational map induced by the coordinate projection $\mathbb{k}^{[n]} \to \mathbb{k}^{\{i,j\}}$. Then the wonderful variety W_L is the closure of $\phi(\mathbb{P}L)$ in $(\mathbb{P}^1)^{\binom{n}{2}}$, which is smooth and can be described as an iterated blow-up of $\mathbb{P}L$ [dCP95, Proposition 1.6] (see also Proposition 3.9 for the agreement of the definition in [dCP95] with ours). In particular W_L is contained in the closure of the image of ϕ, which we denote $X_{[n]}$ and call the *permutohedral variety (of dimension $n - 1$)*.

While almost all matroids are not realizable [Nel18], geometric properties of realizable matroids sometimes persist for all matroids. A landmark example arises from the intersection theory of W_L; because it is a smooth projective variety, its intersection numbers display positivity properties. This was the template for the development of the Hodge theory of matroids [AHK18], which established

the corresponding positivity properties for Chow rings of matroids. Here, we prove cohomological vanishing results for line bundles on W_L, and we show that the positivity properties of sheaf Euler characteristics implied by these vanishing results extend to all matroids.

The line bundles in question are the pullbacks to W_L of nef line bundles on $X_{[n]}$. Because $X_{[n]}$ is a smooth projective toric variety, these have the following combinatorial description; we point to [CLS11] as a standard reference on toric geometry and adopt its conventions. Applying to $X_{[n]}$ the standard dictionary between toric nef divisors and polytopes [CLS11, Chapter 6], one obtains a correspondence between nef toric divisor classes on $X_{[n]}$ and *generalized permutohedra*, which are lattice polytopes $P \subset \mathbb{R}^n$ satisfying the property that every edge of P is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j \in [n]$ (see [BEST23, Section 2.7] and [ACEP20] for details of this correspondence). For example, base polytopes of matroids are exactly generalized permutohedra which are contained in the unit cube. Let $\mathcal{L}_P$ denote the corresponding nef line bundle on $X_{[n]}$. We also write $\mathcal{L}_P$ for the restriction of the line bundle to W_L. Nef line bundles on projective toric varieties are basepoint-free, and the complete linear system of $\mathcal{L}_P$ induces a map $f_P \colon X_{[n]} \to \mathbb{P}(\Bbbk^{P \cap \mathbb{Z}^n})$, where the set of lattice points $P \cap \mathbb{Z}^n$ is identified with the torus-invariant basis of $H^0(X_{[n]}, \mathcal{L}_P)$ [CLS11, Chapter 4.3].

Theorem A. *Let P be a generalized permutohedron in $\mathbb{R}^n$. Then:*

(1) $H^i(W_L, \mathcal{L}_P^{\otimes a}) = 0$ *unless either:* (i) $a \geq 0$ *and* $i = 0$, *or* (ii) $a < 0$ *and* $i = \dim f_P(W_L)$.
(2) *For any $a \geq 0$, the restriction map $H^0(X_{[n]}, \mathcal{L}_P^{\otimes a}) \to H^0(W_L, \mathcal{L}_P^{\otimes a})$ is surjective.*
(3) *We have $Rf_{P*}\mathcal{O}_{W_L} = \mathcal{O}_{f_P(W_L)}$.*

In Section 5.1, we show that the conclusion of Theorem A fails for some other classes of nef or ample line bundles on wonderful varieties. In this sense, Theorem A appears to be sharp. An explicit combinatorial formula for $\dim f_P(W_L)$ is provided in Theorem 4.2. If P is of the maximal dimension, i.e., $(n-1)$-dimensional, then $\dim f_P(W_L) = \dim W_L = r - 1$. We note the following corollary.

Corollary 1.1. *The section ring $\bigoplus_{a \geq 0} H^0(W_L, \mathcal{L}_P^{\otimes a})$ coincides with the homogeneous coordinate ring of $f_P(W_L)$ in $\mathbb{P}(\Bbbk^{P \cap \mathbb{Z}^n})$, and in particular is generated in degree 1. The subvariety $f_P(W_L)$ in $\mathbb{P}(\Bbbk^{P \cap \mathbb{Z}^n})$ is projectively normal and arithmetically Cohen–Macaulay.*

Proof. Because a generalized permutohedron P is a normal lattice polytope (Proposition 3.9), the homogeneous coordinate ring of $\mathbb{P}(\Bbbk^{P \cap \mathbb{Z}^n})$ surjects onto the section ring $\bigoplus_{a \geq 0} H^0(X_{[n]}, \mathcal{L}_P^{\otimes a})$. Therefore (2) implies that the section ring $\bigoplus_{a \geq 0} H^0(W_L, \mathcal{L}_P^{\otimes a})$ is the homogeneous coordinate ring of the subvariety $f_P(W_L)$ in $\mathbb{P}(\Bbbk^{P \cap \mathbb{Z}^n})$. Then (3) implies that the section ring $\bigoplus_{a \geq 0} H^0(W_L, \mathcal{L}_P^{\otimes a})$ coincides with the section ring $\bigoplus_{a \geq 0} H^0(f_P(W_L), \mathcal{O}(a))$ of $\mathcal{O}(1)$ on $\mathbb{P}(\Bbbk^{P \cap \mathbb{Z}^n})$. Because W_L is smooth (and hence normal), the isomorphism $\mathcal{O}_{f_P(W_L)} \simeq f_{P*}\mathcal{O}_{W_L}$ from (3) implies that $f_P(W_L)$ is normal, and hence projectively normal. The projection formula and (3) imply that the natural map from $H^i(f_P(W_L), \mathcal{O}(a))$ to $H^i(W_L, \mathcal{L}_P^{\otimes a})$ is an isomorphism for all i and a. The cohomology vanishing in (1) implies that $f_P(W_L)$ is arithmetically Cohen–Macaulay: see, e.g., [EL23, Proposition 4.5]. $\qquad\square$

Special cases of this corollary recover results or resolve conjectures in the prior literature.

- When P is the convex hull of $\{\mathbf{e}_S : S \subset [n] \text{ and } |S| = n - 1\}$, the homogeneous coordinate ring of $f_P(W_L)$ is known as the *Orlik–Terao algebra* [OT94, Ter02], and Corollary 1.1 recovers a main result of [PS06] that this algebra is Cohen–Macaulay.
- More generally, when P is the convex hull of $\{\mathbf{e}_S : S \subset [n] \text{ and } |S| = n - k\}$ for $1 \leq k \leq n - 1$, the homogeneous coordinate ring of $f_P(W_L)$ is known as the *higher order Orlik–Terao algebra*,

studied extensively in the commutative algebra literature [GST18, TX21, Bur23]. In this case, Corollary 1.1 resolves and generalizes the conjecture of Tohǎneanu that the higher order Orlik–Terao algebra for $k = 2$ is Cohen–Macaulay [Toh21].

- When $P = -P(\mathrm{M})$, the convex hull of $\{-\mathbf{e}_B \, : \, B \text{ a basis of the matroid M of } L\}$, the variety $f_P(W_L)$ is known as *Kapranov's visible contours compactification*, appearing in the study of moduli of hyperplane arrangements [Kap93, HKT06, KT06]. This variety is the log-canonical model of a hyperplane arrangement complement associated to the data of $L \subseteq \Bbbk^{[n]}$. Corollary 1.1 states that this variety is arithmetically Cohen–Macaulay.

We now describe the numerical positivity of sheaf Euler characteristics implied by Theorem A that we will establish for all matroids. Corollary 1.1 implies that the polynomial $a \mapsto \chi(W_L, \mathcal{L}_P^{\otimes a})$ is the Hilbert function of a graded Cohen–Macaulay algebra which is generated in degree 1. The Hilbert functions of such algebras were classified by Macaulay: for a univariate polynomial $p(a)$ of degree d, if one writes

$$\sum_{a \geq 0} p(a) t^a = \frac{h_0^* + h_1^* t + \cdots + h_d^* t^d}{(1-t)^{d+1}},$$

then the polynomial p is the Hilbert function of a graded Cohen–Macaulay algebra that is generated in degree 1 if and only if $(h_0^*, \ldots, h_d^*)$ is a *Macaulay vector* (also known as an M-vector or O-sequence). This means that $h_i^* \geq 0$ for each i and that the sequence $(h_0^*, \ldots, h_d^*)$ satisfies certain explicit inequalities bounding how quickly it can grow; see [BH93, Theorem 4.2.10] for details. Note that $(h_0^*, \ldots, h_d^*)$ are determined by the equation

$$p(a) = \sum_{i=0}^{d} h_i^* \binom{a + d - i}{d}.$$

In particular, we see that $h_d^* = (-1)^d p(-1)$, and that $(-1)^d p(a) \geq 0$ for $a < 0$.

Previous work allows us to state an analogue of the above numerical positivity for all matroids, as we now describe. The proof of [BEST23, Corollary 10.6] gave an explicit combinatorial formula for the K-class $[\mathcal{O}_{W_L}]$ in the Grothendieck group $K_\circ(X_{[n]})$, showing that the class depends only on the matroid of L. In particular, for any given $\mathcal{L} \in \mathrm{Pic}(X_{[n]})$, by the projection formula for $W_L \hookrightarrow X_{[n]}$, the quantity $\chi(W_L, \mathcal{L}|_{W_L}) = \chi(X_{[n]}, \mathcal{O}_{W_L} \otimes \mathcal{L})$ depends only on the matroid realized by L. This suggested the possibility of generalizing the map $\chi(W_L, -)$ to a function whose first argument is a (not necessarily realizable) matroid. The authors of [LLPP24] carried out this program, defining the *matroid K-ring* $K(\mathrm{M})$ of a loopless matroid M, equipped with a surjection $K^\circ(X_{[n]}) \twoheadrightarrow K(\mathrm{M})$ and an Euler characteristic map $\chi(\mathrm{M}, -) \colon K(\mathrm{M}) \to \mathbb{Z}$, so that, when L realizes M, there is an isomorphism $K(\mathrm{M}) \cong K^\circ(W_L)$ identifying the functions $\chi(\mathrm{M}, -)$ and $\chi(W_L, -)$. Then, the authors of [EL23] defined the h^*-vector $h^*(\mathrm{M}, \mathcal{L})$ of a loopless matroid M and a line bundle $\mathcal{L} \in \mathrm{Pic}(X_{[n]})$ via the equality

$$\sum_{a \geq 0} \chi(\mathrm{M}, \mathcal{L}^{\otimes a}) t^a = \frac{h_0^*(\mathrm{M}, \mathcal{L}) + h_1^*(\mathrm{M}, \mathcal{L}) t + \cdots + h_d^*(\mathrm{M}, \mathcal{L}) t^d}{(1-t)^{d+1}},$$

where d is the degree of the polynomial $a \mapsto \chi(\mathrm{M}, \mathcal{L}^{\otimes a})$. When M is realizable and $\mathcal{L}$ is the line bundle associated to a generalized permutohedron, we have explained how Corollary 1.1 implies that $h^*(\mathrm{M}, \mathcal{L})$ is a Macaulay vector. We show this holds more generally, proving [EL23, Conjecture 4.8].

Theorem B. *Let* M *be a loopless matroid on* $[n]$, *and let* P *be a generalized permutohedron in* $\mathbb{R}^n$. *Then, the* h^*-*vector* $h^*(\mathrm{M}, \mathcal{L}_P)$ *is a Macaulay vector.*

An explicit combinatorial formula for the degree d of the polynomial $a \mapsto \chi(\mathrm{M}, \mathcal{L}_P^{\otimes a})$ is given in Theorem 4.2. If P is of maximal dimension $(n-1)$, then $d = \mathrm{rank}(\mathrm{M}) - 1$.

A consequence of Theorem B is a new proof of the 20-year-old f-vector conjecture of Speyer [Spe05], as we now explain. If P is a generalized permutohedron, then the negated polytope $-P = \{-p : p \in P\}$ is as well. Define the invariant $\omega(\mathrm{M}) \in \mathbb{Z}$ of a loopless matroid M of rank r by

$$\omega(\mathrm{M}) = (-1)^{r-n+\dim P(\mathrm{M})} \chi(\mathrm{M}, \mathcal{L}_{-P(\mathrm{M})}^{-1}).$$

The ω invariant is the leading coefficient of Speyer's invariant $g_{\mathrm{M}}(t) \in \mathbb{Z}[t]$, which was defined in [Spe09, FS12] in an attempt to bound the complexity of polyhedral subdivisions of the base polytope $P(\mathrm{M})$ of a matroid M into base polytopes of matroids. Such subdivisions had arisen in the study of Grassmannians and moduli spaces of hyperplane arrangements [Laf03, Kap93, HKT06]; their more recent appearances are surveyed in [Ard18, Section 4.4]. Speyer conjectured an upper bound on the number of faces of each dimension of such a subdivision, the titular f-vector. He reduced the conjecture to showing that the coefficients of $g_{\mathrm{M}}(t)$ were of predictable sign (the manifestation in this context of K-theoretic positivity), and he proved this in [Spe09] for matroids realizable over fields of characteristic zero. The authors of [FSS24] further reduced the conjecture for general matroids to the statement of Corollary 1.2 below, which was first proved in [BF24].

Corollary 1.2. *For any loopless matroid* M*, one has* $\omega(\mathrm{M}) \geq 0$*.*

If M is realizable over a field $\Bbbk$ of characteristic zero and $\dim P(\mathrm{M}) = n - 1$, then $\mathcal{L}_P$ is nef and big, so this follows immediately from the Kawamata–Viehweg vanishing theorem.

Proof of Corollary 1.2. Because $P(\mathrm{M})$ and $-P(\mathrm{M})$ induce the same partition of $[n]$ in the sense of Section 4, Corollary 4.6 implies that the polynomial $a \mapsto \chi(\mathrm{M}, \mathcal{L}_{-P(\mathrm{M})}^{\otimes a})$ has degree $d := r - n + \dim P(\mathrm{M})$. Then $(-1)^d \chi(\mathrm{M}, \mathcal{L}_{-P(\mathrm{M})}^{-1}) = h_d^*(\mathrm{M}, \mathcal{L}_{-P(\mathrm{M})})$, which is nonnegative by Theorem B. $\square$

We sketch the proof of the main theorems in the case when $\mathcal{L}_P$ is ample on $X_{[n]}$. If one can show that the wonderful variety W_L degenerates to a reduced and Cohen–Macaulay union of torus-invariant strata in $X_{[n]}$, then the vanishing results in Theorem A follow from Frobenius splitting techniques. In fact, we produce a collection of K-classes on $X_{[n]}$ such that every subscheme having one of these K-classes is reduced and Cohen–Macaulay: we call these subschemes "kindred." We prove that W_L is kindred. Since W_L always has some torus-invariant degeneration, which shares its K-class, Theorem A follows. As we have explained, this also gives Theorem B when M is realizable. In this way, we avoid the difficult task of explicitly controlling the combinatorics of degenerations of W_L. To extend Theorem B to all matroids, the task remaining is to obtain a kindred subscheme of $X_{[n]}$ whose K-class is the one appropriate to compute $\chi(\mathrm{M}, -)$. We construct a union of torus-invariant subvarieties with the desired property, which we call the "tropical initial degeneration of M," using the tropical degeneration formula of Katz [Kat09, Theorem 10.1].

Organization. In Section 2, we define and exhibit properties of kindred subschemes. There we work in the setting of subschemes of $(\mathbb{P}^1)^\ell$. Applications to our main theorems are obtained via the embedding $X_{[n]} \hookrightarrow (\mathbb{P}^1)^{\binom{n}{2}}$ and coordinate projections thereof. In Section 3, we construct the tropical initial degeneration, relate it to the wonderful variety and to matroid Euler characteristics, and prove our main theorems. In Section 4, we give a formula for the degree of the polynomial $a \mapsto \chi(\mathrm{M}, \mathcal{L}_P^{\otimes a})$.

In Section 5, we collect examples that support the optimality of Theorem A, and we establish similar vanishing theorems for close cousins of wonderful varieties.

Previous work. When P is a Minkowski sum of standard simplices, Theorem B was proved in [EL23]. However, this class of polytopes does not include many of the most interesting cases, such as the ones listed below Corollary 1.1.

In [Liu], Liu extends the techniques of [BF22] to show that, if $\operatorname{char} \Bbbk = 0$ and P is a Minkowski sum of matroid polytopes, then $H^i(W_L, \mathcal{L}_P) = 0$ for $i > 0$, proving part of Theorem A(1) in this case.

Corollary 1.2 was also proved in [BF24] by a completely different method. The authors control the sign of Euler characteristics of the form $\chi(X_{[n]}, [\bigwedge^i \mathcal{Q}_{\mathrm{M}_1}^\vee] \otimes [\bigwedge^j \mathcal{Q}_{\mathrm{M}_2}^\vee])$ for a pair of matroids $\mathrm{M}_1, \mathrm{M}_2$ on $[n]$, where $\mathcal{Q}_{\mathrm{M}}$ is one of the tautological classes of [BEST23]. It follows from [LLPP24, Proposition 5.3] (see also [BEST23, Corollary 10.6]) that

$$\chi(\mathrm{M}_1, \mathcal{L}_{-P(\mathrm{M}_2)}^{-1}) = \chi(\mathrm{M}_1, [\textstyle\bigwedge^{n-r} \mathcal{Q}_{\mathrm{M}_2}^\vee]) = \sum_{i \geq 0} (-1)^i \chi(X_{[n]}, [\textstyle\bigwedge^i \mathcal{Q}_{\mathrm{M}_1}^\vee] \otimes [\textstyle\bigwedge^{n-r} \mathcal{Q}_{\mathrm{M}_2}^\vee]).$$

When $\mathrm{M}_1 = \mathrm{M}_2 = \mathrm{M}$, [FS12, Lemma 6.2] gives that $\chi(X_{[n]}, [\bigwedge^i \mathcal{Q}_{\mathrm{M}}^\vee] \otimes [\bigwedge^{n-r} \mathcal{Q}_{\mathrm{M}}^\vee]) = 0$ unless $i = r$, and so $\omega(\mathrm{M}) = (-1)^{\dim P(\mathrm{M})-n} \chi(X_{[n]}, [\bigwedge^r \mathcal{Q}_{\mathrm{M}}^\vee] \otimes [\bigwedge^{n-r} \mathcal{Q}_{\mathrm{M}}^\vee])$. Using this formula, the positivity result [BF24, Theorem D] then implies Corollary 1.2. However, the techniques of [BF24] cannot be used to prove either Theorem A or Theorem B, nor can the results in this paper be used to deduce the positivity consequences of [BF24, Theorem D].

2. KINDRED SUBSCHEMES

We begin by introducing a remarkable family of connected components of the Hilbert scheme of a product of projective lines. Subschemes in these components, which we call kindred subschemes, have a very close relationship to their degenerations. Kindred subschemes are a special case of subschemes defined by Cartwright–Sturmfels ideals [CDNG15, CDNG20] (Remark 2.13).

A set is *independent* in a matroid M if it is a subset of a basis of M.

Definition 2.1. A closed subscheme X of $(\mathbb{P}^1)^\ell$ is said to be *kindred* if it is empty, or if there is a matroid M on $[\ell]$ such that, for all $(a_1, \ldots, a_\ell) \in \mathbb{Z}^\ell$, we have

$$\chi(X, \mathcal{O}(a_1, \ldots, a_\ell)) = \sum_{I \text{ independent in M}} \prod_{i \in I} a_i.$$

We call the matroid M appearing in Definition 2.1 the *progenitor matroid* of X. It is clear that whether a subscheme X is kindred only depends on the class $[\mathcal{O}_X]$ in the Grothendieck group of coherent sheaves $K_\circ((\mathbb{P}^1)^\ell)$. The function $(a_1, \ldots, a_\ell) \mapsto \chi(X, \mathcal{O}(a_1, \ldots, a_\ell))$ is a polynomial known as the *Snapper polynomial* of X, after [Sna59]. It agrees with the $\mathbb{Z}^\ell$-graded Hilbert polynomial of any

ideal I determining X in the homogeneous coordinate ring $\Bbbk[(\mathbb{P}^1)^\ell] = \Bbbk[x_1, \ldots, x_\ell, y_1, \ldots, y_\ell]$, i.e., for $a_1, \ldots, a_\ell$ sufficiently large, we have

$$\chi(X, \mathcal{O}(a_1, \ldots, a_\ell)) = \dim_\Bbbk(\Bbbk[(\mathbb{P}^1)^\ell]/I)_{(a_1, \ldots, a_\ell)}.$$

Recall that the fundamental class $[Y]$ of a subscheme Y of $(\mathbb{P}^1)^\ell$ is the class in $A_{\dim Y}((\mathbb{P}^1)^\ell)$ given by summing the classes of the top-dimensional irreducible components of Y, with coefficient equal to the multiplicity of that component. More generally, if $\mathcal{F}$ is a coherent sheaf on $(\mathbb{P}^1)^\ell$, then its fundamental class is the sum of the top-dimensional irreducible components of its support, with coefficient equal to the rank of the sheaf at the generic point of that component.

For a subset S of $[\ell]$, let $Y_S = \{(x_1, \ldots, x_\ell) : x_i = 0 \text{ if } i \notin S\} \subset (\mathbb{P}^1)^\ell$. Then $\{[Y_S]\}_{S \subseteq [\ell]}$ is a basis for $A_*((\mathbb{P}^1)^\ell)$. By the next proposition, an application of the Hirzebruch–Riemann–Roch theorem, we can recover the fundamental class of a sheaf from its Snapper polynomial.

Proposition 2.2. [Ful98, Example 15.2.16 & Example 15.1.5] *Let $\mathcal{F}$ be a coherent sheaf on $(\mathbb{P}^1)^\ell$ whose support has dimension d and whose fundamental class is $\sum_{S \in \binom{[\ell]}{d}} c_S [Y_S]$. Then the polynomial $(a_1, \ldots, a_\ell) \mapsto \chi((\mathbb{P}^1)^\ell, \mathcal{F}(a_1, \ldots, a_\ell))$ has degree d, and its degree d part is $\sum_{S \in \binom{[\ell]}{d}} c_S \prod_{i \in S} a_i$.*

Applying Proposition 2.2 when $\mathcal{F} = \mathcal{O}_X$ for some kindred $X \subseteq (\mathbb{P}^1)^\ell$, we have the following formula.

Corollary 2.3. *Let X be a kindred subscheme of $(\mathbb{P}^1)^\ell$ with progenitor matroid M. Then we have*

$$[X] = \sum_{B \text{ basis of } \mathrm{M}} [Y_B] \in A_{\dim X}((\mathbb{P}^1)^\ell).$$

Proposition 2.4. *Let M be a matroid on $[\ell]$, and let $Y_\mathrm{M} = \bigcup_{B \text{ basis}} Y_B$. Then Y_M is a kindred subscheme of $(\mathbb{P}^1)^\ell$ with progenitor matroid M.*

Proof. This is a special case of [EL23, Proposition 2.14], but we sketch an easy direct proof. The homogeneous coordinate ring of Y_M is the quotient of $\Bbbk[x_1, \ldots, x_\ell, y_1, \ldots, y_\ell]$ by the ideal $\langle \prod_{i \in D} y_i : D \subseteq [\ell] \text{ not independent in } \mathrm{M} \rangle$. The $\mathbb{Z}^\ell$-graded Hilbert function of this ring is given by the formula in Definition 2.1 since, for $\boldsymbol{a} \in \mathbb{Z}^\ell_{\geq 0}$, the set $\{\prod_{i \in [\ell]} x_i^{a_i - b_i} y_i^{b_i} : \{i : b_i > 0\} \text{ is independent in } \mathrm{M}\}$ forms a monomial basis for the degree $\boldsymbol{a}$ component of the quotient ring. Because the Hilbert function agrees with the Snapper polynomial (i.e., the multigraded Hilbert polynomial) on points of $\mathbb{Z}^\ell$ which are sufficiently deep in the positive orthant, this implies the result. $\square$

Proposition 2.5. *Let X be a kindred subscheme of $(\mathbb{P}^1)^\ell$ with progenitor matroid M. Then X degenerates to Y_M inside of $(\mathbb{P}^1)^\ell$.*

Proof. Let B be the ℓ-fold product of the group of 2×2 lower triangular matrices acting factorwise on $(\mathbb{P}^1)^\ell$, with unique fixed point given by $(0, \ldots, 0)$. Then B acts on the Hilbert scheme of $(\mathbb{P}^1)^\ell$. Because B is solvable, by the Borel fixed point theorem, the B-orbit of the point corresponding to X in the Hilbert scheme has a B-fixed point in its closure. This gives a degeneration of X to a subscheme Z of $(\mathbb{P}^1)^\ell$ which is fixed by B.

We have $[\mathcal{O}_X] = [\mathcal{O}_Z]$ in $K_\circ((\mathbb{P}^1)^\ell)$, and therefore $[X] = [Z] \in A_{\dim X}((\mathbb{P}^1)^\ell)$. The reduction Z^{red} is a union of B-fixed subvarieties of $(\mathbb{P}^1)^\ell$. The only B-fixed subvarieties of $(\mathbb{P}^1)^\ell$ are the Y_S, for various S. These have linearly independent classes in $A_{\dim X}((\mathbb{P}^1)^\ell)$, so in order to have $[X] = [Z]$, Corollary 2.3 shows that Z must contain Y_M, where M is the progenitor matroid of X. By Proposition 2.4, Y_M is kindred, and so $[\mathcal{O}_{Y_\mathrm{M}}] = [\mathcal{O}_X] = [\mathcal{O}_Z]$. In particular, $Y_\mathrm{M} = Z$. $\square$

Corollary 2.6. *Let X be a kindred subscheme of $(\mathbb{P}^1)^\ell$. Then X is Cohen–Macaulay, pure dimensional, geometrically reduced, and geometrically connected.*

Proof. Let M be the progenitor matroid of X. It is clear that Y_M is pure dimensional, geometrically reduced, and geometrically connected. Furthermore, it is known that Y_M is Cohen–Macaulay: the multihomogeneous coordinate ring of Y_M is equal to (the extension by extra variables of) the Stanley–Reisner ring of the matroid independence complex, which is Cohen–Macaulay [PB80, Theorem 3.2.1]. Therefore the same properties are true for X [Sta25, Tag 045U, Tag 02NM, Tag 0C0D, Tag 055J]. $\quad\square$

Corollary 2.7. *Let X be a kindred subscheme of $(\mathbb{P}^1)^\ell$. Then, for any $a_1, \ldots, a_\ell \geq 0$, we have*
$$H^i(X, \mathcal{O}(a_1, \ldots, a_\ell)) = 0 \text{ for } i > 0,$$
and the restriction map
$$H^0((\mathbb{P}^1)^\ell, \mathcal{O}(a_1, \ldots, a_\ell)) \to H^0(X, \mathcal{O}(a_1, \ldots, a_\ell))$$
is surjective. For any $a_1, \ldots, a_\ell < 0$, we have
$$H^i(X, \mathcal{O}(a_1, \ldots, a_\ell)) = 0 \text{ for } i < \dim X.$$

Proof. By upper semicontinuity [Sta25, Tag 0BDN], it suffices to prove these statements for Y_M. For the surjectivity of the restriction map, one applies upper semicontinuity to the kernel of the surjective map $H^0((\mathbb{P}^1)^\ell, \mathcal{O}(a_1, \ldots, a_\ell)) \otimes \mathcal{O}_X \to \mathcal{O}_X(a_1, \ldots, a_\ell)$ of vector bundles.

When $(\mathbb{P}^1)^\ell$ is viewed as a homogeneous space for $(GL_2)^\ell$, Y_M is a union of Schubert varieties. In particular, Y_M is defined over $\mathrm{Spec}\,\mathbb{Z}$, and the reduction of Y_M modulo any prime p is compatibly Frobenius split in $(\mathbb{P}^1)^\ell$ relative to an ample divisor [BK05, Theorem 2.3.10, Proposition 1.2.1]. The line bundle $\mathcal{O}(a_1, \ldots, a_\ell)$ is defined over the prime field $\mathbb{F}$ of $\mathbb{k}$, and we can check the cohomology vanishing and the surjectivity of the restriction map over $\mathbb{F}$ [Sta25, Tag 02KH]. If the characteristic of $\mathbb{k}$ is positive, then the splitting and Cohen–Macaulayness of Y_M implies the vanishing and the surjectivity: see [BK05, Theorem 1.4.8, Lemma 1.4.7(ii), and Theorem 1.2.9] as well as [Bri03, Section 4]. If $\mathbb{k}$ has characteristic 0, then the desired result follows from upper semicontinuity. $\quad\square$

We will make use of the following standard consequence of the Leray spectral sequence and Serre vanishing. See, for example, [Hyr99, Lemma 2.1].

Proposition 2.8. *Let $f\colon X \to Y$ be a proper morphism, and let $\mathcal{L}$ be an ample line bundle on Y. Then $R^i f_* \mathcal{O}_X = 0$ for all $i > 0$ if and only if $H^i(X, f^*\mathcal{L}^{\otimes n}) = 0$ for all $i > 0$ and all n sufficiently large.*

Proposition 2.9. *Let X be a kindred subscheme of $(\mathbb{P}^1)^\ell$, and let $p\colon (\mathbb{P}^1)^\ell \to (\mathbb{P}^1)^m$ be a coordinate projection. Then $p(X)$ is a kindred subscheme of $(\mathbb{P}^1)^m$ and $Rp_* \mathcal{O}_X = \mathcal{O}_{p(X)}$.*

Proof. We may assume that p is the projection onto the first m coordinates. By Proposition 2.7, the higher cohomology on X of positive twists of $\mathcal{O}(1, \ldots, 1, 0, \ldots, 0) = p^*\mathcal{O}(1, \ldots, 1)$ vanishes. By Proposition 2.8, we have $R^i p_* \mathcal{O}_X = 0$ for $i > 0$.

We proceed to show that the natural map from $\mathcal{O}_{p(X)}$ to $p_* \mathcal{O}_X$ is an isomorphism. Because the higher direct images vanish, we have

(1) $$\chi((\mathbb{P}^1)^m, p_* \mathcal{O}_X \otimes \mathcal{O}(a_1, \ldots, a_m)) = \chi(X, \mathcal{O}(a_1, \ldots, a_m, 0, \ldots, 0))$$

for any $a_1, \ldots, a_m$. Let M be the progenitor of X, and let N denote the restriction of M to $[m]$, i.e., N is the matroid whose independent sets are the independent sets of M which are contained in $[m]$. By

considering the leading term of the polynomial on the right-hand side of (1), Proposition 2.2 implies that the fundamental class of $p_*\mathcal{O}_X$ is $\sum_{B \text{ basis of N}} [Y_B]$. Because the support of $p_*\mathcal{O}_X$ is $p(X)$, this implies that the fundamental class of $p(X)$ is $\sum_{B \text{ basis of N}} [Y_B]$.

As in the proof of Proposition 2.5, we can degenerate $p(X)$ to a subscheme Z which contains $Y_N :=$ $\bigcup_{B \text{ basis of N}} Y_B$. Let $\mathcal{I}$ be the ideal sheaf of Y_N in Z. We have

$$[\mathcal{O}_{p(X)}] - [\mathcal{O}_{Y_N}] = [\mathcal{I}] \in K_\circ((\mathbb{P}^1)^m).$$

On the other hand, because there is an injective map $\mathcal{O}_{p(X)} \to p_*\mathcal{O}_X$ we have $[\mathcal{O}_{p(X)}] = [p_*\mathcal{O}_X] - [p_*\mathcal{O}_X/\mathcal{O}_{p(X)}]$. By comparing the Snapper polynomials and using Proposition 2.4, we have that $[p_*\mathcal{O}_X] = p_*[\mathcal{O}_X] = [\mathcal{O}_{Y_N}]$. We deduce that

$$-[p_*\mathcal{O}_X/\mathcal{O}_{p(X)}] = [\mathcal{I}].$$

For n sufficiently large, we have $\chi((\mathbb{P}^1)^m, p_*\mathcal{O}_X/\mathcal{O}_{p(X)} \otimes \mathcal{O}(n, \ldots, n)) \geq 0$, with equality if and only if $p_*\mathcal{O}_X/\mathcal{O}_{p(X)} = 0$. Similarly, $\chi((\mathbb{P}^1)^m, \mathcal{I} \otimes \mathcal{O}(n, \ldots, n)) \geq 0$, with equality if and only if $\mathcal{I} = 0$. We deduce that $p_*\mathcal{O}_X = \mathcal{O}_{p(X)}$ and $\mathcal{O}_Z = \mathcal{O}_{Y_N}$, so $p(X)$ is kindred.   □

Remark 2.10. It is to obtain Proposition 2.9 that we have included the matroid condition in the definition of kindred subschemes. Proposition 2.5 and its corollaries hold true for the broader class of subschemes whose Snapper polynomial is $\sum_{I \in \mathcal{S}} \prod_{i \in I} a_i$ for any family $\mathcal{S} \subseteq 2^{[\ell]}$ of sets closed under taking subsets, all of whose maximal elements have the same cardinality: let us call such families $\mathcal{S}$ *pure*. In the proof of Proposition 2.9, we see that the fundamental class of the projection of X to $(\mathbb{P}^1)^J$ is a sum over the maxima of the set family $\{I \in \mathcal{S} : I \subseteq J\}$. Requiring this set family to be pure for all $J \subseteq [\ell]$ forces $\mathcal{S}$ to be the independent set family of a matroid (or empty) [Oxl11, Exercise 1.1.3]. In any event, the matroid condition loses us no generality for integral subschemes (Proposition 2.11).

We now give examples of kindred subschemes and discuss related results in the literature. The most important example for us will be Corollary 3.7, to come. Many other examples arise from the following result of Brion [Bri03] (see also [Per14, §4.4] for an exposition in the context of spherical varieties).

Proposition 2.11. *Let X be an integral subvariety of $(\mathbb{P}^1)^\ell$ whose fundamental class is multiplicity-free, i.e., the coefficients in the expansion of $[X]$ in terms of the $[Y_S]$ are all either 0 or 1. Then X is kindred.*

Proof. By [BH20, Theorem 4.6] or [CCRL$^+$20], the fundamental class of X is equal to the fundamental class of Y_M for some matroid M. By [Bri03], X degenerates to a reduced union of the Y_S. As the fundamental class is preserved under this degeneration, X degenerates to Y_M, which is kindred by Proposition 2.4. The degeneration also preserves the class in $K_\circ((\mathbb{P}^1)^\ell)$, so X is kindred.   □

Example 2.12. The wonderful variety W_L, embedded in $(\mathbb{P}^1)^{\binom{n}{2}}$ as in Section 1, has a fundamental class which is multiplicity-free [Li18], and so W_L is a kindred subscheme. This also follows directly from [LLPP24, Corollary 7.5], which computes the Snapper polynomial of W_L: see the proof of Corollary 3.7 below for more detail. As a consequence, any subscheme Z of $X_{[n]}$ with $[\mathcal{O}_Z] = [\mathcal{O}_{W_L}]$ in $K_\circ(X_{[n]})$ defines a kindred subscheme of $(\mathbb{P}^1)^{\binom{n}{2}}$, and in particular is reduced and Cohen–Macaulay.

Many further examples of subvarieties of $(\mathbb{P}^1)^\ell$ whose fundamental class is multiplicity-free can be obtained by taking the closure in $(\mathbb{P}^1)^\ell$ of certain subvarieties of $\mathbb{A}^\ell$. Closures of linear subspaces [AB16] and of certain determinantal varieties [HL24] are multiplicity-free, and so give examples of kindred subschemes. A fact central to [BF24] is that if L_1 and L_2 are linear subspaces of $\mathbb{k}^n$ such that

L_2 is not contained in any coordinate hyperplane, then the closure of the semi-inverted Hadamard product $\{v_1 \cdot v_2^{-1} : v_1 \in L_1,\ v_2 \in L_2 \cap (\Bbbk^*)^n\}$ in $(\mathbb{P}^1)^n$ is multiplicity-free.

Remark 2.13. Kindred subschemes of $(\mathbb{P}^1)^\ell$ are closely related to *Cartwright–Sturmfels* ideals in the $\mathbb{N}^\ell$-graded ring $\Bbbk[(\mathbb{P}^1)^\ell] = \Bbbk[x_1, \ldots, x_\ell, y_1, \ldots, y_\ell]$, the homogeneous coordinate ring of $(\mathbb{P}^1)^\ell$ [CDNG15, CDNG20]. These are, by definition, homogeneous ideals whose multigraded generic initial ideal is radical. It is a consequence of Proposition 2.7 and Proposition 2.5 that the saturated ideal in $\Bbbk[(\mathbb{P}^1)^\ell]$ defining a kindred subscheme is a Cartwright–Sturmfels ideal. Conversely, the subscheme of $(\mathbb{P}^1)^\ell$ which is defined by a Cartwright–Sturmfels ideal whose multidegree is the bases of a matroid is a kindred subscheme. However, the K-theoretic nature of the definition of kindred schemes is more convenient for our purposes.

Remark 2.14. Finally, we remark that in [Bri01] (see also [BF22, Theorem 4.3]), Brion showed that an integral subvariety of $(\mathbb{P}^1)^\ell$ whose multidegree is multiplicity-free is normal and, if the characteristic of $\Bbbk$ is 0, has rational singularities. This complements Proposition 2.9, as in some cases the singularities of a kindred subvariety can be resolved using a projection from a kindred subvariety embedded in a larger product of projective lines. In this manner Proposition 2.9 can be used to replace the invocations of rational singularities in the collapsing arguments of [BF22], thus allowing the characteristic 0 assumption of that work to be removed.

3. VANISHING THEOREMS

In this section we introduce a subscheme of the permutohedral variety $X_{[n]}$ for each loopless matroid M, the *tropical initial degeneration* $\mathrm{ind}_w \mathrm{M}$. It depends on the choice of a "sufficiently generic" weight w. When M is realized by $L \subseteq \Bbbk^n$, the subscheme $\mathrm{ind}_w \mathrm{M}$ will be a Gröbner degeneration of W_L inside of $X_{[n]}$. Recall that there is an embedding of $X_{[n]}$ into $(\mathbb{P}^1)^{\binom{n}{2}}$. We will show that $\mathrm{ind}_w \mathrm{M}$ is a kindred subscheme of $(\mathbb{P}^1)^{\binom{n}{2}}$, and we will use this to prove Theorems A and B.

Let $\Sigma_{[n]}$ be the *permutohedral fan*, the fan of the toric variety $X_{[n]}$, so that cones σ of $\Sigma_{[n]}$ index closed boundary strata $V(\sigma)$ of $X_{[n]}$. To each loopless matroid M on $[n]$, there is an associated subfan Σ_{M} of $\Sigma_{[n]}$ called the *Bergman fan* of M [AK06]. Concretely, $\Sigma_{[n]}$ is the rational fan on $\mathbb{Z}^n/\mathbb{Z}\mathbf{e}_{[n]}$ whose rays are generated by the $\mathbf{e}_S$ for all nonempty proper subsets $S \subset [n]$, and in which $\mathbf{e}_{S_1}, \ldots, \mathbf{e}_{S_k}$ lie in a common cone if and only if, up to reordering, $S_1 \subset \cdots \subset S_k$. The *rank* $\mathrm{rk}_{\mathrm{M}}(S)$ of a set $S \subseteq [n]$ in the matroid M is the size of a largest intersection of S with a basis of M, and a *flat* of M is a subset of $[n]$ maximal with respect to its rank. Then Σ_{M} contains exactly the cones of $\Sigma_{[n]}$ whose ray generators all correspond to flats of M. The dimension of Σ_{M} is $r - 1$, where r is the rank of M.

Throughout, choose some $w \in \mathbb{Q}^n/\mathbb{Q}\mathbf{e}_{[n]}$ which is sufficiently generic, in the sense that for each pair of cones σ and τ of $\Sigma_{[n]}$, if the intersection $\sigma \cap (w + \tau)$ is nonempty, then it is transverse. In particular, if $\dim \sigma + \dim \tau < n - 1$, we require that σ and $w + \tau$ are disjoint.

Definition 3.1. For a loopless matroid M of rank r on $[n]$, let $\mathrm{ind}_w \mathrm{M}$ be the union of those strata $V(\sigma) \subseteq X_{[n]}$ such that σ intersects a maximal cone of $w + \Sigma_{\mathrm{M}}$, where σ is a cone of $\Sigma_{[n]}$ of codimension $r - 1$. If M has loops, then define $\mathrm{ind}_w \mathrm{M}$ to be empty.

Note that $\mathrm{ind}_w \mathrm{M}$ is by definition reduced. The symbol ind stands for "initial degeneration". [1]

---

[1]We have avoided choosing the symbol "in M" to avert any confusion with the notion of the initial matroid of a valuated matroid, which we do not use.

Proposition 3.2. *Let $L \subseteq \Bbbk^n$ be a realization of a loopless matroid M. Then there is a flat degeneration of W_L to $\mathrm{ind}_w \mathrm{M}$ inside of $X_{[n]}$. In particular, $[\mathcal{O}_{W_L}] = [\mathcal{O}_{\mathrm{ind}_w \mathrm{M}}]$ in $K_\circ(X_{[n]})$.*

Proof. Note that $\mathrm{ind}_w \mathrm{M}$ does not change if we scale w, so we may assume that w lies in $\mathbb{Z}^n/\mathbb{Z}\mathbf{e}_{[n]}$, the cocharacter lattice of the torus in $X_{[n]}$. In particular, w defines an action of $\mathbb{G}_m$ on $X_{[n]}$. We claim that the flat limit $\lim_{t\to 0} t \cdot W_L$ is equal to $\mathrm{ind}_w \mathrm{M}$.

In [Kat09, Theorem 10.1], Katz shows that, as w is sufficiently generic, the limit is a union of torus-invariant strata. He also gives a combinatorial description of the strata which occur, and this description matches the definition of $\mathrm{ind}_w \mathrm{M}$ because Σ_{M} is the tropicalization of $\mathbb{P}L \cap \mathbb{P}((\Bbbk^*)^{[n]})$ [Stu02, §9.3]. In particular, this shows that the reduction of $\lim_{t\to 0} t \cdot W_L$ is equal to $\mathrm{ind}_w \mathrm{M}$, so it suffices to show that this limit is reduced.

By Example 2.12, W_L defines a kindred subscheme of $(\mathbb{P}^1)^{\binom{n}{2}}$. In particular, because Euler characteristics are locally constant in proper flat families, any flat degeneration of W_L inside of $(\mathbb{P}^1)^{\binom{n}{2}}$ is kindred. Corollary 2.6 then implies that the flat limit is reduced.

It remains to show that $[\mathcal{O}_{W_L}] = [\mathcal{O}_{\mathrm{ind}_w \mathrm{M}}]$ in $K_\circ(X_{[n]})$. For any class $a \in K^\circ(X_{[n]})$, we have

$$\chi(X_{[n]}, a \cdot [\mathcal{O}_{W_L}]) = \chi(W_L, a) = \chi(\mathrm{ind}_w \mathrm{M}, a) = \chi(X_{[n]}, a \cdot [\mathcal{O}_{\mathrm{ind}_w \mathrm{M}}])$$

because Euler characteristics are preserved by degeneration. By [AP15, Theorem 1.3], the pairing $K^\circ(X_{[n]}) \times K_\circ(X_{[n]}) \to \mathbb{Z}$ given by $(a,b) \mapsto \chi(X_{[n]}, ab)$ is perfect, so $[\mathcal{O}_{W_L}] = [\mathcal{O}_{\mathrm{ind}_w \mathrm{M}}]$. $\qquad\square$

Remark 3.3. Reducedness of the flat limit implies that all multiplicities in Katz' formula equal 1. That is, it is a consequence of the proof that for any codimension $r-1$ cone σ of $\Sigma_{[n]}$, there is at most one maximal cone of $w + \Sigma_{\mathrm{M}}$ which intersects it, and the tropical intersection multiplicity is 1. This fact can also be proved using the agreement of stable intersection and matroid intersection in the sense of Welsh [Spe08, §4] (see also [EHL23, Corollary 1.7]).

Proposition 3.2 implies that Euler characteristic computations on $\mathrm{ind}_w \mathrm{M}$ are equal to the analogous computations on the wonderful variety, and so to the analogous computations in $K(\mathrm{M})$. In order to prove Theorem B, we will need to relate computations on $\mathrm{ind}_w \mathrm{M}$ to computations in $K(\mathrm{M})$ for non-realizable matroids. Our tool to do this will be *valuativity*.

For a set $P \subseteq \mathbb{R}^n$, let $\mathbf{1}_P \colon \mathbb{R}^n \to \mathbb{Z}$ be its indicator function. A function valued in an abelian group, with domain the set of all matroids of rank r on $[n]$, is said to be *valuative* if it factors through the map $\mathrm{M} \mapsto \mathbf{1}_{P(\mathrm{M})}$. Valuative functions form an abelian group under addition.

Proposition 3.4. *The assignment $\mathrm{M} \mapsto [\mathcal{O}_{\mathrm{ind}_w \mathrm{M}}] \in K_\circ(X_{[n]})$ is valuative.*

We prepare for the proof with the following lemma.

Lemma 3.5. *A cone σ of $\Sigma_{[n]}$, of any dimension, intersects $w + \Sigma_{\mathrm{M}}$ if and only if $V(\sigma) \subseteq \mathrm{ind}_w \mathrm{M}$.*

Proof. If $V(\sigma)$ is a component of $\mathrm{ind}_w \mathrm{M}$, then the strata $V(\tau)$ contained in $V(\sigma)$ are those indexed by cones $\tau \supseteq \sigma$. Because σ contains a point v of the relative interior of a face F of $w + \Sigma_{\mathrm{M}}$ and F meets σ transversely, a neighborhood of v within F intersects every such cone τ.

Conversely, suppose σ is a cone of $\Sigma_{[n]}$ of codimension less than $r-1$ meeting $w + \Sigma_{\mathrm{M}}$. By induction on the dimension, it suffices to find a proper face of σ which meets $w + \Sigma_{\mathrm{M}}$. By our transversality assumptions, the intersection A of $w + \Sigma_{\mathrm{M}}$ with the subspace $\mathrm{span}_{\mathbb{R}} \sigma$ equals the support of their stable intersection. Because $w + \Sigma_{\mathrm{M}}$ is a balanced polyhedral complex when every maximal cone is given weight 1, the stable intersection A also has positive weights on its facets making it balanced. The

dimension of A is $\dim \Sigma_{\mathrm{M}} - \operatorname{codim} \sigma > 0$. But σ is a strictly convex cone, and a positive-dimensional balanced complex cannot be contained in a strictly convex cone, so A meets the boundary of σ. $\qquad \square$

Proof of Proposition 3.4. Let M be a matroid of rank r on $[n]$. Let $\mathcal{P}_r$ denote the poset of all torus-orbit closures in $X_{[n]}$ of dimension at most $r-1$, ordered by inclusion. Then $\mathcal{P}_r$ has the intersect-decompose property of [Knu09] and contains the components of $\operatorname{ind}_w \mathrm{M}$, and we can use [Knu09, Theorem 1] to find integers $c_\sigma(\mathrm{M})$ such that

$$[\mathcal{O}_{\operatorname{ind}_w \mathrm{M}}] = \sum_{V(\sigma) \in \mathcal{P}_r} c_\sigma(\mathrm{M})[\mathcal{O}_{V(\sigma)}].$$

The computation uses a recurrence, which we restate to be careful about the effect of using a larger intersect-decompose set than the minimal one. First define

$$i_\sigma(\mathrm{M}) = \begin{cases} 1 & \text{if } V(\sigma) \subseteq \operatorname{ind}_w \mathrm{M}, \\ 0 & \text{otherwise.} \end{cases}$$

Now the $c_\sigma(\mathrm{M})$ are defined by the following recursive property:

$$c_\sigma(\mathrm{M}) = i_\sigma(\mathrm{M}) - \sum_{\substack{V(\sigma') \in \mathcal{P}_r \\ V(\sigma') \supsetneq V(\sigma)}} c_{\sigma'}(\mathrm{M}).$$

The sum on the right side is empty when $\dim V(\sigma) > r - 1$. By descending induction on $\dim V(\sigma)$, if $i_\sigma(\mathrm{M}) = 0$, then $c_\sigma(\mathrm{M}) = 0$. Then considering only those σ for which $i_\sigma(\mathrm{M}) = 1$ yields the Möbius recurrence as Knutson states it.

By Lemma 3.5, $i_\sigma(\mathrm{M}) = 1$ if and only if $\sigma \cap (w + \Sigma_{\mathrm{M}})$ is nonempty. If $\sigma \cap (w + \Sigma_{\mathrm{M}})$ is nonempty, then it is the intersection of two tropically convex sets [Ham15, Proposition 2.14], and therefore is tropically convex itself and hence contractible [DS04, Theorem 2]. Because a polyhedral complex with a bounded face admits a deformation retract onto its bounded subcomplex (see, e.g., the proof of [EMPV25, Lemma 3.4]), for a sufficiently large cube C centered at the origin, the intersection $\sigma \cap (w + \Sigma_{\mathrm{M}})$ deformation retracts to $C \cap \sigma \cap (w + \Sigma_{\mathrm{M}})$. In particular, we have $i_\sigma(\mathrm{M}) = \chi_{\mathrm{top}}(C \cap \sigma \cap (w + \Sigma_{\mathrm{M}}))$, where χ_{top} is the topological Euler characteristic. If A and B are bounded polyhedral sets, then one may find polyhedral complex structures for A, B, $A \cap B$, and $A \cup B$, compatible with inclusions, so $\chi_{\mathrm{top}}(A) + \chi_{\mathrm{top}}(B) = \chi_{\mathrm{top}}(A \cup B) + \chi_{\mathrm{top}}(A \cap B)$. Because polyhedral sets are intersection closed, [Gro78] then implies that the topological Euler characteristic of a bounded polyhedral set is linear in its indicator function. Since the assignment $\mathrm{M} \mapsto \mathbf{1}_{\Sigma_{\mathrm{M}}}$ is valuative [Ham17, Proposition 2.26 & Remark 5.17], [BEST23, Corollary 7.11(ii)], the association $\mathrm{M} \mapsto i_\sigma(\mathrm{M})$ is therefore valuative. The recursive formula then implies that the coefficients $c_\sigma(\mathrm{M})$ are also valuative. $\qquad \square$

For any loopless matroid M, there is a restriction map from $K^\circ(X_{[n]})$ to $K(\mathrm{M})$. In particular, for any class $\xi \in K^\circ(X_{[n]})$, we may consider the Euler characteristic $\chi(\mathrm{M}, \xi)$ of the image of ξ in $K(\mathrm{M})$. When M is realized by L, this map coincides with the restriction map $K^\circ(X_{[n]}) \to K^\circ(W_L)$ followed by the identification $K^\circ(W_L) \cong K(\mathrm{M})$, see [LLPP24, Section 5].

Proposition 3.6. *For any class $\xi \in K^\circ(X_{[n]})$ and any loopless matroid M, we have*

$$\chi(\mathrm{M}, \xi) = \chi(\operatorname{ind}_w \mathrm{M}, \xi).$$

Proof. When M is realizable, this follows from Proposition 3.2. By Proposition 3.4, the function which assigns a matroid M to $\chi(\operatorname{ind}_w \mathrm{M}, \xi)$ is valuative. By [LLPP24, Lemma 6.4], the function which assigns

a matroid M to $\chi(\mathrm{M}, \xi)$ is valuative. If two valuative functions agree on all realizable matroids, then they are equal on all matroids [DF10, Theorem 5.4] (see also [BEST23, Lemma 5.9]). $\qquad \square$

Corollary 3.7. *For every matroid* M, *the embedding of* $\mathrm{ind}_w \mathrm{M}$ *into* $(\mathbb{P}^1)^{\binom{n}{2}}$ *realizes it as a kindred subscheme.*

Proof. If M has loops, then $\mathrm{ind}_w \mathrm{M}$ is empty, so this is automatic. We may therefore assume that M is loopless. Let $\mathcal{L}_{i,j}$ denote the restriction of the $\mathcal{O}(1)$ on the factor of $(\mathbb{P}^1)^{\binom{n}{2}}$ labeled by $\{i,j\}$. In [LLPP24, Corollary 7.5], the authors show that

$$\chi(\mathrm{M}, \bigotimes_{i,j} \mathcal{L}_{i,j}^{\otimes a_{i,j}}) = \sum_S \prod_{\{i,j\} \subset S} a_{i,j},$$

where the sum is over subsets S of $\binom{[n]}{2} := \{\{i,j\} : i,j \in [n], i \neq j\}$ that satisfy the *dragon Hall–Rado condition*: for any nonempty $T \subset S$, it holds that $\mathrm{rk}_\mathrm{M}(\bigcup_{\{i,j\} \in T} \{i,j\}) \geq |T| + 1$. It is known that the subsets which satisfy the dragon Hall–Rado condition are the bases of a matroid on $\binom{[n]}{2}$ called the *Dilworth truncation* of M: see, e.g., [Mas81, §1.1]. Proposition 3.6 implies that the Snapper polynomial of $\mathrm{ind}_w \mathrm{M}$ has the same formula, and so $\mathrm{ind}_w \mathrm{M}$ is a kindred subscheme, with the Dilworth truncation of M as its progenitor matroid. $\qquad \square$

Using Corollary 2.6, we deduce the following result from Corollary 3.7.

Corollary 3.8. $\mathrm{ind}_w \mathrm{M}$ *is Cohen–Macaulay, geometrically connected, and geometrically reduced.*

We already have the tools to prove Theorems A and B for ample line bundles on $X_{[n]}$, but we will need to develop an additional tool to prove the main theorems for nef line bundles. We start with the following fact, the so-called strong normality of generalized permutohedra. Recall that, by our definitions, generalized permutohedra are lattice polytopes.

Proposition 3.9 ([Wel76, Theorem 18.6.3]). *Let* $P_1, \ldots, P_\ell$ *be a collection of generalized permutohedra in* $\mathbb{Z}^n$. *Every lattice point of the Minkowski sum* $P_1 + \cdots + P_\ell$ *can be written as* $p_1 + \cdots + p_\ell$, *with* $p_i \in P_i \cap \mathbb{Z}^n$.

Recall that every nef line bundle on $X_{[n]}$ corresponds to a generalized permutohedron P. As noted after [EHL23, Proposition 3.10], Proposition 3.9 implies that the map $X_{[n]} \to \prod_{i=1}^\ell \mathbb{P}(\Bbbk^{P_i \cap \mathbb{Z}^n})$ induced by the complete linear series of the line bundle associated to each P_i has image isomorphic to the toric variety of the normal fan of $P_1 + \cdots + P_\ell$. In particular, the image of the map $f_P \colon X_{[n]} \to \mathbb{P}(\Bbbk^{P \cap \mathbb{Z}^n})$ is the toric variety X_P of the normal fan of P, and when $P_1, \ldots, P_{\binom{n}{2}}$ is the collection of all segments $\mathrm{conv}\{\mathbf{e}_i, \mathbf{e}_j\}$, we recover the defining embedding $X_{[n]} \hookrightarrow (\mathbb{P}^1)^{\binom{n}{2}}$ of the introduction.

Proposition 3.10. *Let P be a generalized permutohedron, and let* $f_P \colon X_{[n]} \to X_P$ *be the corresponding map. Then f_P is "locally a coordinate projection from* $(\mathbb{P}^1)^{\binom{n}{2}}$," *in the following sense. There is an open cover of X_P by open sets $\{U_\sigma\}$ such that each U_σ admits an open embedding into $(\mathbb{P}^1)^\ell$ for some ℓ. Setting Y_σ to be the closure of U_σ in $(\mathbb{P}^1)^\ell$, we have the following commutative diagram*

$$
\begin{array}{ccccccc}
X_{[n]} & \longleftarrow & f_P^{-1}(U_\sigma) & \longrightarrow & X_{[n]} & \hookrightarrow & (\mathbb{P}^1)^{\binom{n}{2}} \\
\downarrow{\scriptstyle f_P} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\
X_P & \longleftarrow & U_\sigma & \longrightarrow & Y_\sigma & \hookrightarrow & (\mathbb{P}^1)^\ell,
\end{array}
$$

where the map from $X_{[n]}$ to $(\mathbb{P}^1)^{\binom{n}{2}}$ is the defining embedding, and p is a coordinate projection.

Proof. Let σ be a maximal cone of the normal fan of P, and let U_σ be the corresponding open subset of X_P. Let $T_1, \ldots, T_\ell$ be the directions of the edges leaving the vertex of P corresponding to σ. Let Q be the Minkowski sum of the simplices corresponding to $T_1, \ldots, T_\ell$, where the simplex corresponding to an edge direction $\mathbf{e}_i - \mathbf{e}_j$ is $\text{conv}\{\mathbf{e}_i, \mathbf{e}_j\}$. Note that Q is a generalized permutohedron. Let Y_σ be the toric variety corresponding to the normal fan of Q. Then σ is also a cone of the fan of Y_σ, so U_σ can be identified with an open subset of Y_σ. It follows from Proposition 3.9 that there is an embedding of Y_σ into $(\mathbb{P}^1)^\ell$ induced by $T_1, \ldots, T_\ell$. As the embedding of $X_{[n]}$ into $(\mathbb{P}^1)^{\binom{n}{2}}$ is the map induced by taking all $\binom{n}{2}$ of the simplices $\text{conv}\{\mathbf{e}_i, \mathbf{e}_j\}$, this gives the desired diagram. $\square$

Corollary 3.11. *Let P be a generalized permutohedron, and let $f_P \colon X_{[n]} \to X_P$ be the corresponding map. Then, for any matroid M, the natural map $\mathcal{O}_{f_P(\mathrm{ind}_w \mathrm{M})} \to Rf_{P*}\mathcal{O}_{\mathrm{ind}_w \mathrm{M}}$ is an isomorphism, and $f_P(\mathrm{ind}_w \mathrm{M})$ is Cohen–Macaulay. If M is realized by L, then the natural map $\mathcal{O}_{f_P(W_L)} \to Rf_{P*}\mathcal{O}_{W_L}$ is an isomorphism, and $f_P(W_L)$ is Cohen–Macaulay.*

Proof. Let U_σ be an element of the open cover constructed in Proposition 3.10, which is equipped with a locally closed embedding $U_\sigma \hookrightarrow (\mathbb{P}^1)^\ell$. By Proposition 3.7, $\mathrm{ind}_w \mathrm{M}$ is a kindred subscheme of $(\mathbb{P}^1)^{\binom{n}{2}}$. By Proposition 2.9, $p(\mathrm{ind}_w \mathrm{M})$ is a kindred subscheme of $(\mathbb{P}^1)^\ell$, and the natural map from $\mathcal{O}_{p(\mathrm{ind}_w \mathrm{M})}$ to $Rp_*\mathcal{O}_{\mathrm{ind}_w \mathrm{M}}$ is an isomorphism. By Corollary 2.6, $p(\mathrm{ind}_w \mathrm{M})$ is Cohen–Macaulay, so $U_\sigma \cap p(\mathrm{ind}_w \mathrm{M})$ is Cohen–Macaulay. As being Cohen–Macaulay is a local property and $f_P(\mathrm{ind}_w \mathrm{M})$ is covered by open sets of the form $U_\sigma \cap p(\mathrm{ind}_w \mathrm{M})$, this implies that $f_P(\mathrm{ind}_w \mathrm{M})$ is Cohen–Macaulay. Similarly, that the natural map $\mathcal{O}_{f_P(\mathrm{ind}_w \mathrm{M})} \to Rf_{P*}\mathcal{O}_{\mathrm{ind}_w \mathrm{M}}$ is an isomorphism is a local property, so this holds as well. When M is realized by L, since W_L is kindred (Example 2.12), an identical argument shows that the natural map $\mathcal{O}_{f_P(W_L)} \to Rf_{P*}\mathcal{O}_{W_L}$ is an isomorphism and that $f_P(W_L)$ is Cohen–Macaulay. $\square$

Proposition 3.12. *Let P be a generalized permutohedron, and let $f_P \colon X_{[n]} \to X_P$ be the corresponding map. Then $H^i(\mathrm{ind}_w \mathrm{M}, \mathcal{L}_P^{\otimes a}) = 0$ unless $a \geq 0$ and $i = 0$, or $a < 0$ and $i = \dim f_P(\mathrm{ind}_w \mathrm{M})$. For any a, the restriction map from $H^0(X_{[n]}, \mathcal{L}_P^{\otimes a}) \to H^0(\mathrm{ind}_w \mathrm{M}, \mathcal{L}_P^{\otimes a})$ is surjective.*

Proof. The toric variety X_P is equipped with an ample line bundle $\mathcal{O}(1)$ corresponding to the polytope P; this line bundle pulls back to $\mathcal{L}_P$ under f_P. By Corollary 3.11 and the projection formula, the natural map from $H^i(f_P(\mathrm{ind}_w \mathrm{M}), \mathcal{O}(a))$ to $H^i(\mathrm{ind}_w \mathrm{M}, \mathcal{L}_P^{\otimes a})$ is an isomorphism, so it suffices to show the desired vanishing and surjectivity of global sections for $H^i(f_P(\mathrm{ind}_w \mathrm{M}), \mathcal{O}(a))$.

Note that $f_P(\mathrm{ind}_w \mathrm{M})$ is a union of torus-invariant subvarieties in the toric variety X_P. As such, it is defined over $\mathrm{Spec}\,\mathbb{Z}$. By upper semicontinuity, it suffices to show the desired vanishing and surjectivity of global sections after base changing $f_P(\mathrm{ind}_w \mathrm{M})$ to a field of positive characteristic. Over any field of positive characteristic, $f_P(\mathrm{ind}_w \mathrm{M})$ is a compatibly split subscheme of X_P with respect to a certain Frobenius splitting of X_P [Pay09, Proposition 3.2], see also [BK05, Proposition 1.2.1]. As in the proof of Proposition 2.7, the vanishing and surjectivity of global sections then follows from [BK05, Theorem 1.2.8, Theorem 1.2.9], as $f_P(\mathrm{ind}_w \mathrm{M})$ is Cohen–Macaulay by Corollary 3.11. $\square$

Proof of Theorem A. Note that (3) was proved in Corollary 3.11. For parts (1) and (2), we use the degeneration of W_L to $\mathrm{ind}_w \mathrm{M}$ constructed in Proposition 3.2; these then follow from upper semicontinuity and Proposition 3.12. $\square$

Proof of Theorem B. By Proposition 3.6, $\chi(\mathrm{M}, \mathcal{L}_P^{\otimes a}) = \chi(\mathrm{ind}_w \mathrm{M}, \mathcal{L}_P^{\otimes a})$. As $H^i(\mathrm{ind}_w \mathrm{M}, \mathcal{L}_P^{\otimes a}) = 0$ for any $a \geq 0$ and $i > 0$, we see that $\chi(\mathrm{ind}_w \mathrm{M}, \mathcal{L}_P^{\otimes a}) = \dim H^0(\mathrm{ind}_w \mathrm{M}, \mathcal{L}_P^{\otimes a})$ and in particular, the right-hand side is a polynomial in a of some degree d. Using Proposition 3.9 and the results in Proposition 3.12, we

see that the ring $\bigoplus_{a \geq 0} H^0(\mathrm{ind}_w \mathrm{M}, \mathcal{L}_P^{\otimes a})$ is Cohen–Macaulay and generated in degree 1. As described in the introduction, this implies that if we write

$$\sum_{a \geq 0} \dim H^0(\mathrm{ind}_w \mathrm{M}, \mathcal{L}_P^{\otimes a}) t^a = \frac{h_0^* + h_1^* t + \cdots + h_d^* t^d}{(1-t)^{d+1}},$$

then $(h_0^*, \ldots, h_d^*)$ is a Macaulay vector. The result follows. $\square$

4. DIMENSIONS OF IMAGES

Theorem A states that the cohomology of a negative tensor power of a line bundle $\mathcal{L}_P$ on a wonderful variety vanishes except possibly in degree equal to the dimension of $f_P(W_L)$, where f_P is the map to projective space induced by $\mathcal{L}_P$. In Theorem 4.2 below, we give an explicit combinatorial formula for that dimension. Theorem 4.2 is stated in terms of $\mathrm{ind}_w \mathrm{M}$, but if L realizes M, then W_L degenerates to $\mathrm{ind}_w \mathrm{M}$ by Proposition 3.2, and so they have the same class in the Chow groups of $X_{[n]}$. This implies that $\dim f_P(W_L) = \dim f_P(\mathrm{ind}_w \mathrm{M})$ for any generalized permutohedron P since, if A is an ample divisor on $X_{[n]}$, then $\dim W_L - \dim f_P(W_L)$ is the least i such that $f_{P*}(A^i \frown [W_L])$ is nonzero, and similarly for $\mathrm{ind}_w \mathrm{M}$ because it is equidimensional.

For a loopless matroid M, we begin by showing that $\dim f_P(\mathrm{ind}_w \mathrm{M})$ is equal to the degree of the Snapper polynomial $\chi(\mathrm{M}, \mathcal{L}_P^{\otimes a})$, which is used to define the h^* vector, and is also equal to a quantity derived from the *Chow ring $A^\bullet(\mathrm{M})$* of M. This will be used to deduce Theorem 4.2. For a rank r loopless matroid M, let $\int_{\mathrm{M}} : A^{r-1}(\mathrm{M}) \to \mathbb{Z}$ be the degree map as defined in [AHK18, Section 5].

Proposition 4.1. *Let M be a loopless matroid on $[n]$, and let P be a generalized permutohedron in $\mathbb{R}^n$. Then the following quantities are equal:*

    (1) The degree of the polynomial $a \mapsto \chi(\mathrm{M}, \mathcal{L}_P^{\otimes a})$.

    (2) The dimension of $f_P(\mathrm{ind}_w \mathrm{M})$.

    (3) The largest d such that $c_1(\mathcal{L}_P)^d$ is nonzero in $A^\bullet(\mathrm{M})$, i.e., the numerical dimension of $c_1(\mathcal{L}_P)$ in $A^\bullet(\mathrm{M})$.

Proof. Note that the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(\Bbbk^{P \cap \mathbb{Z}^n})$ pulls back to $\mathcal{L}_P$ via f_P. By Proposition 3.6 and Corollary 3.11, we have $\chi(\mathrm{M}, \mathcal{L}_P^{\otimes a}) = \chi(\mathrm{ind}_w \mathrm{M}, \mathcal{L}_P^{\otimes a}) = \chi(f_P(\mathrm{ind}_w \mathrm{M}), \mathcal{O}(a))$. The line bundle $\mathcal{O}(1)$ is ample on $f_P(\mathrm{ind}_w \mathrm{M})$, so its Snapper polynomial has degree equal to the dimension of $f_P(\mathrm{ind}_w \mathrm{M})$, proving that the first quantity is equal to the second quantity.

For the third quantity, we first note from [AHK18, Section 5] that $A^\bullet(\mathrm{M})$ is equipped with a surjection $A^\bullet(X_{[n]}) \to A^\bullet(\mathrm{M})$, such that, for any $\xi \in A^{r-1}(X_{[n]})$, we have $\int_{\mathrm{M}}(\xi) = \xi \frown [\Sigma_{\mathrm{M}}]$. Here $[\Sigma_{\mathrm{M}}] \in A_{r-1}(X_{[n]})$ is the Chow homology class called the *Bergman class* of M, which is the Minkowski weight corresponding to Σ_{M} as a balanced fan.

Let d be the numerical dimension of $c_1(\mathcal{L}_P)$. For $0 \leq k \leq d$, Poincaré duality for $A(\mathrm{M})$ [AHK18, Theorem 6.19] shows that there is some $x_k \in A^{r-1-k}(X_{[n]})$ such that $\int_{\mathrm{M}} x_k \cdot c_1(\mathcal{L}_P)^k \neq 0$. As the ample cone is full dimensional and $A^\bullet(\mathrm{M})$ is generated in degree 1, we may take $x_k = A^{r-1-k}$ for some ample $A \in A^1(X_{[n]})$ and each $0 \leq k \leq d$. Because $f_{P*}([\mathrm{ind}_w \mathrm{M}] \cdot A^{r-1-k})$ is effective and $\mathcal{O}(1)$ is ample, we see that $f_{P*}(A^{r-1-k} \frown [\mathrm{ind}_w \mathrm{M}])$ is nonzero if and only if $c_1(\mathcal{O}(1))^k \frown f_{P*}(A^{r-1-k} \frown [\mathrm{ind}_w \mathrm{M}])$ is nonzero. The latter is equal to $A^{r-1-k} \cdot c_1(\mathcal{L}_P)^k \frown [\mathrm{ind}_w \mathrm{M}]$ by the projection formula, which in turn is equal to $\int_{\mathrm{M}} A^{r-1-k} \cdot c_1(\mathcal{L}_P)^k$ because the fundamental class of $\mathrm{ind}_w \mathrm{M}$ in $A_\bullet(X_{[n]})$ is equal to the Bergman class $[\Sigma_{\mathrm{M}}]$ by Proposition 3.6 and the Hirzebruch–Riemann–Roch theorem, used like it is in Proposition 2.2. Therefore $r - 1 - d$ is the smallest number such that $f_{P*}(A^{r-1-d} \frown [\mathrm{ind}_w \mathrm{M}])$ is

nonzero, and so $r-1-d = \dim \operatorname{ind}_w \mathrm{M} - \dim f_P(\operatorname{ind}_w \mathrm{M})$ since $\operatorname{ind}_w \mathrm{M}$ is equidimensional of dimension $r-1$. Rearranging implies the result. $\qquad \square$

We can now give a combinatorial formula for the quantities in Proposition 4.1. This formula will imply that the numerical dimension of $c_1(\mathcal{L}_P)$ in $A^\bullet(\mathrm{M})$ only depends on the lineality space of the normal fan of P. A generalized permutohedron P in $\mathbb{R}^n$ induces a partition $[n] = S_1 \sqcup \cdots \sqcup S_\ell$, where the sets are the equivalence classes induced by the equivalence relation generated by setting $i \sim j$ if there is an edge of direction $\mathbf{e}_i - \mathbf{e}_j$ in P. The parts of the partition induced by a matroid polytope $P(\mathrm{M})$ are usually called the *connected components* of M.

Theorem 4.2. *Let* M *be a loopless matroid on* $[n]$*, and let* P *be a generalized permutohedron with induced partition* $[n] = S_1 \sqcup \cdots \sqcup S_\ell$*. Then the numerical dimension of* $c_1(\mathcal{L}_P)$ *in* $A^\bullet(\mathrm{M})$ *is the minimum over partitions* $[n] = T_1 \sqcup \cdots \sqcup T_k$ *coarsening* $[n] = S_1 \sqcup \cdots \sqcup S_\ell$ *of* $\sum_{i=1}^k (\mathrm{rk}_{\mathrm{M}}(T_i) - 1)$*.*

We will need a preparatory lemma. For a set $S \subseteq [n]$, write Δ_S for the convex hull of $\{\mathbf{e}_i : i \in S\}$, which is a generalized permutohedron, and let $h_S = c_1(\mathcal{L}_{\Delta_S})$.

Lemma 4.3. *Let* $[n] = S_1 \sqcup \cdots \sqcup S_\ell$ *be a partition, and let* $Q = \sum_{i=1}^\ell \sum_{S \subseteq S_i} \Delta_S$*. Let* d *be the minimum over partitions* $[n] = T_1 \sqcup \cdots \sqcup T_k$ *coarsening* $[n] = S_1 \sqcup \cdots \sqcup S_\ell$ *of* $\sum_{i=1}^k (\mathrm{rk}_{\mathrm{M}}(T_i) - 1)$*. Then* $\int_{\mathrm{M}} c_1(\mathcal{L}_Q)^d h_{[n]}^{r-1-d} > 0$*, and the numerical dimension of* $c_1(\mathcal{L}_Q)$ *is* d*.*

Proposition 4.4. [BES24, Theorem 5.2.4] *For a multiset* $\{J_1, \ldots, J_{r-1}\}$ *of subsets of* $[n]$ *and a loopless matroid* M *on* $[n]$*, then* $\int_{\mathrm{M}} h_{J_1} \cdots h_{J_{r-1}}$ *is equal to* 1 *if* $\{J_1, \ldots, J_{r-1}\}$ *satisfies the* dragon Hall–Rado condition*:*

$$\mathrm{rk}_{\mathrm{M}}\left(\bigcup_{i \in I} J_i\right) \geq |I| + 1 \quad \text{for all nonempty } I \subseteq [r-1],$$

and is 0 *otherwise.*

Proof of Lemma 4.3. To prove that $\int_{\mathrm{M}} c_1(\mathcal{L}_Q)^d h_{[n]}^{r-1-d} > 0$, it suffices to check that there is some multiset $\{S_{i_1}, \ldots, S_{i_d}\}$ that satisfies the dragon Hall–Rado condition after adding $r - 1 - d$ copies of $[n]$. Let a_i be the multiplicity of S_i in such a family. This holds for $(a_i) \in \mathbb{N}^\ell$ if and only if

$$\sum_{i \in I} a_i \leq \mathrm{rk}_{\mathrm{M}}\left(\bigcup_{i \in I} S_i\right) - 1$$

for every nonempty $I \subseteq [\ell]$. By [Edm70, Theorem 8] the nonnegative integer vectors (a_i) satisfying these bounds are the lattice points of a polymatroid in Edmonds' sense. The maximum attained by $\sum_{i=1}^\ell a_i$ is given by the evaluation of its rank function at $[\ell]$, which, from the description in [Edm70, Theorem 8], is exactly d. It follows from [LLPP24, Corollary 7.5], which gives an explicit formula for the Snapper polynomial $\chi(\mathrm{M}, \bigotimes_S \mathcal{L}_{\Delta_S}^{\otimes a_S})$ in terms of the dragon Hall–Rado condition, that the degree of the polynomial $a \mapsto \chi(\mathrm{M}, \mathcal{L}_Q^{\otimes a})$ is d, and so the numerical dimension of $c_1(\mathcal{L}_Q)$ is d by Proposition 4.1. $\qquad \square$

As the last item of preparation we recall a result implicit in the proof of [EL23, Proposition 4.11].

Proposition 4.5. *Let* D_1 *and* D_2 *be the restrictions of nef classes from* $A^1(X_{[n]})$ *to* $A^1(\mathrm{M})$*, where* M *has rank* r*. Let* E *be the restriction of an effective class in* $A^1(X_{[n]})$*, and assume that* $D_2 + E$ *is the restriction of a nef class from* $A^1(X_{[n]})$*. Then, for each* $0 \leq a \leq r - 1$*,*

$$\int_{\mathrm{M}} D_1^a D_2^{r-1-a} \leq \int_{\mathrm{M}} D_1^a (D_2 + E)^{r-1-a}.$$

Proof of Theorem 4.2. Let Q be the Minkowski sum $\sum_{i=1}^{\ell} \sum_{S \subseteq S_i} \Delta_S$, and let d and e be the numerical dimensions of $c_1(\mathcal{L}_P)$ and $c_1(\mathcal{L}_Q)$, respectively. By Lemma 4.3, it suffices to show that $d = e$.

As Q is the product of permutohedra, one from each $\mathbb{R}^{S_i}$, the image of $X_{[n]}$ under f_Q is $X_{S_1} \times \cdots \times X_{S_\ell}$. The partition $[n] = S_1 \sqcup \cdots \sqcup S_\ell$ has the property that P is a product $P_1 \times \cdots \times P_\ell$ of generalized permutohedra $P_i \subset \mathbb{R}^{S_i}$, each of maximal dimension $|S_i| - 1$ [AA23, Section 5.1]. Thus, the map f_P factors through f_Q, and so $\dim f_P(\mathrm{ind}_w \mathrm{M}) \leq \dim f_Q(\mathrm{ind}_w \mathrm{M})$, i.e., $d \leq e$.

The generalized permutohedron P defines a nef and big line bundle on $X_{S_1} \times \cdots \times X_{S_\ell}$. In particular, by [Laz04, Corollary 2.2.7], we can write $m c_1(\mathcal{L}_P) = c_1(\mathcal{L}_Q) + E$ for some $m > 0$, where E is the pullback of an effective divisor from $X_{S_1} \times \cdots \times X_{S_\ell}$ (and so is effective). By Lemma 4.3, $\int_{\mathrm{M}} c_1(\mathcal{L}_Q)^e h_{[n]}^{r-1-e} > 0$. By Proposition 4.5, we have

$$0 < \int_{\mathrm{M}} c_1(\mathcal{L}_Q)^e h_{[n]}^{r-1-e} \leq \int_{\mathrm{M}} (c_1(\mathcal{L}_Q) + E)^e h_{[n]}^{r-1-e} = m^e \int_{\mathrm{M}} c_1(\mathcal{L}_P)^e h_{[n]}^{r-1-e}.$$

In particular, $c_1(\mathcal{L}_P)^e \neq 0$, so $e \leq d$. □

We record a case in which the formula of Theorem 4.2 is particularly easy to evaluate.

Corollary 4.6. *If* M *is a matroid of rank* r *on* $[n]$ *and* P *a generalized permutohedron such that the induced partition* $S_1 \sqcup \cdots \sqcup S_\ell$ *of* P *coarsens the induced partition of* $P(\mathrm{M})$ *(into connected components), then the numerical dimension of* $c_1(\mathcal{L}_P)$ *equals* $r - \ell = r - n + \dim P$.

Proof. The rank function rk_{M} is additive on unions of connected components [Oxl11, Fact 4.2.13]. Any partition $T_1 \sqcup \cdots \sqcup T_k$ coarsening the induced partition of P also coarsens the partition into connected components of M, and therefore

$$\sum_{i=1}^{k} (\mathrm{rk}_{\mathrm{M}}(T_i) - 1) = \mathrm{rk}_{\mathrm{M}}([n]) - k = r - k$$

which is minimized by maximizing k, i.e., taking $T_\bullet$ and $S_\bullet$ identical with $k = \ell$. □

While the Snapper polynomial of $\mathcal{L}_P$ on a matroid is valuative, we caution that the h^* vector is not valuative because of the varying powers of $1 - t$ that depend on the dimension, as computed in this section. Explicitly, let M be the rank 2 uniform matroid on $[4]$ (i.e., with all sets of size 2 as bases), let M_1 and M_2 be rank 2 matroids on $[4]$ with unique non-basis $\{1, 2\}$ and $\{3, 4\}$, respectively, and let M_{12} be the rank 2 matroid on $[4]$ with both of these as non-bases. One can verify that with $P = P(\mathrm{M}_{12})$, the h^* vectors do not respect the valuative relation $\mathbf{1}_{P(\mathrm{M})} = \mathbf{1}_{P(\mathrm{M}_1)} + \mathbf{1}_{P(\mathrm{M}_2)} - \mathbf{1}_{P(\mathrm{M}_{12})}$.

5. DISCUSSION AND EXAMPLES

In this section, we give some examples of nef divisors which do not satisfy the conclusion of Theorem A and Theorem B, and we discuss a few extensions of the main theorems.

5.1. **Examples.** In the Hodge theory of matroids [AHK18] and more generally in tropical Hodge theory [ADH23], a positivity property known as the "Kähler package" is established for the set of *strictly convex divisor classes* [AHK18, Definition 4.1 & 4.2] (see also [ADH23, Definition 5.1]), which contains, in general strictly, the pullbacks of ample divisors on the permutohedral variety. In contrast, the following example shows that the conclusion of Theorem B may fail for the line bundle of a "strictly convex" divisor class.

Example 5.1. For $n \geq 2$, let $[n, \overline{n}] := \{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$, and define $L = \mathbb{C}^3 \hookrightarrow \mathbb{C}^{[n, \overline{n}]}$ by $(x, y, z) \mapsto (x - z, x - 2z, \ldots, x - nz, y - z, y - 2z, \ldots, y - nz)$. The corresponding hyperplane arrangement on $\mathbb{P}L = \mathbb{P}^2$ consists of two families of parallel lines $\{\ell_i := V(x - iz)\}_{i \in [n]}$ and $\{\ell_{\overline{j}} := V(y - jz)\}_{\overline{j} \in [\overline{n}]}$. The wonderful variety W_L is the blow-up of $n^2 + 2$ points consisting of $[0 : 1 : 0] = \bigcap_{i \in [n]} \ell_i$, $[1 : 0 : 0] = \bigcap_{\overline{j} \in [\overline{n}]} \ell_{\overline{j}}$, and $\ell_i \cap \ell_{\overline{j}}$ for every pair $(i, \overline{j})$. Let $x_{[n]}$, $x_{[\overline{n}]}$, and $x_{i,\overline{j}}$ respectively denote the exceptional divisors, and let x_i be the strict transform of ℓ_i for $i \in [n, \overline{n}]$. Denote by ∂W_L the set of divisors $\{x_{i,\overline{j}}\}_{i \in [n], \overline{j} \in [\overline{n}]} \cup \{x_{[n]}, x_{[\overline{n}]}\} \cup \{x_i\}_{i \in [n, \overline{n}]}$. For a divisor D on W_L, one can translate [ADH23, Definition 5.1] to the statement that D is "strictly convex" if and only if D is $\mathbb{R}$-linearly equivalent to a positive linear combination of ∂W_L and the intersection product $D.E$ is positive for all $E \in \partial W_L$.

Let α denote the pullback of the hyperplane class along the blow-down $W_L \to \mathbb{P}^2$, and define a divisor class $D \in A^1(W_L)$ by

$$D := (n + 2)\alpha - x_{[n]} - x_{[\overline{n}]} - \sum_{i \in [n], \overline{j} \in [\overline{n}]} x_{i,\overline{j}}.$$

Since $\alpha = \sum_{i \in [n]} x_{i,\overline{k}} + x_{[\overline{n}]} = \sum_{i \in [n]} x_{k,\overline{i}} + x_{[n]}$ in $A^1(W_L)$ for any fixed $k \in [n]$, one verifies that D is $\mathbb{R}$-linearly equivalent to a positive linear combination of the divisors in ∂W_L. Moreover, one verifies that $D.E = 1$ for all $E \in \partial W_L$, so D is "strictly convex." By Riemann–Roch for surfaces, one computes

$$\chi(W_L, \mathcal{O}_{W_L}(aD)) = \frac{D.D}{2}a^2 - \frac{D.K_{W_L}}{2}a + \chi(W_L, \mathcal{O}_{W_L}) = (2n + 1)a^2 - \frac{(n + 1)(n - 4)}{2}a + 1.$$

From the relation $\chi(W_L, \mathcal{O}_{W_L}(aD)) = h_0^*\binom{a+2}{2} + h_1^*\binom{a+1}{2} + h_2^*\binom{a}{2}$, we find that

$$(h_0^*, h_1^*, h_2^*) = \left(1, \frac{-n^2 + 7n + 2}{2}, \frac{(n + 1)n}{2}\right)$$

in this case. In particular, the h^*-vector is not a Macaulay vector for $n \geq 6$: $n = 6$ gives $(1, 4, 21)$, $n = 7$ gives $(1, 1, 28)$, and $n \geq 8$ gives $h_1^* < 0$. One can verify directly using submodular functions that the divisor class D is indeed not a pullback of a nef divisor on the permutohedral variety when $n > 3$.

The next example shows that the conclusion of Theorem B may fail for a line bundle even if there is a tensor power that is isomorphic to $\mathcal{L}_P$ for some generalized permutohedron P.

Example 5.2. Let $\Bbbk$ be a field of characteristic 2, and let $L \subseteq \Bbbk^7$ be a 3-dimensional subspace whose intersections with the coordinate hyperplanes of $\Bbbk^7$ are, for some choice of coordinates on L, the 7 planes defined over $\mathbb{F}_2$. The matroid realized by L is known as the *Fano matroid*. Then W_L is obtained by blowing up the 7 $\mathbb{F}_2$-rational points of $\mathbb{P}^2$. Let D be the anti-canonical divisor of W_L, so $D = \mathcal{O}(3) \otimes \mathcal{O}(-E_1 - \cdots - E_7)$, where the E_i are the exceptional divisors. The line bundle $\mathcal{O}(2D)$ is the restriction of the line bundle $\mathcal{L}_{-\Delta_{[7]}}$ on $X_{[7]}$ corresponding to the divisor class usually denoted by β.

However, $\mathcal{O}(D)$ is not the restriction of a nef line bundle from $X_{[7]}$. That is, there is no integer-valued submodular function $\mathrm{rk} \colon 2^{[7]} \to \mathbb{Z}$ taking value 0 at the empty set and singletons, -1 at the three-element flats of the Fano matroid (the triples of coordinates of $\mathbb{P}(\Bbbk^7)$ vanishing at $\mathbb{F}_2$-rational points of $\mathbb{P}L$), and -3 at $[7]$. We sketch a verification. First, rk takes value at most -2 on sets of size 5, because these are unions of two three-element flats. The assumptions imply rk is nonincreasing, so $\mathrm{rk}(S) \in \{0, -1\}$ when $|S| = 2$. If $\mathrm{rk}(S) = -1$, submodularity forces that $\mathrm{rk}([7] \setminus S) = -2$ and, for any $T \subseteq [7]$, that $\mathrm{rk}(T) = \mathrm{rk}(T \cap S) + \mathrm{rk}(T \setminus S)$. When T is a three-element flat meeting S in one element, this shows $\mathrm{rk}(T \setminus S) = -1$. Applying submodularity to two of these differences and one three-element flat disjoint from S gives the contradiction $\mathrm{rk}([7] \setminus S) \leq -3$. Otherwise $\mathrm{rk}(S) = 0$ whenever $|S| = 2$.

Then applying submodularity to a set of size 5 and two three-element flats that meet it in two elements implies the contradiction $\mathrm{rk}([7]) \leq -4$.

Let x, y, z be coordinates on $\mathbb{P}^2$. The dimension of the space of sections $H^0(W_L, \mathcal{O}(D))$ is 3, and the sections $a = xy(x + y), b = xz(x + z), c = yz(y + z)$ form a basis for $H^0(W_L, \mathcal{O}(D))$. The sections define a map $\pi \colon W_L \to \mathbb{P}^2$, which is generically 2 to 1. Let d be the section $xyz(x + y)(x + z)(y + z)$ of $\mathcal{O}(2D)$. Then $H^0(W_L, \mathcal{O}(2D))$ is generated by $\mathrm{Sym}^2 H^0(W_L, \mathcal{O}(D))$ and d. The complete linear system of $H^0(W_L, \mathcal{O}(2D))$ defines a map f from W_L to $\mathbb{P}^6$, whose image is the variety usually called the *reciprocal plane* [PS06], and this map factors through $\mathbb{P}(1, 1, 1, 2)$. This map contracts the strict transforms of the $\mathbb{F}_2$-rational lines on $\mathbb{P}^2$ to obtain 7 A_1 surface singularities. As A_1 surface singularities are rational and Gorenstein, we see that $Rf_* \mathcal{O}_{W_L} = \mathcal{O}_{f(W_L)}$.

Viewing $f(W_L)$ as a hypersurface in $\mathbb{P}(1, 1, 1, 2)$, we see that the section ring is $\Bbbk[a, b, c, d]/(d^2 - abc(a + b + c))$, which is Gorenstein. This describes $f(W_L)$ as a double cover of $\mathbb{P}^2$, branched along the quartic $abc(a + b + c) = 0$. We see that $\pi_* \mathcal{O}_{W_L} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ and $R^i \pi_* \mathcal{O}_{W_L} = 0$ for $i > 0$, and that the h^* vector of $\mathcal{O}(D)$ is $(1, 0, 1)$. This is not a Macaulay vector.

The following example shows that Theorem A(1) does not extend to restrictions to W_L of globally generated vector bundles on $X_{[n]}$.

Example 5.3. In [BEST23], the authors introduced a vector bundle $\mathcal{Q}_L$ on $X_{[n]}$ associated to a linear subspace L of $\Bbbk^n$. This bundle $\mathcal{Q}_L$ is globally generated, and it is equipped with a section that transversely cuts out W_L [BEST23, Theorem 7.10], and so the restriction of $\mathcal{Q}_L$ to W_L is identified with the normal bundle of W_L in $X_{[n]}$. As such, $H^0(W_L, \mathcal{Q}_L)$ is identified with the tangent space to $[\mathcal{O}_{W_L}]$ in the Hilbert scheme of $X_{[n]}$, and $H^1(W_L, \mathcal{Q}_L)$ is the obstruction space to deforming W_L inside of $X_{[n]}$ [FGI+05, Proposition 6.5.2]. Let $\Bbbk = \mathbb{C}$, and let M be the matroid of rank 3 on $[12]$ appearing in [CL23, Theorem 4.7]. This matroid has the property that Gr_M, the locally closed subset of $\mathrm{Gr}(3, 12)$ which consists of realizations of M, is a 12-dimensional scheme with 2 singular points, which have tangent spaces of dimension 13. One can show that if L is a realization of M, then an open neighborhood of $[\mathcal{O}_{W_L}]$ in the Hilbert scheme of $X_{[12]}$ is isomorphic to an open neighborhood of Gr_M containing the point $[L]$. In particular, if L corresponds to one of the smooth points of Gr_M, then $\dim H^0(W_L, \mathcal{Q}_L) = 12$, but if L corresponds to one of the singular points of Gr_M, then $\dim H^0(W_L, \mathcal{Q}_L) = 13$ and $\dim H^1(W_L, \mathcal{Q}_L) > 0$.

5.2. **Vanishing theorems for other varieties.** We now explain how Theorem A can be used to deduce vanishing theorems for some variations on wonderful varieties. Given a loopless matroid M, a *building set* $\mathcal{G}$ is a subset of the lattice of flats of M which satisfies some combinatorial conditions, given in [FK04, Definition 4.4]. Recall that Δ_S denotes the convex hull of $\{\mathbf{e}_i : i \in S\}$. Set

$$P_{\mathcal{G}} = \sum_{S \colon \mathrm{cl}_M(S) \in \mathcal{G}} \Delta_S,$$

where the sum denotes Minkowski sum and cl_M sends a set S to the smallest flat containing it. Note that $P_{\mathcal{G}}$ is a generalized permutohedron. The *wonderful variety of* $\mathcal{G}$, denoted $W_{L,\mathcal{G}}$, is the image of W_L under the map from $X_{[n]}$ to $X_{P_{\mathcal{G}}}$. If $\mathcal{G}$ contains $[n]$, the combinatorial conditions on the building set $\mathcal{G}$ guarantee that $W_{L,\mathcal{G}}$ is smooth and can be described as an iterated blow-up of $\mathbb{P}L$ [dCP95, Section 4.1].

By Theorem A(3) and the projection formula, if $\mathcal{L}$ is any line bundle on $W_{L,\mathcal{G}}$, then the cohomology does not change when we pull back $\mathcal{L}$ along the map from W_L to $W_{L,\mathcal{G}}$. In particular, Theorem A implies the following result.

Theorem 5.4. *Let $L \subseteq \Bbbk^n$ be a linear subspace which is not contained in any coordinate hyperplane, and let $\mathcal{G}$ be a building set on the lattice of flats of the matroid M. Let $\mathcal{L}$ be the restriction of a nef line bundle on $X_{P_{\mathcal{G}}}$ to $W_{L,\mathcal{G}}$ of numerical dimension d. Then $H^i(W_{L,\mathcal{G}}, \mathcal{L}^{\otimes a}) = 0$ unless either $a \geq 0$ and $i = 0$, or $a < 0$ and $i = d$, the restriction map $H^0(X_{P_{\mathcal{G}}}, \mathcal{L}^{\otimes a}) \to H^0(W_{L,\mathcal{G}}, \mathcal{L}^{\otimes a})$ is surjective, and the ring $\bigoplus_{a \geq 0} H^0(W_{L,\mathcal{G}}, \mathcal{L}^{\otimes a})$ is Cohen–Macaulay and generated in degree 1.*

In particular, Theorem 5.4 proves vanishing theorems for line bundles on $\overline{M}_{0,n}$ [dCP95, Section 4.2] and wonderful compactifications of subspace arrangements (see [CHL$^+$22]).

In [BHM$^+$22], the authors introduced the *augmented wonderful variety* W_L^{aug} of a linear subspace $L \subseteq \Bbbk^n$. This variety is constructed as an iterated blow-up of the projective completion $\mathbb{P}(L \oplus \Bbbk)$, and it can also be defined as the closure of L in the stellahedral toric variety X_{St_n}, a certain toric compactification of $\Bbbk^n$. We use Theorem A to deduce the following vanishing theorem for W_L^{aug}.

Theorem 5.5. *Let $L \subseteq \Bbbk^n$ be a linear subspace, and let $\mathcal{L}$ be the restriction of a nef line bundle from X_{St_n} to W_L^{aug} of numerical dimension d. Then $H^i(W_L^{\mathrm{aug}}, \mathcal{L}^{\otimes a}) = 0$ unless either $a \geq 0$ and $i = 0$, or $a < 0$ and $i = d$, the restriction map $H^0(X_{\mathrm{St}_n}, \mathcal{L}^{\otimes a}) \to H^0(W_L^{\mathrm{aug}}, \mathcal{L}^{\otimes a})$ is surjective, and the ring $\bigoplus_{a \geq 0} H^0(W_L^{\mathrm{aug}}, \mathcal{L}^{\otimes a})$ is Cohen–Macaulay and generated in degree 1.*

Proof. By replacing $\Bbbk^n$ with the smallest coordinate subspace containing L, we can reduce to the case when L is not contained in any coordinate hyperplane. As described in [BHM$^+$22, Remark 4.1], W_L^{aug} arises as a stage in the iterated blow-up construction of the wonderful variety of $L \oplus \Bbbk \subseteq \Bbbk^{[n] \cup \{0\}}$. Set

$$P = \sum_{S \ni 0} \Delta_S \subset \mathbb{R}^{[n] \cup \{0\}},$$

where the sum denotes Minkowski sum. Then the stellahedral toric variety X_{St_n} is isomorphic to X_P. Let f denote the map from $X_{[n+1]}$ to X_P, so W_L^{aug} is isomorphic to $f(W_{L \oplus \Bbbk})$. The result then follows from Theorem A by pulling back to $W_{L \oplus \Bbbk}$. $\qquad\square$

One can similarly deduce an analogue of Theorem B for augmented K-rings of matroids [LLPP24].

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