A DECOMPOSITION THEOREM FOR LEFSCHETZ MODULES

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ABSTRACT. A Lefschetz module is a module over a graded algebra A that satisfies analogues of Poincaré duality, the Hard Lefschetz property, and the Hodge–Riemann relations with respect to an open convex cone $\mathcal{K}$ in the degree one part of A. We analyze its decomposition into indecomposable modules over subrings of A that are generated by elements in the closure of $\mathcal{K}$, establishing structural results that parallel the decomposition theorem for morphisms of complex projective varieties. We use our theorems to recover key statements in combinatorial Hodge theory and to illuminate the Hodge-theoretic aspects of the decomposition theorem in algebraic geometry.

1. INTRODUCTION

Let $A = \bigoplus_{k \geqslant 0} A^k$ be a finite dimensional commutative graded algebra over $\mathbb{R}$, and let $\mathcal{K}_A$ be a nonempty open convex cone in A^1. Let $M = \bigoplus_{k \geqslant 0} M^k$ be a finite dimensional graded A-module equipped with a symmetric bilinear form $\mathcal{Q}\colon M \times M \to \mathbb{R}$ that is *A-invariant*:

$$\mathcal{Q}(ax, y) = \mathcal{Q}(x, ay) \text{ for all } a \in A \text{ and all } x, y \in M.$$

We say that $(M, \mathcal{Q})$ is a *Lefschetz module* of degree d over $(A, \mathcal{K}_A)$ if it satisfies the following three properties, called the *Kähler package* for $(\mathcal{Q}, \mathcal{K}_A)$:

(PD) The induced bilinear pairing between the graded pieces

$$\mathcal{Q}\colon M^i \times M^j \longrightarrow \mathbb{R}$$

is nondegenerate if $i + j = d$, and it is zero if $i + j \neq d$ (*Poincaré duality*).

(HL) For each nonnegative $k \leqslant \frac{d}{2}$ and $\eta \in \mathcal{K}_A$, the linear map

$$M^k \longrightarrow M^{d-k}, \qquad x \longmapsto \eta^{d-2k} x$$

is an isomorphism of vector spaces (*Hard Lefschetz property*).

(HR) For each nonnegative $k \leqslant \frac{d}{2}$ and $\eta \in \mathcal{K}_A$, the symmetric bilinear form

$$M^k \times M^k \longrightarrow \mathbb{R}, \qquad (x_1, x_2) \longmapsto (-1)^k \mathcal{Q}\Big(x_1, \eta^{d-2k} x_2\Big)$$

is positive definite on the kernel of the linear map

$$M^k \longrightarrow M^{d-k+1}, \qquad x \longmapsto \eta^{d-2k+1} x$$

(*Hodge–Riemann relations*).

A classic example of a Lefschetz module of degree d is given by $\bigoplus_{k\geqslant 0} H^{k,k}(X,\mathbb{R})$, where X is a compact Kähler manifold of dimension d, and $H^{k,k}(X,\mathbb{R}) = H^{k,k} \cap H^{2k}(X,\mathbb{R})$. Here

$$\mathcal{Q} = \text{the Poincaré pairing on } \bigoplus_{k\geqslant 0} H^{k,k}(X,\mathbb{R}) \text{ and } \mathcal{K}_A = \text{the Kähler cone in } H^{1,1}(X,\mathbb{R}).$$

In this case, the Kähler package for $(\mathcal{Q},\mathcal{K}_A)$ is a consequence of the Hodge theory of harmonic forms [Huy05, Chapter 3].[1] Another prominent class of examples comes from the intersection cohomology of a complex projective variety[2] and, assuming Grothendieck's standard conjectures on algebraic cycles [Gro69], the ring of algebraic cycles modulo homological equivalence on a smooth projective variety.[3] Other examples of Lefschetz modules include the combinatorial intersection cohomology of a convex polytope [Kar04], the reduced Soergel bimodule of a Coxeter group element [EW14], the Chow ring, the augmented Chow ring, and the conormal Chow ring of a matroid [AHK18, BHM+22, ADH23], as well as the intersection cohomology of a matroid [BHM+b], and the cohomology of Kähler tropical varieties [AP20, AP25]. See Example 3.6 for a discussion of Lefschetz modules that do not arise from Kähler geometry or algebraic geometry.

In the remainder of this paper, we suppose that $(M,\mathcal{Q})$ is a Lefschetz module of degree d over $(A,\mathcal{K}_A)$ and deduce a number of structural results. Let B be a subalgebra of A generated by a subset of the closure $\overline{\mathcal{K}}_A \subseteq A^1$, and set

$$\mathcal{K}_B \coloneqq \text{the nonempty open convex cone given by the interior of } \overline{\mathcal{K}}_A \cap B^1 \text{ in } B^1.$$

For a graded B-module N, we define the shifted module $N[-k]$ as the direct sum of the vector spaces N^{i-k} placed in degree i. This is a graded B-module in a natural way.

We work in the category of finite dimensional graded B-modules, and we show that an arbitrary decomposition of M into indecomposable objects has a number of remarkable properties that we call the *decomposition package*. Choose any decomposition

$$M \simeq \bigoplus_{\alpha} \bigoplus_{k} N_\alpha[-k]^{\oplus m(\alpha,k)} \tag{1.1}$$

where $N_\alpha = N_\alpha^0 \oplus \cdots \oplus N_\alpha^{d(\alpha)}$ are indecomposable graded B-modules satisfying

$$N_\alpha^0 \neq 0, \;\; N_\alpha^{d(\alpha)} \neq 0, \;\; \text{and } \; N_\alpha \not\simeq N_\beta \text{ as graded } B\text{-modules for all } \alpha \neq \beta.$$

By the Krull–Schmidt theorem applied to the category of finite dimensional graded B-modules [Ati56], the collection $\{N_\alpha\}$ and the multiplicities $m(\alpha,k)$ are independent of the choice of decomposition of M. We then have the following properties.

[1]The Hodge decomposition splits $H(X,\mathbb{C})$ into Lefschetz modules with complex coefficients. For formulations of our main results over $\mathbb{C}$ instead of $\mathbb{R}$, see Remark 2.1 and Theorem 2.12.

[2]A detailed discussion of the known proofs of the Hard Lefschetz property and the Hodge–Riemann relations in this case can be found in [dCM09a, Section 3].

[3]Some of these Lefschetz modules can be defined over a subfield $\mathbb{K} \subseteq \mathbb{R}$. For formulations of our main results over $\mathbb{K}$ and their applications, see Remark 2.2, Proposition 2.8, and Example 3.3.

Theorem 1.1 (Simplicity)**.** If $d(\alpha) \leqslant d(\beta) + 2k$, then

$$\mathrm{Hom}_B(N_\alpha, N_\beta[-k]) = \begin{cases} \mathbb{R},\ \mathbb{C}, \text{ or } \mathbb{H}, & \text{if } k = 0 \text{ and } \alpha = \beta, \\ 0 & \text{if otherwise.} \end{cases}$$

Each of the three finite dimensional division algebras over the real numbers $\mathbb{R}$ arises as the endomorphism ring of some N_α; see Example 3.1 for the case of the complex numbers $\mathbb{C}$ and Example 3.2 for the case of the quaternions $\mathbb{H}$.

Theorem 1.2 (Duality)**.** For each α, up to a nonzero constant multiple, there is a unique nonzero B-invariant symmetric bilinear form $\mathcal{Q}_\alpha$ on N_α satisfying the orthogonality relation

$$\mathcal{Q}_\alpha(N^i_\alpha, N^j_\alpha) = 0 \text{ unless } i + j = d(\alpha).$$

This B-invariant symmetric bilinear form $\mathcal{Q}_\alpha$ is nondegenerate.

We show that, for each α, the bilinear form $\mathcal{Q}_\alpha$ gives N_α the structure of a Lefschetz module of degree $d(\alpha)$ over $(B, \mathcal{K}_B)$.

Theorem 1.3 (Hard Lefschetz)**.** For each nonnegative $k \leqslant \frac{d(\alpha)}{2}$ and $\ell \in \mathcal{K}_B$, the linear map

$$N^k_\alpha \longrightarrow N^{d(\alpha)-k}_\alpha, \qquad x \longmapsto \ell^{d(\alpha)-2k} x$$

is an isomorphism of vector spaces.

Theorem 1.4 (Hodge–Riemann)**.** There exists a unique $\epsilon_\alpha \in \{\pm 1\}$ such that, for each nonnegative $k \leqslant \frac{d(\alpha)}{2}$ and $\ell \in \mathcal{K}_B$, the symmetric bilinear form

$$N^k_\alpha \times N^k_\alpha \longrightarrow \mathbb{R}, \qquad (x_1, x_2) \longmapsto (-1)^k \epsilon_\alpha \mathcal{Q}_\alpha\Big(x_1, \ell^{d(\alpha)-2k} x_2\Big)$$

is positive definite on the kernel of the linear map

$$N^k_\alpha \longrightarrow N^{d(\alpha)-k+1}_\alpha, \qquad x \longmapsto \ell^{d(\alpha)-2k+1} x.$$

Throughout the paper, an important role will be played by an increasing filtration

$$P = \Big(0 \subseteq P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{2d} = M\Big)$$

called the *perverse filtration* of M over B, defined as follows: Choose an element $\ell \in \mathcal{K}_B$, and choose a decomposition of M into a direct sum of cyclic graded $\mathbb{R}[\ell]$-modules of the form

$$\bigoplus_{k=0}^{d(x)} \mathrm{span}(\ell^k x),$$

where x is an element of degree $n(x)$ in M. We then define P_j as the subspace of M spanned by those summands with $d(x) + 2n(x) \leqslant j$. Alternatively, we can describe the perverse filtration as

$$P_j \cap M^k = \sum_c \ell^{k-c} M \cap \mathrm{ann}_M(\ell^{j+1-k-c}) \cap M^k. \tag{1.2}$$

It follows from Theorem 1.3 that the perverse filtration P is independent of the choice of $\ell \in \mathcal{K}_B$. Indeed, we can refine the given decomposition of M into indecomposable graded B-modules to a decomposition into indecomposable graded $\mathbb{R}[\ell]$-modules and observe that, for each j, we have

$$P_j = \bigoplus_{d(\alpha)+2k \leqslant j} N_\alpha[-k]^{\oplus m(\alpha,k)}. \tag{1.3}$$

In particular, each P_j is a graded B-submodule of M, so multiplication by elements of B preserves the perverse filtration:

$$bP_j \subseteq P_j \text{ for all } j \text{ and all } b \in B.$$

Let $\mathrm{Gr} \coloneqq \bigoplus_j P_j/P_{j-1}$ be the associated graded of the perverse filtration. Note that

$$\mathrm{Gr} = \bigoplus_{i,j} \mathrm{Gr}^{i,j}, \text{ where } \mathrm{Gr}^{i,j} \coloneqq (M^i \cap P_j)/(M^i \cap P_{j-1}),$$

and that Gr has the structure of a graded B-module such that $B^k \,\mathrm{Gr}^{i,j} \subseteq \mathrm{Gr}^{i+k,j}$. By construction, M and Gr are isomorphic as graded B-modules. We set

$$V_\alpha \coloneqq \bigoplus_k V_\alpha^k, \text{ where } V_\alpha^k \coloneqq \mathrm{Hom}_B(N_\alpha[-k], \mathrm{Gr}^{\bullet, d(\alpha)+2k}).$$

Let $\mathbb{D}_\alpha$ be the division algebra $\mathrm{Hom}_B(N_\alpha, N_\alpha)$. Note that N_α and V_α are modules over $\mathbb{D}_\alpha$. It follows from Theorem 1.1 and Theorem 1.3 that for each j and each α, the natural map

$$N_\alpha \otimes_{\mathbb{D}_\alpha} V_\alpha^k \longrightarrow \mathrm{Gr}^{\bullet,j}$$

is injective when $d(\alpha) + 2k = j$. Therefore, $\mathrm{Gr}^{\bullet,j}$ admits a decomposition

$$\mathrm{Gr}^{\bullet,j} = \bigoplus_{\alpha,\, d(\alpha)+2k=j} N_\alpha \otimes_{\mathbb{D}_\alpha} V_\alpha^k.$$

In particular, $\dim_{\mathbb{D}_\alpha} V_\alpha^k = m(\alpha, k)$. It follows that $\mathrm{Gr}^{\bullet,j}$ has a unique "isotypic" decomposition, where the summands are given by the sum of all submodules of $\mathrm{Gr}^{\bullet,j}$ which are isomorphic to $N_\alpha[-k]$ (equivalently, the sum of all submodules that are quotients of $N_\alpha[-k]$).

We now formulate analogues of the relative Hard Lefschetz theorem and the relative Hodge–Riemann relations using Gr. By Lemma 4.1, we have $aP_j \subseteq P_{j+2k}$ for each $a \in A^k$, so we have an A-module structure $*\colon A \times \mathrm{Gr} \to \mathrm{Gr}$ such that

$$A^k * \mathrm{Gr}^{i,j} \subseteq \mathrm{Gr}^{i+k,j+2k} \text{ for all } k \text{ and } i, j.$$

It is straightforward to check that

$$\mathcal{Q}(M^i \cap P_j, M^{d-i} \cap P_{2d-j-1}) = 0 \text{ for all } i, j, \tag{1.4}$$

see Lemma 4.7. We therefore get an induced symmetric bilinear form

$$\underline{\mathcal{Q}}\colon \mathrm{Gr}^{i,j} \times \mathrm{Gr}^{d-i,2d-j} \longrightarrow \mathbb{R} \quad \text{for all } i, j.$$

Note that $\underline{\mathcal{Q}}$ is both A-invariant and B-invariant:

$$\underline{\mathcal{Q}}(a * x, y) = \underline{\mathcal{Q}}(x, a * y) \quad \text{and} \quad \underline{\mathcal{Q}}(bx, y) = \underline{\mathcal{Q}}(x, by) \text{ for all } a \in A \text{ and } b \in B.$$

Let $\mathcal{K}_{A/B}$ be the open convex cone $\mathcal{K}_A + B^1$ in A^1.

Theorem 1.5 (Relative Hard Lefschetz)**.** For each nonnegative $j \leqslant d$ and $\eta \in \mathcal{K}_{A/B}$, the linear map

$$\mathrm{Gr}^{\bullet,j} \longrightarrow \mathrm{Gr}^{\bullet+d-j,2d-j}, \qquad x \longmapsto \eta^{d-j} * x$$

is an isomorphism of vector spaces.

Note that, for each $\eta \in A^1$, there is a map $V_\alpha^k \longrightarrow V_\alpha^{k+1}$ induced by $\eta * -$.

Corollary 1.6. For each α and $\eta \in \mathcal{K}_{A/B}$ and for each nonnegative $k \leqslant (d - d(\alpha))/2$, there is an isomorphism of vector spaces

$$V_\alpha^k \longrightarrow V_\alpha^{d-d(\alpha)-k}$$

induced by $\eta^{d-d(\alpha)-2k} * -$. In particular, for each α, the multiplicities $m(\alpha, k) = \dim_{\mathbb{D}_\alpha} V_\alpha^k$ form a symmetric and unimodal sequence.

For elements $\eta \in \mathcal{K}_{A/B}$ and $\ell \in \mathcal{K}_B$, and nonnegative integers $j \leqslant d$ and $i \leqslant j/2$, set

$$\mathrm{Prim}^{i,j} \coloneqq \ker(\eta^{d-j+1} \colon \mathrm{Gr}^{i,j} \to \mathrm{Gr}^{i+d-j+1,2d-j+2}) \cap \ker(\ell^{j-2i+1} \colon \mathrm{Gr}^{i,j} \to \mathrm{Gr}^{j-i+1,j}).$$

It follows from Theorem 1.3 and Theorem 1.5 that there is a direct sum decomposition

$$\mathrm{Gr} = \bigoplus_{\substack{j \leqslant d \\ i \leqslant j/2}} \bigoplus_{\substack{s \leqslant d-j \\ t \leqslant j-2i}} \eta^s * \left(\ell^t \, \mathrm{Prim}^{i,j}\right). \tag{1.5}$$

Theorem 1.7 (Relative Hodge–Riemann relations)**.** For each $\eta \in \mathcal{K}_{A/B}$ and $\ell \in \mathcal{K}_B$ and for each nonnegative $j \leqslant d$ and $i \leqslant d/2$, the symmetric bilinear form on $\mathrm{Gr}^{i,j}$ defined by

$$(x, y) \longmapsto (-1)^i \underline{\mathcal{Q}}(x, \eta^{d-j} * \ell^{j-2i} y)$$

is positive definite when restricted to the subspace $\mathrm{Prim}^{i,j}$.

Corollary 1.8. For each $j \leqslant d$ and $\eta \in \mathcal{K}_{A/B}$, the B-module

$$\ker(\eta^{d-j+1} \colon \mathrm{Gr}^{\bullet,j} \longrightarrow \mathrm{Gr}^{\bullet+d-j+1,2d-j+2})$$

equipped with the form $\underline{\mathcal{Q}}(x, \eta^{d-j} * y)$ is a Lefschetz module of degree j over $(B, \mathcal{K}_B)$.

Corollary 1.9. For each $k \leqslant d$ and $\ell \in \mathcal{K}_B$, the A-module

$$\ker(\ell^{k+1} \colon \mathrm{Gr}^{\bullet,2\bullet+k} \longmapsto \mathrm{Gr}^{\bullet+k+1,2\bullet+k})$$

equipped with the form $\underline{\mathcal{Q}}(x, \ell^k y)$ is a Lefschetz module of degree $d - k$ over $(A, \mathcal{K}_{A/B})$.

Let R be the graded subalgebra of A consisting of all elements that preserve the perverse filtration on M. As observed before, M and Gr are isomorphic as graded B-modules. In fact, a stronger statement holds.

Theorem 1.10 (Decomposition)**.** As graded R-modules, M is isomorphic to Gr.

The proof of Theorem 1.10 produces a canonical isomorphism, whose construction depends on the choice of $\eta \in \mathcal{K}_{A/B}$.[4]

In Section 2, we give a number of applications of the above theorems to matroids, polytopes, and projective varieties. Motivated by combinatorial applications, many authors have shown that certain modules are Lefschetz [McM93, Kar04, EW14, AHK18, Kar19, BHM$^+$22, ADH23, BHM$^+$b]. Our results, especially Theorem 1.3 and Theorem 1.4, give a tool to produce more Lefschetz modules from a given Lefschetz module, and we show that several of the above results can be easily deduced from our work.

Our results can be used to study Lefschetz modules coming from geometry as well. In Section 2.4.1, we use our main theorems to verify Grothendieck's standard conjectures on algebraic cycles [Gro69] in new cases. In Section 2.4.3, by combining our results with [BBD82], we recover a result of Saito [Sai88, Sai90] and de Cataldo–Migliorini [dCM05] that provides a polarized pure Hodge structure on the intersection cohomology of a complex projective variety.

The proofs of our main theorems are inspired by the beautiful work of de Cataldo and Migliorini on the decomposition theorem [dCM02, dCM05, dCM09b]. Although our inductive strategy resembles theirs, important differences arise. For instance, while their argument relies on the Lefschetz hyperplane theorem and the simplicity of certain perverse sheaves, these tools are not available in our setting.

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2. THE DECOMPOSITION PACKAGE IN PRACTICE

In this section, we discuss some variations on the main theorems and give applications.

2.1. **Coefficient fields.**

We provide some remarks on the extensions of the main theorems to cases where the coefficients of M are different.

Remark 2.1. The results of this paper can be extended to modules with complex coefficients. Let A and $\mathcal{K}_A$ be as in Section 1, and set $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$. Let $M = \bigoplus_{k \geqslant 0} M^k$ be a finite dimensional graded $A_{\mathbb{C}}$-module endowed with a Hermitian form $\mathcal{Q} \colon M \times M \to \mathbb{C}$ that is A-invariant:

$$\mathcal{Q}(ax, y) = \mathcal{Q}(x, \bar{a}y) \ \text{ for all } a \in A_{\mathbb{C}} \text{ and all } x, y \in M.$$

[4] Following ideas of Deligne [Del94], de Cataldo produces in [dC13] several other distinguished isomorphisms using the choice of η in the geometric setting.

We say that $(M, \mathcal{Q})$ is a *complex Lefschetz module* of degree d over $(A, \mathcal{K}_A)$ if it satisfies the Kähler package for $(\mathcal{Q}, \mathcal{K}_A)$, namely, it satisfies (PD) for the induced complex valued pairings $\mathcal{Q}\colon M^i \times M^j \to \mathbb{C}$, it satisfies (HL), and (HR) holds for all $\eta \in \mathcal{K}_A$ and $k \leqslant d/2$: the Hermitian form

$$M^k \times M^k \longrightarrow \mathbb{C}, \qquad (x_1, x_2) \longmapsto (-1)^k \mathcal{Q}\Big(x_1, \eta^{d-2k} x_2\Big)$$

is positive definite on the kernel of the map $M^k \to M^{d-k+1}$ given by multiplication by η^{d-2k+1}.

Let B and $\mathcal{K}_B$ be as in Section 1, and set $B_{\mathbb{C}} = B \otimes_{\mathbb{R}} \mathbb{C}$. We choose a decomposition of M as in (1.1), requiring that each N_α is an indecomposable $B_{\mathbb{C}}$-module which is nonzero in degrees 0 and $d(\alpha)$. Let $\overline{N}_\alpha$ be the complex vector space N_α endowed with the $B_{\mathbb{C}}$-module structure in which each element $b \in B_{\mathbb{C}}$ acts as the complex conjugate $\bar{b}$ on N_α. Then the analogue of Theorem 1.1 holds in the sense that, when $d(\alpha) \leqslant d(\beta) + 2k$, then

$$\mathrm{Hom}_B(N_\alpha, \overline{N}_\beta[-k]) = \begin{cases} \mathbb{C} & \text{if } k = 0 \text{ and } \alpha = \beta, \\ 0 & \text{if otherwise.} \end{cases}$$

The induced form $\mathcal{Q}_\alpha$ on N_α obtained from the above result is Hermitian, and it is well-defined up to multiplication by a nonzero constant in $\mathbb{C}$. Moreover, for each α, there is a unique $\epsilon_\alpha \in \mathbb{S}^1$, the unit circle in $\mathbb{C}$, such that $\epsilon_\alpha \mathcal{Q}_\alpha$ gives N_α the structure of a complex Lefschetz module of degree $d(\alpha)$ over $(B, \mathcal{K}_B)$. In other words, the analogues of Theorem 1.3 and 1.4 hold.

The perverse filtration in this setting is defined similarly, taking the spans over $\mathbb{C}$. Therefore the P_j are $B_{\mathbb{C}}$-modules, and so Gr is a graded $B_{\mathbb{C}}$-module with $B^k_{\mathbb{C}} \,\mathrm{Gr}^{i,j} \subseteq \mathrm{Gr}^{i+k,j}$. Moreover, Gr admits a similar $A_{\mathbb{C}}$-module structure. In addition, the orthogonality property (1.4) holds, inducing a Hermitian form

$$\underline{\mathcal{Q}}\colon\ \mathrm{Gr}^{\bullet,\bullet} \times \mathrm{Gr}^{d-\bullet, 2d-\bullet} \longrightarrow \mathbb{C}.$$

This form is both $A_{\mathbb{C}}$-invariant and $B_{\mathbb{C}}$-invariant:

$$\underline{\mathcal{Q}}(a * x, y) = \underline{\mathcal{Q}}(x, \bar{a} * y) \ \text{ and } \ \underline{\mathcal{Q}}(bx, y) = \underline{\mathcal{Q}}(x, \bar{b}y) \ \text{ for all } a \in A_{\mathbb{C}} \text{ and } b \in B_{\mathbb{C}}.$$

With these modifications, the statements of Theorems 1.5 and 1.7, as well as Corollaries 1.6 and 1.8 and 1.9 still hold. Finally, let R be the graded subalgebra of $A_{\mathbb{C}}$ consisting of all elements which preserve the perverse filtration. Then the statement of Theorem 1.10 still holds. $\diamond$

For example, let X be a connected compact Kähler manifold of complex dimension d with real cohomology $H(X) = \bigoplus_{k \geqslant 0} H^k(X, \mathbb{R})$. The orientation on X induces an isomorphism $\deg\colon H^{2d}(X, \mathbb{R}) \to \mathbb{R}$. Let $\mathcal{Q}$ be the symmetric bilinear form on $H(X)$ defined by $(x, y) \mapsto \deg(xy)$, i.e., the Poincaré pairing. For each nonnegative integer k, the kth cohomology with complex coefficients of X is equipped with a Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{\substack{p,q \geqslant 0 \\ p+q=k}} H^{p,q}.$$

For each k, let $H^{k,k}(X, \mathbb{R})$ be the intersection of $H^{2k}(X, \mathbb{R})$ with $H^{k,k}$. Let $A = \bigoplus_{k \geqslant 0} H^{k,k}(X, \mathbb{R})$, so $A_{\mathbb{C}} = \bigoplus_{k \geqslant 0} H^{k,k}$, and let $\mathcal{K}_A$ be the Kähler cone in $H^{1,1}(X, \mathbb{R})$. For each integer n, let M_n be the graded A-module defined by $M_n = \bigoplus_{k\geqslant 0} H^{k+n,k}$. We endow M_n with the Hermitian form

$$\mathcal{Q}_n \colon M_n \times M_n \longrightarrow \mathbb{C}, \quad \mathcal{Q}_n(\alpha,\beta) = \mathrm{i}^n \mathcal{Q}(\alpha,\bar{\beta}), \quad \text{where } \mathrm{i} = \sqrt{-1}.$$

Then each $(M_n, \mathcal{Q}_n)$ is a complex Lefschetz module of degree $d-n$ over $(A, \mathcal{K}_A)$. We refer to Section 2.4.2 for more on Lefschetz modules with a Hodge structure.

Remark 2.2. Some Lefschetz modules have natural $\mathbb{Q}$-structures. For example, this is the case for the cohomology of a connected smooth complex projective variety if the cohomology is generated by algebraic cycles, the Chow ring of a matroid, or the reduced Soergel bimodule of a Weyl group element. $\diamond$

2.2. **Applications to matroids.** In recent years, significant progress in matroid theory has been achieved by proving that certain combinatorially defined modules are Lefschetz, see [AHK18, ADH23, BHM+b]. Several of these results can be recovered or extended using the main theorems of this paper. Let M be a matroid of rank d on $E = \{1,\dots,n\}$. For each $0 \leqslant k \leqslant d$, let $\mathcal{L}^k(\mathrm{M})$ denote the set of flats of M of rank k.

Theorem 2.3. For any $k \leqslant j \leqslant d-k$, we have $|\mathcal{L}^k(\mathrm{M})| \leqslant |\mathcal{L}^j(\mathrm{M})|$.

Theorem 2.3 was conjectured in [DW74, DW75], proved for realizable matroids in [HW17], and established in full generality in [BHM+b]. Its best known special case is the de Bruijn–Erdős theorem on point-line incidences in projective planes [dBE48]:

> *Let E be a set of points in a projective plane that is not contained in any line. Then there are at least $|E|$ distinct lines that intersect E in at least two points.*

We now show how Theorem 2.3 can be deduced from the main results of this paper.

Let $H(\mathrm{M})$ denote the *graded Möbius algebra* of M. As a graded vector space, we have

$$H(\mathrm{M}) = \bigoplus_{k=0}^{d} \Bigg(\bigoplus_{F \in \mathcal{L}^k(\mathrm{M})} \mathbb{R} y_F \Bigg),$$

where the degree of y_F is the rank of F. The multiplication is given by the formula

$$y_F y_G = \begin{cases} y_{F\vee G} & \text{if } \mathrm{rk}(F) + \mathrm{rk}(G) = \mathrm{rk}(F \vee G), \\ 0 & \text{if otherwise.} \end{cases}$$

Note that $H(\mathrm{M})$ is generated as an algebra by the elements y_i for each rank 1 flat i, and that the dimension of $H^k(\mathrm{M})$ is the number of rank k flats of M. In [BHM+22], the authors construct a graded algebra $A(\mathrm{M})$, called the *augmented Chow ring* of M, which contains $H(\mathrm{M})$ as a subalgebra. By [BHM+22, Theorem 1.3], the augmented Chow ring $A(\mathrm{M})$ is equipped with an open cone $\mathcal{K}_{A(\mathrm{M})}$ in its degree one part and a map $\deg\colon A(\mathrm{M}) \to \mathbb{R}$ such that the pairing $\mathcal{Q}(x,y) = \deg(xy)$ gives $(A(\mathrm{M}), \mathcal{Q})$ the structure of a Lefschetz module of degree d over $A(\mathrm{M})$ with respect to $\mathcal{K}_{A(\mathrm{M})}$. Furthermore, each y_i lies in the closure of $\mathcal{K}_{A(\mathrm{M})}$.

Choose a decomposition of $A(\mathrm{M})$ into indecomposable graded $H(\mathrm{M})$-modules. Let $IH(\mathrm{M})$ denote the unique summand which intersects the degree 0 part of $A(\mathrm{M})$, so $IH(\mathrm{M})$ contains $H(\mathrm{M})$. By the Krull–Schmidt theorem applied to the category of finite dimensional graded $H(\mathrm{M})$-modules [Ati56], $IH(\mathrm{M})$ is well-defined up to isomorphism of graded $H(\mathrm{M})$-modules. Let $\mathcal{K}_{H(\mathrm{M})}$ denote the interior of $\overline{\mathcal{K}}_{A(\mathrm{M})} \cap H(\mathrm{M})$, which contains the set of all positive linear combinations of the y_i, i.e., $\left\{ \sum_{i=1}^n c_i y_i : c_i > 0 \text{ for all } i \right\}$.

Proof of Theorem 2.3. The main theorems of this paper show that $(IH(\mathrm{M}), \mathcal{Q}|_{IH(\mathrm{M})})$ is a Lefschetz module over $(H(\mathrm{M}), \mathcal{K}_{H(\mathrm{M})})$ of degree d. Theorem 1.3 states that multiplication by $(\sum y_i)^{d-2k}$ induces an isomorphism from $IH^k(\mathrm{M})$ to $IH^{d-k}(\mathrm{M})$. In particular, multiplication by $(\sum y_i)^{d-2k}$ induces an injection from $H^k(\mathrm{M})$ to $H^{d-k}(\mathrm{M})$, as multiplication by $\sum y_i$ preserves the submodule $H(\mathrm{M})$ of $IH(\mathrm{M})$. This implies the desired inequality. □

In [BHM$^+$b], the proof of Theorem 2.3 was achieved by giving an explicit construction of $IH(\mathrm{M})$ as a submodule of $A(\mathrm{M})$ and proving directly that it is a Lefschetz module. One of the other main results of *loc. cit.*, [BHM$^+$b, Theorem 1.9], is the identification of the Hilbert series of $IH(\mathrm{M})$ with the recursively-defined *Z-polynomial* of a matroid [PXY18]. In particular, this proves that the coefficients of the Z-polynomial are nonnegative, which is not clear from the definition. Furthermore, if $\mathfrak{m}$ is the ideal of positively graded elements in $H(\mathrm{M})$, then the Hilbert series of $IH(\mathrm{M})/\mathfrak{m}IH(\mathrm{M})$ is the *Kazhdan–Lusztig polynomial* of M, a recursively-defined polynomial which was introduced in [EPW16], and so the coefficients of the Kazhdan–Lusztig polynomial of M are nonnegative as well.

One key tool in the computation of the Hilbert series of $IH(\mathrm{M})$ and $IH(\mathrm{M})/\mathfrak{m}IH(\mathrm{M})$ in [BHM$^+$b] is the following statement [BHM$^+$b, Lemma 6.2]. For a flat F of M, let M_F denote the contraction of M at F.

Proposition 2.4. As graded vector spaces, $y_F IH(\mathrm{M})$ is isomorphic to $IH(\mathrm{M}_F)[-\operatorname{rk}(F)]$.

This result is proved in *loc. cit.* only after giving an explicit construction of $IH(\mathrm{M})$. In the forthcoming work [BHM$^+$a], the authors give a direct algebraic proof of Proposition 2.4. Using Proposition 2.4, we can compute the Hilbert function of $IH(\mathrm{M})$ and $IH(\mathrm{M})/\mathfrak{m}IH(\mathrm{M})$ without further input. We begin with a general result about Lefschetz modules.

Proposition 2.5. Let $(M, \mathcal{Q})$ be an indecomposable Lefschetz module of degree d over $(A, \mathcal{K}_A)$. If d is even and M is not concentrated in degree $d/2$, then for any nonzero $x \in M^{d/2}$, there is some $a \in A$ of positive degree such that ax is nonzero.

That is, an indecomposable Lefschetz module has no socle in its middle degree unless it is concentrated in middle degree.

Proof of Proposition 2.5. Let S be the subspace of $M^{d/2}$ consisting of elements such that, for all $a \in A$ of positive degree, we have $ax = 0$. Note that S is a graded A-submodule of M. Choose some $\eta \in \mathcal{K}_A$. For any nonzero $x \in S$, (HR) gives that $(-1)^{d/2}\mathcal{Q}(x,x) > 0$ as $\eta x = 0$. In particular, the restriction of $\mathcal{Q}$ to S is definite and therefore nondegenerate. Therefore, if we set $M' = \{y \in M : \mathcal{Q}(x,y) = 0 \text{ for all } x \in S\}$, then $M = M' \oplus S$. Note that M' is a nonzero graded A-submodule as M is not concentrated in degree $d/2$. Because M is indecomposable, this implies that $S = 0$. □

We can use Proposition 2.5 to recover a key vanishing statement from [BHM+b, Proposition 1.7].

Proposition 2.6. The Hilbert series of $IH(\mathrm{M})/\mathfrak{m}IH(\mathrm{M})$ has degree less than $d/2$.

Proof. Note that (HL) implies that $\{x \in IH(\mathrm{M}) : \mathfrak{m}x = 0\}$ vanishes in degree strictly less than $d/2$. As $IH(\mathrm{M})$ is indecomposable by definition, Proposition 2.5 implies that $\{x \in IH(\mathrm{M}) : \mathfrak{m}x = 0\}$ vanishes in degree $d/2$. The bilinear form $\mathcal{Q}$ induces a perfect pairing between $IH(\mathrm{M})/\mathfrak{m}IH(\mathrm{M})$ and $\{x \in IH(\mathrm{M}) : \mathfrak{m}x = 0\}$, which gives the result. □

From Proposition 2.4 and Proposition 2.6, one can then use elementary algebraic considerations to derive a recursion for the Hilbert series of $IH(\mathrm{M})$ and $IH(\mathrm{M})/\mathfrak{m}IH(\mathrm{M})$. This recursion implies that the Hilbert series coincide with the Z-polynomial and Kazhdan–Lusztig polynomial of M, see [BHM+b, Proof of Theorems 1.2 and 1.3].

Additionally, the main theorems can be applied in a number of other settings related to matroids. For example, in [BHM+22, Theorem 1.4], the authors study the decomposition of the Chow ring and augmented Chow ring of a matroid as a module over the subring generated by a particular element in the boundary of $\overline{\mathcal{K}}$. Theorem 1.5 and Theorem 1.7 immediately give consequences for this decomposition. The authors also study the decomposition of the (augmented) Chow ring of a matroid as a module over the (augmented) Chow ring of a matroid deletion [BHM+22, Theorem 1.1 and 1.2]. As the (augmented) Chow ring of the matroid deletion is a subring which is generated by elements in $\overline{\mathcal{K}}$, Theorem 1.5 and Theorem 1.7 immediately give consequences for this decomposition.

2.3. **Applications to polytopes.** For a full-dimensional polytope in a real vector space W, the associated normal fan is a polyhedral fan in the dual space which is equipped with a strictly convex piecewise-linear function. Such a fan is called a *projective* fan. If the polytope has rational coordinates with respect to some $\mathbb{Q}$-structure on W, one can then associate a projective toric variety to this projective fan. Beginning with the work of Stanley [Sta80, Sta87], the fact that the intersection cohomology of a projective toric variety is a Lefschetz module has been used to deduce several combinatorial inequalities satisfied by polytopes, corresponding to the nonnegativity and unimodality of the intersection cohomology Betti numbers of the toric variety.

However, it is not always possible to choose a $\mathbb{Q}$-structure on W so that the polytope has rational coordinates. In this case, there is no associated toric variety. It was an outstanding open problem to show that the same inequalities hold for all polytopes until it was resolved by Karu

[Kar04], building on the earlier works [McM93, BBFK02, BL03]. This was done by constructing an analogue of the intersection cohomology purely in terms of the projective fan and proving that it is a Lefschetz module. In what follows, we discuss applications of the main theorems of this paper to this setting.

Let Σ be a projective fan in a real vector space V of dimension d. A *piecewise polynomial* function on Σ is a real-valued function on V whose restriction to any cone of Σ is equal to the restriction of a polynomial function on V. Let $A(\Sigma)$ be the ring of piecewise polynomial functions on Σ modulo the ideal generated by globally linear functions. In [BBFK02, BL03], a graded $A(\Sigma)$-module $IH(\Sigma)$, called the intersection cohomology of Σ, is constructed. When Σ is rational, $IH(\Sigma)$ is identified with the intersection cohomology of the corresponding toric variety.

The module $IH(\Sigma)$ is constructed using the theory of sheaves on the poset of cones of the fan. For each cone σ of Σ, the authors of [BBFK02, BL03] construct a sheaf $\mathcal{L}_\sigma$ on the poset of cones of Σ whose global sections form a module over the ring of piecewise polynomial functions on Σ. Let $\mathfrak{o}$ be the cone $\{0\}$ of Σ. We obtain $IH(\Sigma)$ by tensoring the global sections of $\mathcal{L}_{\mathfrak{o}}$ with $A(\Sigma)$. When the fan is rational and complete, we obtain the intersection cohomology of the torus-orbit closure corresponding to σ by tensoring the global sections of $\mathcal{L}_\sigma$ with $A(\Sigma)$, with the grading so that it is supported in degrees $\{0, \ldots, d - \dim \sigma\}$. The construction of $\mathcal{L}_\sigma$ is valid for any fan, even one that is not necessarily projective or complete. Let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a proper map of fans. Then there is an analogue of the decomposition theorem: let $\tilde{\mathfrak{o}}$ denote the cone $\{0\}$ of $\widetilde{\Sigma}$. Then $R^i\pi_*\mathcal{L}_{\tilde{\mathfrak{o}}} = 0$ for $i > 0$ [BL03, Theorem 5.6], and

$$\pi_*\mathcal{L}_{\tilde{\mathfrak{o}}} \xrightarrow{\sim} \oplus_{\sigma\in\Sigma}\mathcal{L}_\sigma \otimes W_\sigma, \tag{2.1}$$

where $W_\sigma = \oplus W_\sigma^i$ is a graded vector space [BL03, Theorem 2.2 and 2.6]. The sheaf $\mathcal{L}_{\mathfrak{o}}$ appears as a summand, which implies that, if Σ is projective, $IH(\Sigma)$ is an $A(\Sigma)$-module summand of $IH(\widetilde{\Sigma})$.

Let $\mathcal{K}$ be the cone of strictly convex piecewise linear functions on Σ. In [Kar04], Karu proved that if Σ is projective, then $IH(\Sigma)$, which is equipped with an $A(\Sigma)$-invariant bilinear form (see [BL03, Section 6]), is a Lefschetz module of degree d over $(A(\Sigma), \mathcal{K})$. When Σ is simplicial, this was previously proved by McMullen [McM93].

Since $IH(\Sigma)$ can be decomposable as a graded $A(\Sigma)$-module, we cannot use the main theorems to recover Karu's result. Indeed, let $\widetilde{\Sigma}$ be a projective simplicial fan which refines Σ, which exists by [CLS11, Theorem 6.1.8 and 11.1.9]. Then $A(\widetilde{\Sigma})$ is a Lefschetz module [McM93] and $IH(\Sigma)$ is a graded $A(\Sigma)$-module summand of $A(\widetilde{\Sigma})$. But Theorem 1.3 cannot be applied to $IH(\Sigma)$ because it may not be indecomposable over the subring of $A(\Sigma)$ which is generated in degree 1.

In order to apply the results of this paper, we need to know that a filtration on $IH(\widetilde{\Sigma})$ which is defined using sheaves on fans agrees with the perverse filtration, which is defined using a strictly convex piecewise linear function on Σ. This first filtration, which was introduced and studied in [Kar19, Section 4], is defined using the decomposition in (2.1). We let $\tau^{\leqslant p} \oplus_{\sigma\in\Sigma} \mathcal{L}_\sigma \otimes W_\sigma$ denote the sum of summands of the form $\mathcal{L}_\sigma \otimes W^i_\sigma$ where $i \leqslant \dim\sigma - d + p$. Because $\widetilde{\Sigma}$ is simplicial, the global sections of $\pi_*\mathcal{L}_{\tilde{0}}$ is the ring of piecewise polynomial functions on $\widetilde{\Sigma}$, and so it is equipped with a surjective map to $A(\widetilde{\Sigma})$. We can then define an increasing filtration on $A(\widetilde{\Sigma})$ by the images of the global sections of $\tau^{\leqslant p} \oplus_{\sigma\in\Sigma} \mathcal{L}_\sigma \otimes W_\sigma$.

If one knows that this first filtration agrees with the perverse filtration associated to a strictly convex piecewise linear function on Σ (and, in particular, is independent of the choice of decomposition (2.1)), then it follows that $IH(\Sigma)$ is an $A(\Sigma)$-module summand of $\mathrm{Gr}^{\bullet,d}$. Furthermore, if η is a strictly convex piecewise linear function on $\widetilde{\Sigma}$, then it follows that $IH(\Sigma)$ is contained in $\ker\eta$ due to the semisimplicity of the appropriate category of sheaves on Σ, see [Kar19, Theorem 2.4]. Corollary 1.8 then implies that $IH(\Sigma)$ is a Lefschetz module.

That these two filtrations coincide is essentially equivalent to (HL) for $IH(\Sigma)$ and the intersection cohomology of star fans of Σ, so it is not easy to establish directly. However, the results of this paper can be used to simplify Karu's proof of this statement in [Kar04]. Karu's argument proceeds by induction on dimension, first proving (HL) for $IH(\Sigma)$ using (HR) for fans of dimension at most $d-1$, and then proving (HR) for $IH(\Sigma)$. After one proves (HL) for $IH(\Sigma)$, the main theorems of this paper imply that $IH(\Sigma)$ is a Lefschetz module, obviating the need for the second step.

In [Kar19], Karu proved an analogue of the relative Hard Lefschetz theorem for sheaves on fans. We show that it is possible to deduce this result from the main theorems of this paper. We need to use that the intersection cohomology of a projective fan is a Lefschetz module [Kar04]. However, if Σ is simplicial, then $IH(\Sigma) = A(\Sigma)$ and $A(\Sigma)$ is generated in degree 1. In this case, which is the one relevant for one of the main combinatorial applications of the relative Hard Lefschetz property in this setting, the unimodality of local h-polynomials of regular subdivisions [Sta92, KS16], the argument below does not require Karu's results. The argument in [LS25, Proof of Theorem 1.9] can also be adapted to deduce the unimodality of local h-polynomial of regular subdivisions from the main theorems without needing the theory of sheaves on fans.

Given a proper map $\pi\colon \widetilde{\Sigma} \to \Sigma$ of fans, we say that a piecewise linear function on the ambient vector space of $\widetilde{\Sigma}$ is *relatively strictly convex* if it is strictly convex on the inverse image of each cone of Σ. We say that π is projective if there is a relatively strictly convex function.

Theorem 2.7. Let Σ be a fan, let $\pi\colon \widetilde{\Sigma} \to \Sigma$ be a projective map of fans, with $\widetilde{\Sigma}$ in a vector space of dimension d and Σ in a vector space of dimension e. Choose a decomposition of $\pi_*\mathcal{L}_{\tilde{0}}$ as in (2.1). Let η be a relatively strictly convex piecewise linear function on $\widetilde{\Sigma}$. Then for any $\sigma \in \Sigma$ and each nonnegative $j \leqslant (d-e-\dim\sigma)/2$, multiplication by $\eta^{d-e-\dim\sigma-2j}$ induces an isomorphism from W^j_σ to $W^{d-e-\dim\sigma-j}_\sigma$.

It follows from the semisimplicity of the appropriate category of sheaves [Kar19, Theorem 2.4], which is a consequence of (HL), that multiplication by η maps W^j_σ into W^{j+1}_σ, so multiplication by $\eta^{d-e-\dim\sigma-2j}$ does indeed induce a map from W^j_σ to $W^{d-e-\dim\sigma-j}_\sigma$.

Proof. We first consider the case when Σ is projective. Then $\widetilde{\Sigma}$ is projective as well, and the discussion above identifies the map $\eta^{d-e-\dim\sigma-2j}\colon W_\sigma^j \to W_\sigma^{d-e-\dim\sigma-j}$ with a map induced by the map $\eta^{d-e-\dim\sigma-2j}{}_*\colon \mathrm{Gr}^{\bullet,e-\dim\sigma+2j} \to \mathrm{Gr}^{\bullet+d-e-\dim\sigma-2j,2d-e+\dim\sigma-2j}$. That this is an isomorphism then follows from Theorem 1.5.

We now reduce the general case to the case where Σ is projective. Because π_* is a pushforward map of sheaves between topological spaces, we can replace Σ by the fan consisting of σ and all of its faces and replace $\widetilde{\Sigma}$ by the inverse image of σ. Choose a projective fan Σ' containing Σ, which is possible by choosing a completion (using [EI06]) and then refining it to make it projective. There is a projective fan $\widetilde{\Sigma}'$ containing $\widetilde{\Sigma}$ which maps to Σ'. For example, if y is a strictly convex piecewise linear function on the support of Σ', we can take the cones of $\widetilde{\Sigma}'$ to be the loci where $\pi^{-1}(y)+\eta$ is linear. We have then reduced to the case already proved. ☐

2.4. **Applications to projective varieties.** Finally, we give some applications of the main theorems to projective varieties. These applications come in two flavors. Assuming Grothendieck's standard conjectures on algebraic cycles [Gro69], the ring of cycles modulo numerical equivalence on a smooth projective variety is a Lefschetz module, see Proposition 2.8. That this holds is known unconditionally in some cases, for example for smooth projective varieties over $\mathbb{C}$ for which the Hodge conjecture is known, and for several classes of varieties over fields of arbitrary characteristic [Ito05]. If this is known for a smooth projective variety X, then we can apply the main theorems to any map from X to a projective variety Y, taking B to be the subring generated by the pullbacks of ample divisor classes on Y. In some cases, this can be used to deduce some of the standard conjectures for Y. This is closely related to the work of Corti and Hanamura [CH00, CH07], who, assuming several conjectures, develop a version of the decomposition theorem for Chow groups.

We also explain how the results of this paper can be used to give easier proofs of several celebrated results about the intersection cohomology of algebraic varieties and the Hodge-theoretic nature of the decomposition theorem. The main theorems give a purely algebraic version of the decomposition theorem, but the results of [BBD82] can be used to show that, in the setting of a map $X \to Y$ of complex projective varieties, this is related to the usual decomposition theorem for perverse sheaves, see Proposition 2.13. Using only the main theorems and results which were available when [BBD82] was written, we can show that the geometrically-defined perverse filtration on the cohomology of X is by Hodge substructures and that the relative Hodge–Riemann relations hold, i.e., the primitive pieces have polarized Hodge structures. With a little more geometric input, we can show that the intersection cohomology of a projective variety carries a polarized pure Hodge structure. This result, which attracted considerable attention in the 70s and 80s primarily using L_2 methods [Zuc79, Zuc83, Che80, HP85, CKS87, KK87], was finally proved using M. Saito's theory of mixed Hodge modules [Sai88, Sai90].

2.4.1. *Algebraic cycles.* We begin by stating the forms of the standard conjectures on algebraic cycles that we will use, see [Gro69]. For a connected smooth projective variety X of dimension

d over an algebraically closed field of arbitrary characteristic, let $A_{\mathrm{num}}(X)$ denote the Chow ring of cycles modulo numerical equivalence with real coefficients, which is equipped with an isomorphism $\deg\colon A^d_{\mathrm{num}}(X) \to \mathbb{R}$. Then we have the following versions of the standard conjecture A and the standard conjecture of Hodge type.

Conjecture $\mathrm{A}(X)$**.** Let $\eta \in A^1_{\mathrm{num}}(X)$ be the class of an ample divisor. Then, for any $k \leqslant d/2$, multiplication by η^{d-2k} induces an isomorphism from $A^k_{\mathrm{num}}(X)$ to $A^{d-k}_{\mathrm{num}}(X)$.

Conjecture $\mathrm{Hdg}(X)$**.** Let $\eta \in A^1_{\mathrm{num}}(X)$ be the class of an ample divisor. Then the bilinear form on $A^k_{\mathrm{num}}(X)$ given by $(x_1, x_2) \mapsto (-1)^k \deg(\eta^{d-2k} x_1 x_2)$ is positive definite on the kernel of multiplication by η^{d-2k+1}.

For a connected smooth projective variety X, let $\mathcal{K}_X$ be the cone in $A^1_{\mathrm{num}}(X)$ given by positive real linear combinations of classes of ample divisors, and let $\mathcal{Q}_X$ be the symmetric bilinear form on $A_{\mathrm{num}}(X)$ given by $(x_1, x_2) \mapsto \deg(x_1 x_2)$.

Proposition 2.8. Assume that $\mathrm{A}(X)$ and $\mathrm{Hdg}(X)$ hold for all connected smooth projective varieties. Let X be a connected smooth projective variety of dimension d. Then $(A_{\mathrm{num}}(X), \mathcal{Q}_X)$ is a Lefschetz module of degree d over $(A_{\mathrm{num}}(X), \mathcal{K}_X)$.

Proof. We induct on the dimension of X. Let η be an element of $\mathcal{K}_X$, and suppose $g \in A^k_{\mathrm{num}}(X)$ is a nonzero element of the kernel of multiplication by η^{d-2k} for some $k \leqslant d/2$. Write $\eta = \sum a_i [D_i]$, where the $[D_i]$ are the classes of smooth connected ample divisors and the a_i are positive real numbers. Then we have

$$0 = \mathcal{Q}_X(g, \eta^{d-2k} g) = \sum a_i \mathcal{Q}_X(g, [D_i]\eta^{d-2k-1} g).$$

Let ι_i^* denote the restriction map $A_{\mathrm{num}}(X) \to A_{\mathrm{num}}(D_i)$. By the projection formula,

$$\mathcal{Q}_X(g, [D_i]\eta^{d-2k-1} g) = \mathcal{Q}_{D_i}(\iota_i^*(g), \iota_i^*(\eta^{d-2k-1} g)).$$

We claim that $\iota_i^*(g)$ is nonzero. Indeed, A(X) implies that $[D_i]^{d-2k} g$ is nonzero, so there is some class $h_i \in A^k_{\mathrm{num}}(X)$ with

$$\mathcal{Q}_X(h_i, [D_i]^{d-2k} g) = \mathcal{Q}_{D_i}(\iota_i^*(h_i), \iota_i^*([D_i]^{d-2k-1})\iota_i^*(g))$$

nonzero, implying the claim. As $\iota_i^*(\eta^{d-2k} g) = 0$ and $A_{\mathrm{num}}(D_i)$ is a Lefschetz module by induction, (HR) implies that $(-1)^k \mathcal{Q}_X(g, [D_i]\eta^{d-2k-1} g) > 0$. Applying this for all i contradicts the fact that $\mathcal{Q}_X(g, \eta^{d-2k} g) = 0$.

We have verified (HL) for any $\eta \in \mathcal{K}_X$. This implies that, for any $\eta \in \mathcal{K}_X$ and $k \leqslant d/2$, the form on $A^k_{\mathrm{num}}(X)$ given by $(x_1, x_2) \mapsto \mathcal{Q}_X(x_1, \eta^{d-2k} x_2)$ is nondegenerate. In particular, the signature of this form does not change as we vary η within $\mathcal{K}_X$. As (HR) for η is equivalent to the signature of this form being $\sum_{i=0}^k (-1)^i (\dim A^i_{\mathrm{num}}(X) - \dim A^{i-1}_{\mathrm{num}}(X))$, that this holds when η is the class of an ample divisor implies the result. $\square$

That $A_{\mathrm{num}}(X)$ is a Lefschetz module is sufficient for many applications of the standard conjectures. For example, if X is a smooth projective variety over the algebraic closure of a finite field, then $A_{\mathrm{num}}(X \times X)$ being a Lefschetz module implies the Riemann hypothesis for X, see [Kle94].

Before applying our results to Chow rings modulo numerical equivalence, we will need a few results on Lefschetz modules which are rings. Let A be a graded $\mathbb{R}$-algebra equipped with an isomorphism $\deg\colon A^d \to \mathbb{R}$ and a nonempty open convex cone $\mathcal{K}_A$ in A^1. Let $\mathcal{Q}_A$ be the symmetric bilinear form defined by $(x_1, x_2) \mapsto \deg(x_1 x_2)$. Suppose that $(A, \mathcal{Q}_A)$ is a Lefschetz module of degree d over $(A, \mathcal{K}_A)$.

Lemma 2.9. Let $y_1, \ldots, y_e$ be elements of $\overline{\mathcal{K}}_A$, and suppose that the product $y_1 \cdots y_e$ is nonzero. Let $\eta \in \mathcal{K}_A$. Then $\deg(\eta^{d-e} y_1 \cdots y_e) > 0$.

Proof. By iteratively applying Lemma 4.6, $A/\operatorname{ann}(y_1 \cdots y_e)$, equipped with the bilinear form $\mathcal{Q}$ described in Section 4.3, is a Lefschetz module of degree $d-e$ over $(A, \mathcal{K}_A)$. Because $y_1 \cdots y_e \neq 0$, 1 is a nonzero element of $A/\operatorname{ann}(y_1 \cdots y_e)$. Then (HR), for $k = 0$, implies that $\mathcal{Q}(1, \eta^{d-e}) = \deg(\eta^{d-e} y_1 \cdots y_e)$ is positive. ☐

We will consider a graded subring B of A for which B^e is 1-dimensional, and $B^s = 0$ for $s > e$. Choose an isomorphism $\deg_B\colon B^e \to \mathbb{R}$, and let $\mathcal{Q}_B$ be the symmetric bilinear form given by $\mathcal{Q}_B(x_1, x_2) = \deg_B(x_1 x_2)$. We will assume that $\mathcal{Q}_B$ is nondegenerate; whether this happens is independent of the choice of isomorphism $\deg_B$.

Proposition 2.10. Suppose that B is generated as a ring by elements of $\overline{\mathcal{K}}_A$. Set $\mathcal{K}_B$ to be the interior of $B^1 \cap \overline{\mathcal{K}}_A$. Then there is a unique $\epsilon \in \{\pm 1\}$ such that $(B, \epsilon \mathcal{Q}_B)$ is a Lefschetz module of degree e over $(B, \mathcal{K}_B)$.

Proof. Because $\mathcal{K}_B$ is open in B^1, B is generated by elements $y_1, \ldots, y_s$ of $\mathcal{K}_B$. Let η be a class in $\mathcal{K}_A$, and set $\ell = y_1 + \cdots + y_s$. Then

$$\deg(\eta^{d-e} \ell^e) = \sum_{(i_1, \ldots, i_e)} \deg(\eta^{d-e} y_{i_1} \cdots y_{i_e}).$$

Each term in this sum either vanishes (if $y_{i_1} \cdots y_{i_e} = 0$) or is positive by Lemma 2.9. Because B is generated by $y_1, \ldots, y_s$, at least one term is positive. So we deduce that $\ell^e \neq 0$, and so $\ell^e \notin \operatorname{ann}(\eta^{d-e})$. Therefore $B^e \cap \operatorname{ann}(\eta^{d-e}) = 0$.

The image of B under the quotient map $A \to A/\operatorname{ann}(\eta^{d-e})$ is isomorphic to B. Indeed, the nondegeneracy of $\mathcal{Q}_B$ implies that every nonzero ideal in B intersects B^e. As $B^e \cap \operatorname{ann}(\eta^{d-e}) = 0$, this implies the claim that $B \cap \operatorname{ann}(\eta^{d-e}) = 0$. By Lemma 4.6, $A/\operatorname{ann}(\eta^{d-e})$ is a Lefschetz module over $(A, \mathcal{K}_A)$. We may replace A by $A/\operatorname{ann}(\eta^{d-e})$, and all the hypotheses are still satisfied. We may therefore assume that $d = e$.

Up to a nonzero constant, the restriction of $\mathcal{Q}_A$ to B is $\mathcal{Q}_B$, and so the restriction of $\mathcal{Q}_A$ to B is nondegenerate. This implies that B is a B-module summand of A, and it is clearly indecomposable as a B-module. The main results of this paper then show that $(B, \mathcal{Q}_A|_B)$ is a Lefschetz module over $(B, \mathcal{K}_B)$. We set ϵ to be the sign of the constant relating $\mathcal{Q}_A|_B$ to $\mathcal{Q}_B$. □

Theorem 2.11. Let $X \to Y$ be a map of connected smooth projective varieties, and assume that $(A_{\text{num}}(X), \mathcal{Q}_X)$ is a Lefschetz module of degree $\dim X$ over $(A_{\text{num}}(X), \mathcal{K}_X)$. Suppose that $A_{\text{num}}(Y)$ is generated as a ring in degree 1. Then $(A_{\text{num}}(Y), \mathcal{Q}_Y)$ is a Lefschetz module of degree $\dim Y$ over $(A_{\text{num}}(Y), \mathcal{K}_Y)$.

Proof. Because the ample cone of Y is open, $A^1_{\text{num}}(Y)$ is generated by $\mathcal{K}_Y$. Then the hypotheses of Proposition 2.10 hold, so it remains to check that the constant ϵ appearing there is 1. But this follows from the fact that if ℓ is an ample class on Y, then $\mathcal{Q}_Y(1, \ell^{\dim Y}) > 0$. □

2.4.2. *Lefschetz modules with Hodge structures.* We now discuss the behavior of the decomposition package with respect to Hodge structures.

Let A and M be as in Section 1, and set $M_{\mathbb{C}} = M \otimes_{\mathbb{R}} \mathbb{C}$. We say that M is a *graded A-module with a Hodge structure* if the following holds:

(HS1) Each graded piece $M^k_{\mathbb{C}}$ is equipped with a pure Hodge structure of weight k, and the elements of A^l act as morphisms of Hodge structures of bidegree (l, l), that is,

$$M^k_{\mathbb{C}} = \bigoplus_{p+q=k} M^{p,q}, \quad \overline{M^{p,q}} = M^{q,p}, \quad \text{and} \quad A^l M^{p,q} \subseteq M^{l+p,l+q}.$$

Now let A, M, $\mathcal{K}_A$ and $\mathcal{Q}$ be as in Section 1, and set $M_{\mathbb{C}} = M \otimes_{\mathbb{R}} \mathbb{C}$. We say that $(M, \mathcal{Q})$ is a *Lefschetz module endowed with a pure Hodge structure* of degree d, or simply a *Lefschetz module with Hodge structure*, if M is a graded A-module with a Hodge structure in the sense of (HS1) and, moreover, the following holds:

(HS2) For each integer n, let $M_n = \bigoplus_{k \geqslant 0} M^{k+n,k}$, and endow M_n with the Hermitian form

$$\mathcal{Q}_n \colon M_n \times M_n \longrightarrow \mathbb{C}, \quad \mathcal{Q}_n(\alpha, \beta) = \mathrm{i}^n \mathcal{Q}(\alpha, \bar{\beta}).$$

Then $(M_n, \mathcal{Q}_n)$ is a complex Lefschetz module of degree $d - n$ over $(A, \mathcal{K}_A)$.

The degrees in a Lefschetz module with Hodge structure are doubled compared to ordinary Lefschetz modules, e.g., we have $\dim M^i = \dim M^{2d-i}$ in a Lefschetz module with Hodge structure of degree d. Note that the category of finite dimensional graded A-modules with Hodge structures is an abelian category in which every object has finite length, so the Krull–Schmidt theorem applies to it.

Let N be a graded A-module with a Hodge structure, and consider the Hodge decomposition $N^k_{\mathbb{C}} = \bigoplus_{p+q} N^{p,q}$ of its graded pieces. For each integer n, set $N_n = \bigoplus_{k \geqslant 0} N^{k+n,k}$. Note that $\overline{N}_n = N_{-n}$. Moreover, N_0 and $N_n \oplus \overline{N}_n$ for $n > 0$ are defined over $\mathbb{R}$, i.e., there are graded

A-modules $L_0, L_1, \dots$ such that $L_{0,\mathbb{C}} \simeq N_0$ and $L_{n,\mathbb{C}} \simeq N_n \oplus \overline{N}_n$ for $n > 0$. Then each L_n is a graded A-module with a Hodge structure, and we have $N = \oplus_{n\geqslant 0} L_n$.

An object N is indecomposable in the category of finite dimensional graded A-modules with Hodge structures if and only if there is a unique $n \geqslant 0$ such that N_n is nonzero, and N_n is an indecomposable graded $A_{\mathbb{C}}$-module. If $n = 0$, this is equivalent to L_0 being an indecomposable graded A-module. If N is an indecomposable Lefschetz module with Hodge structure with $n = 0$, then we say that N is of *Hodge–Tate type*.

Let B and $\mathcal{K}_B$ as in Section 1. Choose a decomposition as in (1.1), where each N_α is indecomposable as a graded B-module with a Hodge structure. Let $N^k_{\alpha,\mathbb{C}} = \bigoplus_{p+q=k} N^{p,q}_\alpha$ be the Hodge decomposition of the graded piece $N^k_{\alpha,\mathbb{C}}$. For each α, one of the following happens: Either N_α is of Hodge–Tate type, in which case, N_α is an indecomposable B-module. Or, $N_{\alpha,\mathbb{C}} = N_{\alpha,n} \oplus \overline{N}_{\alpha,n}$ for a positive integer $n > 0$ with $N_{\alpha,n} = \bigoplus_{k\geqslant 0} N^{k+n,k}_\alpha$, and $N_{\alpha,n}$ is an indecomposable $B_{\mathbb{C}}$-module. Applying Theorems 1.1, 1.2, 1.3, and 1.4 and their complex analogues in Remark 2.1, we infer the existence of $\mathcal{Q}_\alpha$ and $\epsilon_\alpha \in \{\pm 1\}$ such that $(N_{\alpha,n}, \epsilon_\alpha \mathcal{Q}_n)$ is a Lefschetz module over $(B, \mathcal{K}_B)$ with Hodge structure.

We can define the perverse filtration on M, using the decomposition into indecomposable graded B-modules with Hodge structures, except taking into account the degree doubling. I.e., a summand $N_\alpha[-k]$, where N_α is a Lefschetz module with Hodge structure of degree $d(\alpha)$ and so is supported in degrees $0, 2, \dots, 2d(\alpha)$, first appears in $P_{d(\alpha)+k}$.

Theorem 2.12. Consider a decomposition of M into indecomposable B-modules with Hodge structure as in (1.1). Then for each α, $(N_\alpha, \epsilon_\alpha \mathcal{Q}_\alpha)$ is a Lefschetz module with Hodge structure over $(B, \mathcal{K}_B)$. Moreover, the perverse filtration is by Hodge substructures, and the graded pieces of the perverse filtration inherit pure Hodge structures. For each $\eta \in \mathcal{K}_{A/B}$ and $\ell \in \mathcal{K}_B$, the primitive pieces of Gr carry polarized Hodge structures.

Proof. We already discussed the proof of the first statement. The second assertion follows from this, in view of (1.3) and the definition of Gr. The last assertion follows from the version of Theorem 1.7 for complex Lefschetz modules. □

2.4.3. *The Hodge theory of the decomposition theorem.* Let X be a complex projective (integral) variety of dimension d. For a local system $\mathcal{L}$ on a Zariski open subset U of X, let $IC(U, \mathcal{L})$ denote the intersection cohomology perverse sheaf in $D^b_c(X)$, the bounded derived category of constructible sheaves on X. If $\mathcal{L}$ is a simple local system, then $IC(U, \mathcal{L})$ is a simple object in the category of perverse sheaves on X [BBD82, Theorem 4.3.1]. Let $IH(X, \mathcal{L}) = \mathcal{H}(IC(U, \mathcal{L})[-d])$ be the hypercohomology of $IC(U, \mathcal{L})$, shifted so that its support is contained in $\{0, \dots, 2d\}$.

Let $H(X)$ denote the subring of $H^\bullet(X; \mathbb{R})$ generated by the Chern classes of line bundles on X. Then $IH(X, \mathcal{L})$ is a graded module over $H(X)$. Let $\mathcal{K}(X)$ denote the open convex cone in $H^2(X)$ generated by the first Chern classes of ample line bundles.

Now let $f\colon X \to Y$ be a map from X to a projective variety Y, and assume that X is smooth. The decomposition theorem of Beilinson, Bernstein, Deligne, and Gabber [BBD82, Theorem 6.2.5] states that there is an isomorphism in $D^b_c(Y)$

$$f_*\mathbb{R}_X \simeq \bigoplus_i IC(Z_i, \mathcal{L}_i)[-d-e_i],$$

where $\mathbb{R}_X$ is the constant sheaf on X, the Z_i are smooth connected locally closed subvarieties of Y of dimension d_i, each $\mathcal{L}_i$ is a simple local system on Z_i, and the e_i are integers. Taking hypercohomology, for each k we obtain a direct sum decomposition

$$H^k(X;\mathbb{R}) \simeq \bigoplus_i IH^{k-d-e_i+d_i}(Z_i, \mathcal{L}_i). \tag{2.2}$$

Both $H^k(X;\mathbb{R})$ and each $IH(Z_i, \mathcal{L}_i)$ are graded $H(Y)$-modules, and the above isomorphism is an isomorphism of graded $H(Y)$-modules.

Because X is smooth, the main theorems of Hodge theory give $H^\bullet(X;\mathbb{R})$ the structure of a Lefschetz module with Hodge structure over $H(X)$, see, e.g., [Huy05, Chapter 3]. We will consider the decomposition of $H^\bullet(X;\mathbb{R})$ as a module over the image of $H(Y)$ in $H(X)$.

The perverse truncation functors on $D^b_c(Y)$ induce a filtration $P'_\bullet$ on $H^k(X;\mathbb{R})$, which we call the geometric perverse filtration. In terms of the chosen decomposition (2.2), this filtration is given by

$$P'_j = \bigoplus_{e_i \leqslant j} IH(Z_i, \mathcal{L}_i).$$

We also have a perverse filtration $P_\bullet$ on the Lefschetz module $H^\bullet(X;\mathbb{R})$, obtained by decomposing $H^\bullet(X;\mathbb{R})$ in the category of graded $H(Y)$-modules with Hodge structures.

Proposition 2.13. The geometric perverse filtration is equal to the perverse filtration on $H^\bullet(X;\mathbb{R})$.

Proof. Let $\ell \in H^2(Y)$ be an ample class. By [BBD82, Theorem 6.2.10], as ℓ restricts to an ample class on the closure of each Z_i, multiplication by ℓ^{d_i-k} induces an isomorphism from $IH^k(Z_i, \mathcal{L}_i)$ to $IH^{2d_i-k}(Z_i, \mathcal{L}_i)$ for each $k \geqslant 0$ and each i. In particular, refining the decomposition (2.2) into a decomposition of graded $\mathbb{R}[\ell]$-modules, we see that the summand corresponding to $IC(Z_i, \mathcal{L}_i)[-d-e_i]$ is contained in P_{e_i}, and it intersects P_{e_i-1} trivially. $\square$

In particular, this implies that the geometric perverse filtration on $H^\bullet(X;\mathbb{R})$ is "Hodge-theoretic." More precisely, the main theorems for Lefschetz modules with Hodge structures, especially the relative Hodge–Riemann relations, give the following.

Corollary 2.14. The associated graded pieces of the geometric perverse filtration are endowed with pure Hodge structures, and for any ample classes $\eta \in H(X)$ and $\ell \in H(Y)$, the primitive pieces $\mathrm{Prim}^{i,j} \coloneqq \ker(\eta^{d-j+1}\colon \mathrm{Gr}^{i,j} \to \mathrm{Gr}^{i+2d-2j+2,2d-j+2}) \cap \ker(\ell^{j-i+1}\colon \mathrm{Gr}^{i,j} \to \mathrm{Gr}^{2j-i+2,j})$ carry polarized Hodge structures.

Corollary 2.14 was first proved by Saito [Sai88, Sai90] using his theory of mixed Hodge modules, and a simpler proof was given by de Cataldo and Migliorini [dCM05]. The argument above deduces Corollary 2.14 as a formal consequence of the results of [BBD82].

We now explain how the main theorems can be used to put a polarized pure Hodge structure on the intersection cohomology of a complex projective variety. This was first accomplished by Saito [Sai88, Sai90]. Another proof was given by de Cataldo and Migliorini [dCM05], and our approach is similar to theirs.

Let Y be a complex projective variety of dimension d, and let $IH(Y)$ denote the intersection cohomology of Y with respect to the trivial local system. Let $H(Y)$ be the subring of $H^\bullet(Y;\mathbb{R})$ generated by the first Chern classes of line bundles, and let $\mathcal{K}_Y$ denote the open cone in $H^2(Y)$ generated by the first Chern classes of ample line bundles. Note that Verdier duality induces a nondegenerate $H(Y)$-invariant symmetric bilinear form $\mathcal{Q}$ on $IH(Y)$.

Proposition 2.15. There is a pure Hodge structure on $IH(Y)$ so that it is a Lefschetz module with Hodge structure over $(H(Y), \mathcal{K}_Y)$.

Proof. Let $f\colon X \to Y$ be a projective resolution of singularities. By the decomposition theorem [BBD82, Theorem 6.2.5], $IC(Y)$ is a summand of $f_*\underline{\mathbb{R}}_X[d]$. Decompose the Lefschetz module with Hodge structure $H^\bullet(X;\mathbb{R})$ over $H(Y)$. By Proposition 2.13, $IH(Y) \subseteq P_d$ and $IH(Y) \cap P_{d-1} = 0$, so $IH(Y)$ can be identified with a subspace of $\mathrm{Gr}^{\bullet,d}$.

Let ${}^{\mathfrak{p}}\mathcal{H}^k(f_*\underline{\mathbb{R}}[d])$ denote the kth perverse cohomology of $f_*\underline{\mathbb{R}}[d]$, so $IC(Y)$ is a summand of ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\underline{\mathbb{R}}[d])$. Because ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\underline{\mathbb{R}}[d])$ is a semisimple perverse sheaf, it has a canonical decomposition into "isotypic components." Because f is birational, there is a unique summand of ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\underline{\mathbb{R}}[d])$ which is isomorphic to $IC(Y)$. As the hypercohomology of ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\underline{\mathbb{R}}[d])$ is identified with $\mathrm{Gr}^{\bullet,d}$, this identification of $IH(Y)$ with a subspace of $\mathrm{Gr}^{\bullet,d}$ is canonical.

Let η be an ample class on X. Multiplication by η induces a map from ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\underline{\mathbb{R}}[d])$ to ${}^{\mathfrak{p}}\mathcal{H}^2(f_*\underline{\mathbb{R}}[d])$. Because f is birational, ${}^{\mathfrak{p}}\mathcal{H}^2(f_*\underline{\mathbb{R}}[d])$ does not contain any summand isomorphic to $IC(Y)$. Because $IC(Y)$ is a simple perverse sheaf, it is therefore killed by multiplication by η. This implies that $IH(Y)$ is contained in $\ker\eta \subset \mathrm{Gr}^{\bullet,d}$.

By Corollary 2.14, this identifies $IH(Y)$ with a summand of a Lefschetz module with Hodge structure over $H(Y)$. By a variant of Lemma 6.3 for Lefschetz modules with Hodge structures, it suffices to prove that $IH(Y)$ is a Hodge substructure of $\mathrm{Gr}^{\bullet,d}$. If $H^\bullet(X;\mathbb{R})$ is of Hodge–Tate type, then this is automatic. In general, this is accomplished by a geometric argument of de Cataldo and Migliorini [dCM05, Proof of Theorem 2.2.1], which we now sketch.

One proves the following more general statement: let $g\colon W \to Z$ be a map between projective varieties, with W smooth of dimension d. For each k, there is a canonical isomorphism ${}^{\mathfrak{p}}\mathcal{H}^k(g_*\underline{\mathbb{R}}_W[d]) \simeq \bigoplus_i IC(Z_i, \mathcal{L}_i)$, where each $\mathcal{L}_i$ is a semisimple local system and $\overline{Z}_i \neq \overline{Z}_j$ for $i \neq j$. This canonically identifies $IH(Z_i, \mathcal{L}_i)$ with a subspace of $\mathrm{Gr}^{\bullet,d+k}$, which is the hypercohomology of ${}^{\mathfrak{p}}\mathcal{H}^k(g_*\underline{\mathbb{R}}_W[d])$. We claim that, for each k and each i, $IH(Z_i, \mathcal{L}_i)$ is a Hodge substructure of $\mathrm{Gr}^{\bullet,d+k}$. As $IC(Y)$ is the unique summand which is supported on all of Y, this implies the result.

We induct on $\dim W$. By slicing by an ample divisor on either W or Z and using a version of Proposition 4.10 or Proposition 4.12 for Lefschetz modules with Hodge structures, we reduce to proving this general statement for $\mathrm{Gr}^{d,d}$.

Let Z_1 be the dense stratum in Z. Because the $IC(Z_i, \mathcal{L}_i)$ are semisimple perverse sheaves, the subspaces $IH^{\dim Z_i}(Z_i, \mathcal{L}_i)$ of $\mathrm{Gr}^{d,d}$ are orthogonal to each other with respect to the bilinear form $\mathcal{Q}$. As $\mathcal{Q}$ is compatible with the Hodge structures, the orthogonal complement of a Hodge substructure is a Hodge substructure, so it suffices to show that, for each $j \neq 1$, $\oplus_{i \neq j} IH^{\dim Z_i}(Z_i, \mathcal{L}_i)$ is a Hodge substructure of $\mathrm{Gr}^{d,d}$.

Choose some $j \neq 1$, and let V be a resolution of singularities of $\overline{g^{-1}(Z_j)}$. The map from V to Z induces a perverse filtration on $H^\bullet(V;\mathbb{R})$, and there is a pullback map $\mathrm{Gr}^{d,d} H^\bullet(W;\mathbb{R}) \to \mathrm{Gr}^{d,d} H^\bullet(V;\mathbb{R})$. This is a map of Hodge structures, and $IH^{\dim Z_i}(Z_i, \mathcal{L}_i)$ is in the kernel for each $i \neq j$. The key computation is that the restriction of this map to $IH^{\dim Z_j}(Z_j, \mathcal{L}_j)$ is injective. This holds because $IC(Z_j, \mathcal{L}_j)$ is a summand of the pushforward of the appropriately shifted constant sheaf on V to Z. $\square$

3. Examples

By Theorem 1.1, the endomorphism ring of an indecomposable Lefschetz module is a finite dimensional division algebra over $\mathbb{R}$. We begin with two examples illustrating that both the field of complex numbers $\mathbb{C}$ and the quaternions $\mathbb{H}$ can occur.

Example 3.1. Let M^0 be the 4-dimensional real vector space with basis e_1, e_2, e_3, e_4, and let M^1 be the 4-dimensional real vector space with basis $e_1^*, e_2^*, e_3^*, e_4^*$. Set $M \coloneqq M^0 \oplus M^1$, and let $\mathcal{Q}$ be the symmetric bilinear form defined by

$$\mathcal{Q}(e_i, e_j^*) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if otherwise.} \end{cases}$$

A linear map $M^0 \to M^1$ is given by a 4×4 matrix, and $\mathcal{Q}$ is invariant under this map if and only if this matrix is symmetric. Set $A^0 \coloneqq \mathbb{R}$ and

$$A^1 \coloneqq \left\{ \begin{pmatrix} X & dY \\ dY^t & X \end{pmatrix} \middle| \, X = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \, Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \, a, b, c, d \in \mathbb{R} \right\}.$$

The direct sum $A \coloneqq A^0 \oplus A^1$ has the structure of a graded algebra over $\mathbb{R}$. Note that M is an A-module and that $\mathcal{Q}$ is an A-invariant symmetric bilinear form on M. Setting $\mathcal{K}_A$ to be the intersection of A^1 with the cone of positive definite matrices gives $(M, \mathcal{Q})$ the structure of a Lefschetz module of degree 1 over $(A, \mathcal{K}_A)$.

The ring of graded A-module endomorphisms of M is identified with the space of matrices that commute with all matrices in A^1. It is easy to check that this space of matrices is spanned by the identity and the matrix

$$J := \begin{pmatrix} 0_2 & I_2 \\ -I_2 & 0_2 \end{pmatrix}, \quad \text{where } 0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

As the algebra spanned by the identity and J is isomorphic to the complex numbers $\mathbb{C}$, we see that M is indecomposable and has endomorphism ring isomorphic to $\mathbb{C}$. $\diamond$

Example 3.2. Let M^0 be the 8-dimensional real vector space with basis $e_1, e_2, \ldots, e_8$, and let M^1 be the 8-dimensional real vector space with basis $e_1^*, e_2^*, \ldots, e_8^*$. Set $M := M^0 \oplus M^1$, and let $\mathcal{Q}$ be the symmetric bilinear form defined by

$$\mathcal{Q}(e_i, e_j^*) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if otherwise.} \end{cases}$$

A linear map $M^0 \to M^1$ is given by an 8×8 matrix, and $\mathcal{Q}$ is invariant under this map if and only if this matrix is symmetric. Set $A^0 := \mathbb{R}$ and

$$A^1 := \left\{ \begin{pmatrix} X & eY & dY & fY \\ eY^t & X & fY & dY^t \\ dY^t & fY^t & X & eY \\ fY^t & dY & eY^t & X \end{pmatrix} \middle| \; X = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; a, b, c, d, e, f \in \mathbb{R} \right\}.$$

The direct sum $A := A^0 \oplus A^1$ has the structure of a graded algebra over $\mathbb{R}$. Let $\mathcal{K}_A$ be the intersection of A^1 with the cone of positive definite matrices. Note that $(M, \mathcal{Q})$ is a Lefschetz module of degree 1 over $(A, \mathcal{K}_A)$. A lengthy computation shows that the space of matrices that commute with A^1 is spanned by the identity, J, K, and JK, where

$$J := \begin{pmatrix} 0_2 & 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & 0_2 & I_2 \\ -I_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & -I_2 & 0_2 & 0_2 \end{pmatrix}, \text{ and } K := \begin{pmatrix} 0_2 & 0_2 & 0_2 & I_2 \\ 0_2 & 0_2 & -I_2 & 0_2 \\ 0_2 & I_2 & 0_2 & 0_2 \\ -I_2 & 0_2 & 0_2 & 0_2 \end{pmatrix}.$$

As the algebra spanned by the identity, J, K, and JK is isomorphic to the quaternions $\mathbb{H}$, we see that M is indecomposable and has endomorphism ring isomorphic to $\mathbb{H}$. $\diamond$

The next example shows that the conclusion of Theorem 1.2 may fail if one works with coefficients in $\mathbb{Q}$ instead of $\mathbb{R}$.

Example 3.3. Let M^0 be the 2-dimensional rational vector space with basis e_1, e_2, and let M^1 be the 2-dimensional rational vector space with basis e_1^*, e_2^*. Set $M := M^0 \oplus M^1$, and let $\mathcal{Q}$ be the symmetric bilinear form defined by

$$\mathcal{Q}(e_i, e_j^*) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if otherwise.} \end{cases}$$

A linear map $M^0 \to M^1$ is given by a 2×2 matrix, and $\mathcal{Q}$ is invariant under this map if and only if this matrix is symmetric. Set $A^0 \coloneqq \mathbb{Q}$ and

$$A^1 \coloneqq \left\{ \begin{pmatrix} a & b \\ b & a + 2b \end{pmatrix} \,\middle|\, a, b \in \mathbb{Q} \right\}.$$

The direct sum $A \coloneqq A^0 \oplus A^1$ has the structure of a graded algebra over $\mathbb{Q}$. Note that M is an A-module and that $\mathcal{Q}$ is an A-invariant symmetric bilinear form on M. Let $\mathcal{K}_A$ be the intersection of $A^1 \otimes_{\mathbb{Q}} \mathbb{R}$ with the cone of positive definite matrices. Then $(M \otimes_{\mathbb{Q}} \mathbb{R}, \mathcal{Q})$ has the structure of a Lefschetz module of degree 1 over $(A \otimes_{\mathbb{Q}} \mathbb{R}, \mathcal{K}_A)$.

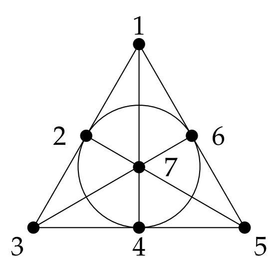
The ring of graded A-module endomorphisms of M is identified with the space of matrices that commute with all matrices in A^1. It is easy to check that this space of matrices is spanned by the identity and the matrix

$$D \coloneqq \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The ring of graded endomorphisms of M is isomorphic to $\mathbb{Q}[\sqrt{2}]$, and hence M is an indecomposable graded A-module, but there is an extra A-invariant symmetric bilinear form given by $(x, y) \mapsto \mathcal{Q}(Dx, y)$. Note that $M \otimes_{\mathbb{Q}} \mathbb{R}$ is decomposable over $A \otimes_{\mathbb{Q}} \mathbb{R}$, and each of its indecomposable summands has an essentially unique nonzero A-invariant symmetric bilinear form. $\diamond$

We give examples of the decomposition package.

Example 3.4. The Fano matroid F_7 is the matroid on seven elements whose bases are the three-element subsets that are not collinear in the following picture of the Fano plane:



Let A be the graded Möbius algebra of F_7, and let $\mathcal{K}_A$ be the set of positive linear combinations of the generators $y_1, \ldots, y_7$. One can check that the symmetric bilinear pairing on A given by the multiplication gives A the structure of a Lefschetz module over $(A, \mathcal{K}_A)$. This is a special case of the statement that the intersection cohomology module of a matroid is a Lefschetz module over the graded Möbius algebra [BHM+b].

Let B be the subalgebra of A generated by the elements y_1, y_3, y_5, y_7. There are infinitely many different decompositions of $M \coloneqq A$ into indecomposable graded B-modules, each of which has three summands. One of the three summands is common to all the decompositions, the one-dimensional summand spanned by

$$\eta \coloneqq y_1 - y_2 + y_3 - y_4 + y_5 - y_6 + y_7.$$

For example, $M = S_1 \oplus S_2 \oplus S_3$, where S_1, S_2, S_3 are graded B-submodules of M given by

$$\begin{aligned}
&S_1^3 = \mathrm{span}(y_{12334567}), && S_2^3 = 0, && S_3^3 = 0,\\
&S_1^2 = \mathrm{span}(y_{123}, y_{147}, y_{156}, y_{257}, y_{345}, y_{367}), && S_2^2 = 0, && S_3^2 = \mathrm{span}(y_2\eta),\\
&S_1^1 = \mathrm{span}(y_1, y_3, y_5, y_7, y_2, y_4 - y_6), && S_2^1 = \mathrm{span}(\eta), && S_3^1 = 0,\\
&S_1^0 = \mathrm{span}(y_\varnothing), && S_2^0 = 0, && S_3^0 = 0.
\end{aligned}$$

We also have $M = T_1 \oplus T_2 \oplus T_3$, where T_1, T_2, T_3 are graded B-submodules of M given by

$$\begin{aligned}
&T_1^3 = \mathrm{span}(y_{12334567}), && T_2^3 = 0, && T_3^3 = 0,\\
&T_1^2 = \mathrm{span}(y_{123}, y_{147}, y_{156}, y_{257}, y_{345}, y_{367}), && T_2^2 = 0, && T_3^2 = \mathrm{span}(y_4\eta),\\
&T_1^1 = \mathrm{span}(y_1, y_3, y_5, y_7, y_4, y_2 - y_6), && T_2^1 = \mathrm{span}(\eta), && T_3^1 = 0,\\
&T_1^0 = \mathrm{span}(y_\varnothing), && T_2^0 = 0, && T_3^0 = 0.
\end{aligned}$$

Note that $S_1 \simeq T_1$ and $S_3 \simeq T_3$ as graded B-modules, and $S_2 = T_2$. One implication of Theorems 1.2 and 1.4 is that, for any decomposition of M into indecomposable graded B-modules

$$M = N_1 \oplus N_2 \oplus N_3 \ \text{ with } \ \dim N_2 = \dim N_3 = 1,$$

the restriction of the Poincaré pairing $\mathcal{Q}$ on M to N_1 is nondegenerate, and $(N_1, \mathcal{Q})$ satisfies the Hard Lefschetz property and the Hodge–Riemann relations with respect to $(B, \mathcal{K}_B)$. ⋄

Example 3.5. Let A be the graded Möbius algebra of F_7 as above, and let B be the subalgebra of A generated by the elements y_1, $y_3 + y_5$, and $y_2 + y_4 + y_6 + y_7$. In this case, $M \coloneqq A$ decomposes uniquely[5] into indecomposable graded B-modules $N_1 \oplus N_2$, where

$$\begin{aligned}
&N_1^3 = \mathrm{span}(y_{12334567}), && N_2^3 = 0,\\
&N_1^2 = \mathrm{span}(y_{147}, y_{246}, y_{345}, y_{123} + y_{156}, y_{257} + y_{367}), && N_2^2 = \mathrm{span}(y_{123} - y_{156}, y_{257} - y_{367}),\\
&N_1^1 = \mathrm{span}(y_1, y_4, y_7, y_3 + y_5, y_2 + y_6), && N_2^1 = \mathrm{span}(y_3 - y_5, y_2 - y_6),\\
&N_1^0 = \mathrm{span}(y_\varnothing), && N_2^0 = 0.
\end{aligned}$$

That N_2 is an indecomposable graded B-module follows from the property that

$$N_2^2 = \mathrm{span}(y_1, y_3 + y_5, y_2 + y_4 + y_6 + y_7) \cdot \xi \ \text{ for any nonzero } \xi \in N_2^1.$$

That N_2 satisfies the Hodge–Riemann relations with respect to $\mathcal{K}_B$ is equivalent to the statement

$$\begin{pmatrix} a + 3c & a - c \\ a - c & a + 2b + c \end{pmatrix} \text{ is positive definite for any } a, b, c > 0.$$

[5]In general, a decomposition of M into indecomposable graded B-modules is unique up to isomorphism of its summands, but not necessarily uniquely determined, as illustrated in Example 3.4. In the present case, however, the uniqueness follows from the fact that every summand shares the same middle degree, or, equivalently, that there is an element of $\mathcal{K}_B$ satisfying the Hard Lefschetz theorem on M, and so the perverse filtration is trivial.

In the previous example, each indecomposable Lefschetz module appearing in the decomposition of M had a lowest-degree nonzero homogeneous component of dimension 1. In the current example, we see that indecomposable Lefschetz modules with lowest-degree components of dimension 2 may appear in the decomposition of M. $\diamond$

While the main examples of Lefschetz modules come from algebraic geometry, there are Lefschetz modules which provably do not appear inside the cohomology of a compact Kähler manifold or a smooth projective variety.

Example 3.6. Consider the cubic polynomial in three variables

$$f(w_1, w_2, w_3) = 14w_1^3 + 6w_1^2 w_2 + 24w_1^2 w_3 + 12w_1 w_2 w_3 + 6w_1 w_3^2 + 3w_2 w_3^2,$$

and let A be the graded algebra over $\mathbb{R}$ cogenerated by f. In other words, A is the unique quotient of $\mathbb{R}[x_1, x_2, x_3]$ equipped with a map $\deg\colon A^3 \to \mathbb{R}$ such that the bilinear maps $A^0 \times A^3 \to \mathbb{R}$ and $A^1 \times A^2 \to \mathbb{R}$ are nondegenerate and

$$\deg((w_1 x_1 + w_2 x_2 + w_3 x_3)^3) = f(w_1, w_2, w_3).$$

Using that f is a *Lorentzian polynomial* in the sense of [BH20], one can check that A is a Lefschetz module over $(A, \mathbb{R}^3_{>0})$. However, A is not a subquotient of the ring of real (p,p) forms on a 3-dimensional compact Kähler manifold nor of the Chow ring modulo numerical equivalence of a 3-dimensional smooth projective variety over an algebraically closed field because f does not satisfy the *reverse Khovanskii–Tessier inequality* of [LX17, JL23], see [Huh23, Example 14]. $\diamond$

4. The perverse filtration

In this section, we establish some fundamental properties of the perverse filtration $0 \subseteq P_0 \subseteq \cdots \subseteq P_{2d} = M$. As mentioned in the introduction, Theorem 1.3 implies that the perverse filtration can be defined using any B-module decomposition of M. However, we will not be able to prove this until after establishing the main theorems.

Choose some $\ell \in \mathcal{K}_B$. For $e \geqslant 0$, let $C_e = \mathbb{R}[\ell]/(\ell^{e+1})$ be the cyclic graded $\mathbb{R}[\ell]$-module which is generated by $1 \in C_e^0$. Fix a decomposition of M into indecomposable $\mathbb{R}[\ell]$-modules:

$$M \simeq \bigoplus_{e,k} C_e[-k]^{\oplus m(e,k)}. \tag{4.1}$$

Recall that P_j is spanned by the summands $C_e[-k]^{\oplus m(e,k)}$ with $e + 2k \leqslant j$. From the alternative description of the perverse filtration in (1.2), we see that the perverse filtration is independent of the choice of $\mathbb{R}[\ell]$-module decomposition.

4.1. Module structures.

The following lemma endows Gr with a natural A-module structure, with multiplication denoted by $*\colon A \times \mathrm{Gr} \to \mathrm{Gr}$ such that $A^s * \mathrm{Gr}^{i,j} \subseteq \mathrm{Gr}^{i+s,j+2s}$.

Lemma 4.1. For each s and j, $A^s P_j \subseteq P_{j+2s}$.

Proof. By (1.2), $P_j \cap M^k$ is spanned by elements of the form $\ell^{k-c}m$, where $m \in M^c$ is an element which satisfies $\ell^{j+1-k-c}m = 0$. Then for any $a \in A^s$, the element

$$a \cdot \ell^{k-c}m = \ell^{s+k-(s+c)}a \cdot m$$

lies in $\ell^{(s+k)-(s+c)}M \cap \operatorname{ann}_M(\ell^{j+2s+1-(s+k)-(s+c)}) \cap M^{s+k}$, which is contained in $P_{j+2s} \cap M^{s+k}$ by (1.2), as required. ☐

We now show that $\mathrm{Gr}^{\bullet,j}$ is a B-module for each j. The following lemma is automatic from the description of the perverse filtration in terms of the chosen $\mathbb{R}[\ell]$-module decomposition.

Lemma 4.2. For each j, we have $\ell P_j \subseteq P_j$.

We will need the following result, which was proved in the setting of variations of polarized Hodge structures in [CK82, Theorem 3.3]. By [Cat08, Theorem 3.1], this implies the following result for Lefschetz modules.

Proposition 4.3. The perverse filtration is independent of the choice of $\ell \in \mathcal{K}_B$.

Proof. By [CK82, Theorem 3.3] and [Cat08, Theorem 3.1], the weight filtration $W_{-d} \subseteq W_{-d+1} \subseteq \cdots \subseteq W_{d-1} \subseteq W_d = M$ associated to ℓ is independent of the choice of $\ell \in \mathcal{K}_B$. The statement now follows by observing that $P_j = \sum_{i+2k \leqslant j} W_i \cap M^k$. ☐

As $\mathcal{K}_B$ is open, this implies that the action of B^1 preserves the perverse filtration. Because B is generated in degree 1, we deduce the following, which gives each $\mathrm{Gr}^{\bullet,j}$ the structure of a graded B-module.

Corollary 4.4. For any $b \in B$ and any j, $bP_j \subseteq P_j$.

From the description of the perverse filtration in terms of the chosen $\mathbb{R}[\ell]$-module decomposition, we have an identification

$$\mathrm{Gr}^{\bullet,j} \simeq \bigoplus_{e+2k=j} C_e[-k]^{\oplus m(e,k)}.$$

In particular, we have the following corollary.

Corollary 4.5. For any $\ell \in \mathcal{K}_B$, each $0 \leqslant j \leqslant 2d$, and each $i \leqslant j/2$, the map $\ell^{j-2i}\colon \mathrm{Gr}^{i,j} \to \mathrm{Gr}^{j-i,j}$ is an isomorphism.

4.2. **The 1-dimensional summands.** We give a concrete description of a submodule of Gr, which we will make use of later. Let

$$N = \bigoplus_k C_0[-k]^{\oplus m(0,k)} \subseteq M$$

be the direct sum of the 1-dimensional $\mathbb{R}[\ell]$-module summands in the chosen $\mathbb{R}[\ell]$-module decomposition of M. The restriction of the perverse filtration to N coincides with the filtration induced by the grading on M, and so N is identified with its image in Gr. Then N admits the following simple description:

$$N = \frac{\mathrm{ann}_M(\ell)}{\ell M \cap \mathrm{ann}_M(\ell)}\,.$$

In fact, N coincides with the kernel of the map $\mathrm{Gr}^{\bullet,2\bullet} \to \mathrm{Gr}^{\bullet+1,2\bullet}$ induced by multiplication by ℓ. It is therefore an A-submodule of Gr, and the above is an equality of A-modules.

4.3. **The descent lemma.** Given $a \in A^k$, we consider the A-module $M_a \coloneqq M/\mathrm{ann}_M(a)$. Let $\varphi\colon M \to M_a$ be the quotient map. This A-module is equipped with an A-invariant bilinear form $\mathcal{Q}_a\colon M_a \times M_a \to \mathbb{R}$ which is defined by the property

$$\mathcal{Q}_a(\varphi(x), \varphi(y)) = \mathcal{Q}(x, ay) \qquad \forall\, x, y \in M.$$

In particular, $\mathcal{Q}_a\colon M_a^i \times M_a^j \to \mathbb{R}$ vanishes unless $i + j = d - k$. By construction, $\mathcal{Q}_a\colon M_a^i \times M_a^{d-k-i} \to \mathbb{R}$ is nondegenerate. The next result from [CKS87, Lemma 1.16] and [Cat08, Theorem 3.2], which is known as the *descent lemma*, will be crucial in what follows. It will allow us to prove statements by induction on the degree of the Lefschetz module.

Lemma 4.6. Let γ be an element in the closure of $\mathcal{K}_A$. Then $M_\gamma = M/\mathrm{ann}_M(\gamma)$ equipped with the bilinear form $\mathcal{Q}_\gamma$ described above is a Lefschetz module of degree $d-1$ over $(A, \mathcal{K}_A)$.

We will frequently consider $M_{\ell^p} = M/\mathrm{ann}(\ell^p)$ for a nonnegative integer p. By repeatedly applying Lemma 4.6, we deduce that M_{ℓ^p}, equipped with the bilinear form $\mathcal{Q}_{\ell^p}$, is a Lefschetz module of degree $d-p$ over $(A, \mathcal{K}_A)$.

The chosen decomposition (4.1) of M into indecomposable $\mathbb{R}[\ell]$-modules induces a decomposition of M_{ℓ^p} into indecomposable $\mathbb{R}[\ell]$-modules: we have

$$M_{\ell^p} \simeq \bigoplus_{\substack{e,k \\ e \geqslant p}} C_{e-p}[-k]^{\oplus m(e,k)}. \tag{4.2}$$

4.4. **The bilinear form on** Gr**.** We construct a nondegenerate symmetric bilinear form on Gr.

Lemma 4.7. For all i and j, we have

$$\mathcal{Q}(M^i \cap P_j, M^{d-i} \cap P_{2d-j-1}) = 0.$$

Proof. Note that $M^i \cap P_j$ is spanned by elements of the form $\ell^{i-c}m$, where $m \in M^c$ with $\ell^{e+1}m = 0$ for some e with $e + 2c \leqslant j$. Similarly, $M^{d-i} \cap P_{2d-j-1}$ is spanned by elements of the form $\ell^{d-i-c'}m'$ with $\ell^{e'+1}m' = 0$ for some e' with $e' + 2c' \leqslant 2d - j - 1$. Then

$$\mathcal{Q}(\ell^{i-c}m, \ell^{d-i-c'}m') = \mathcal{Q}(\ell^{d-c-c'}m, m') = \mathcal{Q}(m, \ell^{d-c-c'}m').$$

This vanishes if either $e+1 \leqslant d-c-c'$ or $e'+1 \leqslant d-c-c'$. Adding the two equations $e+2c \leqslant j$ and $e' + 2c' \leqslant 2d - j - 1$ implies that one of these must hold. $\square$

Lemma 4.7 implies that $\mathcal{Q}$ descends to a symmetric bilinear form

$$\underline{\mathcal{Q}}\colon \operatorname{Gr}^{i,j} \times \operatorname{Gr}^{d-i,2d-j} \longrightarrow \mathbb{R}.$$

We extend $\underline{\mathcal{Q}}$ to a map $\operatorname{Gr} \times \operatorname{Gr} \to \mathbb{R}$ by setting $\underline{\mathcal{Q}}(\operatorname{Gr}^{i,j} \times \operatorname{Gr}^{i',j'}) = 0$ unless $i + i' = d$ and $j + j' = 2d$.

Recall from Section 4.2 that N is the subspace of M spanned by the 1-dimensional summands in the chosen $\mathbb{R}[\ell]$-module decomposition, and that it can be identified with $\operatorname{ann}_M(\ell)/(\ell M \cap \operatorname{ann}_M(\ell))$, which is naturally a subspace of Gr. Since the perverse filtration on N coincides with the filtration induced by the grading, the restriction of the forms $\mathcal{Q}$ and $\underline{\mathcal{Q}}$ to N can be identified.

Lemma 4.8. The restriction of $\mathcal{Q}$ to N is nondegenerate.

Proof. We need to show that each element $m \in \operatorname{ann}_M(\ell)$ which has $\mathcal{Q}(m, n) = 0$ for all $n \in \operatorname{ann}_M(\ell)$ lies in ℓM. Note that $\operatorname{ann}_M(\ell)$ is the orthogonal complement of ℓM with respect to $\mathcal{Q}$. Then (PD) implies that ℓM is the orthogonal complement of $\operatorname{ann}_M(\ell)$. □

In particular, we deduce that $\dim N^i = \dim N^{d-i}$. Recall that $m(e,k)$ is the multiplicity of $C_e[-k]$ in the $\mathbb{R}[\ell]$-module decomposition of M.

Proposition 4.9. The form $\underline{\mathcal{Q}}$ is nondegenerate.

Proof. We first show that $\dim \operatorname{Gr}^{i,j} = \dim \operatorname{Gr}^{d-i,2d-j}$ for all i and j. It follows from Lemma 4.8 that $m(0,k) = m(0, d-k)$. By (4.2), $m(e,k)$ is equal to the multiplicity of $C_0[-k]$ in M_{ℓ^e}. Using Lemma 4.6 and applying Lemma 4.8 to M_{ℓ^e}, we deduce that $m(e,k) = m(e, d-e-k)$. This implies that $\dim \operatorname{Gr}^{i,j} = \dim \operatorname{Gr}^{d-i,2d-j}$.

Choose a basis for M^k which is compatible with the perverse filtration on M^k. Lemma 4.7 and the above equality of dimensions implies that the matrix representing $\mathcal{Q}$ with respect to this basis is triangular. As $\mathcal{Q}$ is nondegenerate, this implies that $\underline{\mathcal{Q}}$ is nondegenerate. □

4.5. **Applications of the descent lemma.** For $\gamma \in \overline{\mathcal{K}}_A$, let $P_{\gamma,\bullet}$ be the perverse filtration on M_γ, defined using our chosen element $\ell \in \mathcal{K}_B$. Let Gr_γ be the associated graded of M_γ with respect to the perverse filtration. Let $\psi\colon M \to M_\ell$ be the quotient map associated to ℓ.

Proposition 4.10. The quotient map $\psi\colon M \to M_\ell$ satisfies $\psi(P_j) = P_{\ell,j-1}$ for all j. The induced map $\operatorname{Gr}\psi\colon \operatorname{Gr}^{i,j} \to \operatorname{Gr}_\ell^{i,j-1}$ is an isomorphism if $i < j/2$.

Proof. If $M \simeq \bigoplus_{e,k} C_e[-k]^{\oplus m(e,k)}$ is the fixed $\mathbb{R}[\ell]$-module decomposition of M, then $\psi(C_e[-k]) = C_{e-1}[-k]$, where C_{-1} is interpreted as 0. By (4.2),

$$M_\ell \simeq \bigoplus_{e,k} C_{e-1}[-k]^{\oplus m(e,k)}$$

is an $\mathbb{R}[\ell]$-module decomposition of M_ℓ. As $C_e[-k]$ lies in P_{e+2k} and $C_{e-1}[-k]$ lies in $P_{\ell,e+2k-1}$, this implies that $\psi(P_j) = P_{\ell,j-1}$.

In terms of the chosen decomposition, we have

$$\mathrm{Gr}^{\bullet,j} = \bigoplus_{e+2k=j} C_e[-k]^{\oplus m(e,k)} \quad \text{and} \quad \mathrm{Gr}_\ell^{\bullet,j-1} = \bigoplus_{e+2k=j} \psi(C_e[-k]^{\oplus m(e,k)}).$$

The restriction of ψ to $C_e[-k]$ is an isomorphism in degree i if $i < e+k$. Since $j/2 = e/2+k$, we see that $i < e+k$, giving the result. $\square$

From the definition of the forms $\underline{\mathcal{Q}}$ on Gr and $\underline{\mathcal{Q}}_\ell$ on Gr_ℓ, obtained in terms of lifting to M and M_ℓ, we get the following compatibility.

Corollary 4.11. For all $x \in \mathrm{Gr}^{i,j}$ and $y \in \mathrm{Gr}^{d-i-1,2d-j}$, we have

$$\underline{\mathcal{Q}}_\ell(\mathrm{Gr}\,\psi(x), \mathrm{Gr}\,\psi(y)) = \underline{\mathcal{Q}}(x, \ell y).$$

Choose some $\eta \in \mathcal{K}_A$, and let $M_\eta = M/\operatorname{ann}_M(\eta)$. Let $\varphi\colon M \to M_\eta$ be the quotient map. Note that (HL) implies that φ is an isomorphism in degree less than $d/2$.

Proposition 4.12. The map $\varphi\colon M \to M_\eta$ satisfies $\varphi(P_j) \subseteq P_{\eta,j}$. The induced map $\mathrm{Gr}\,\varphi\colon \mathrm{Gr}^{i,j} \to \mathrm{Gr}_\eta^{i,j}$ is injective if $j < d$.

Proof. That $\varphi(P_j) \subseteq P_{\eta,j}$ follows from the description of the perverse filtration in (1.2). Choose some $\ell \in \mathcal{K}_B$, and note that Corollary 4.5 implies that $\mathrm{Gr}^{\bullet,j}$ and $\mathrm{Gr}_\eta^{\bullet,j}$ are representations of $\mathfrak{sl}_2$, where ℓ acts as a raising operator. Furthermore, the map $\mathrm{Gr}\,\varphi$ is a map of representations of $\mathfrak{sl}_2$. In particular, in order to check that it is injective for $j < d$, it suffices to check that it is injective when restricted to $\ker(\ell^{j-2i+1}\colon \mathrm{Gr}^{i,j} \to \mathrm{Gr}^{j-i+1,j})$ for each $i \leqslant j/2$.

If $i < j/2$, then, using Proposition 4.10, we may replace M by $M/\operatorname{ann}_M(\ell^{j-2i})$; by Lemma 4.6, this is still a Lefschetz module. So we may assume that $j = 2i$. By Section 4.2, we have an identification of A-modules

$$\bigoplus_i \ker(\ell\colon \mathrm{Gr}^{i,2i} \to \mathrm{Gr}^{i+1,2i}) = \frac{\operatorname{ann}_M(\ell)}{\ell M \cap \operatorname{ann}_M(\ell)}\,.$$

Suppose we have $x \in \ker(\ell\colon \mathrm{Gr}^{i,2i} \to \mathrm{Gr}^{i+1,2i})$, with $i < d/2$. Then $\mathrm{Gr}\,\varphi(x) = 0$ if and only if x is in the kernel of the map

$$\frac{\operatorname{ann}_M(\ell)}{\ell M \cap \operatorname{ann}_M(\ell)} \longrightarrow \frac{\operatorname{ann}_{M_\eta}(\ell)}{\ell M_\eta \cap \operatorname{ann}_{M_\eta}(\ell)}\,.$$

I.e., lifting x to $\tilde{x} \in M$, we have $\varphi(\tilde{x}) = \ell \cdot y$ for some $y \in M_\eta^{i-1}$. Using (PD), we can identify $M_\eta[-1]$ with $\eta M \subseteq M$. Then this means that $\eta \cdot \tilde{x} = \eta \cdot \ell \cdot y$ in M^{i+1}. By (HL), as $i < d/2$, this implies that $\tilde{x} = \ell \cdot y$, so x vanishes in $\mathrm{Gr}^{i,2i}$, as required. $\square$

From the definitions of the bilinear forms $\underline{\mathcal{Q}}$ and $\underline{\mathcal{Q}}_\eta$, we have the following compatibility.

Corollary 4.13. For all $x \in \mathrm{Gr}^{i,j}$ and $y \in \mathrm{Gr}^{d-i-1,2d-j-2}$, we have

$$\underline{\mathcal{Q}}_\eta(\mathrm{Gr}\,\varphi(x), \mathrm{Gr}\,\varphi(y)) = \underline{\mathcal{Q}}(x, \eta * y).$$

5. Relative Hard Lefschetz, Relative Hodge–Riemann, and Decomposition

In this section, we prove the relative Hard Lefschetz property and the relative Hodge-Riemann relations, stated in Theorem 1.5 and Theorem 1.7, as well as the Decomposition Theorem 1.10. We first prove Theorem 1.5 and Theorem 1.7 simultaneously by induction on the degree d of the Lefschetz module M, and then we deduce Theorem 1.10.

Proposition 5.1. Suppose that Theorems 1.5 and 1.7 hold for every Lefschetz module of degree at most $d-1$. Then, for all $0 \leqslant j \leqslant d$ and for all $\eta \in \mathcal{K}_A$, the linear map

$$\mathrm{Gr}^{\bullet,j} \longrightarrow \mathrm{Gr}^{\bullet+d-j,2d-j}, \qquad x \longmapsto \eta^{d-j} * x$$

is an isomorphism.

Proof. We first show that the map is injective. Choose some $\ell \in \mathcal{K}_B$. By Corollary 4.5, there are $\mathfrak{sl}_2$ actions on $\mathrm{Gr}^{\bullet,j}$ and $\mathrm{Gr}^{\bullet+d-j,2d-j}$ where ℓ acts as the raising operator, and $\eta^{d-j}*$ is a map of representations of $\mathfrak{sl}_2$. In order to check that this map is injective, it suffices to check that it is injective on $\ker(\ell^{j-2i+1}\colon \mathrm{Gr}^{i,j} \to \mathrm{Gr}^{j-i+1,j})$ for each $i \leqslant j/2$. If $j = d$, then $\eta^{d-j}*$ is the identity, so we may assume $j < d$. Suppose we have some element

$$x \in \ker(\ell^{j-2i+1}\colon \mathrm{Gr}^{i,j} \to \mathrm{Gr}^{j-i+1,j}) \cap \ker(\eta^{d-j}*\colon \mathrm{Gr}^{i,j} \to \mathrm{Gr}^{i+d-j,2d-j}).$$

Let $\varphi\colon M \to M_\eta = M/\operatorname{ann}_M(\eta)$ be the quotient map, and let $\mathrm{Gr}\,\varphi\colon \mathrm{Gr} \to \mathrm{Gr}_\eta$ be the associated graded, see Proposition 4.12. By construction, we have

$$\mathrm{Gr}\,\varphi(x) \in \ker(\ell^{j-2i+1}\colon \mathrm{Gr}_\eta^{i,j} \to \mathrm{Gr}_\eta^{j-i+1,j}) \cap \ker(\eta^{d-j-1}*\colon \mathrm{Gr}_\eta^{i,j} \to \mathrm{Gr}_\eta^{i+d-j,2d-j}).$$

I.e., $\mathrm{Gr}\,\varphi(x)$ lies in the (i,j)th graded piece of the primitive part of Gr_η. By induction on the degree d, we have

$$(-1)^i \underline{\mathcal{Q}}_\eta(\mathrm{Gr}\,\varphi(x), \eta^{d-j-1} * \ell^{j-2i}\,\mathrm{Gr}\,\varphi(x)) \geqslant 0,$$

with equality if and only if $\mathrm{Gr}\,\varphi(x) = 0$. By Corollary 4.13, we have

$$\underline{\mathcal{Q}}_\eta(\mathrm{Gr}\,\varphi(x), \eta^{d-j-1} * \ell^{j-2i}\,\mathrm{Gr}\,\varphi(x)) = \underline{\mathcal{Q}}(x, \eta^{d-j} * \ell^{j-2i} x) = 0,$$

so we get that $\mathrm{Gr}\,\varphi(x) = 0$. By the injectivity part of Proposition 4.12, we get that $x = 0$.

We have shown that $\eta^{d-j}*$ is injective. Multiplying by ℓ^{j-2i} and using Corollary 4.5, we have an injection

$$\eta^{d-j} * \ell^{j-2i}\colon \mathrm{Gr}^{i,j} \hookrightarrow \mathrm{Gr}^{d-i,2d-j}.$$

By Proposition 4.9, $\dim \mathrm{Gr}^{i,j} = \dim \mathrm{Gr}^{d-i,2d-j}$, so this map is an isomorphism. This proves that $\eta^{d-j}*$ is an isomorphism. □

Once we know that the conclusion of Proposition 5.1 holds for a Lefschetz module M, then we may decompose Gr into primitive pieces as in (1.5).

Proposition 5.2. Suppose that Theorem 1.5 and Theorem 1.7 hold for every Lefschetz module of degree at most $d-1$. If $(i,j) \neq (d/2,d)$, then for any $\eta \in \mathcal{K}_A$ and any $\ell \in \mathcal{K}_B$, the symmetric bilinear form on $\mathrm{Gr}^{i,j}$ defined by

$$(x,y) \longmapsto (-1)^i \underline{\mathcal{Q}}(x, \eta^{d-j} * \ell^{j-2i} y)$$

is positive definite when restricted to $\mathrm{Prim}^{i,j}$.

Proof. Let $x \in \mathrm{Prim}^{i,j}$ be a nonzero element. First suppose that $j < d$. Then, by Proposition 4.12, there is an injective map $\mathrm{Gr}\,\varphi\colon \mathrm{Gr}^{i,j} \to \mathrm{Gr}^{i,j}_{\eta}$. Because Theorem 1.7 holds for M_η, noting that $\mathrm{Gr}\,\varphi(x)$ remains primitive, we have that

$$(-1)^i \underline{\mathcal{Q}}_\eta(x, \eta^{d-j-1} * \ell^{j-2i} x) > 0.$$

The result then follows from Corollary 4.13. The case when $j = d$, but $i < d/2$ is identical, except using Proposition 4.10 and Corollary 4.11. □

We now deal with the remaining primitive part, $\mathrm{Prim}^{d/2,d}$, in the case when d is even. We do this by an analysis of the signature of the restriction of $\underline{\mathcal{Q}}$ to $\mathrm{Gr}^{d/2,\bullet}$. A different approach to the same problem is given in [dCM05, Section 5.4]. We begin by computing the signature of the restriction of $\underline{\mathcal{Q}}$ to $\mathrm{Gr}^{d/2,\bullet}$.

Proposition 5.3. The signature of the restriction of $\underline{\mathcal{Q}}$ to $\mathrm{Gr}^{d/2,\bullet}$ is

$$\sum_{i=0}^{d/2} (-1)^i (\dim M^i - \dim M^{i-1}).$$

Proof. Let $b_j = \dim \mathrm{Gr}^{d/2,j}$. Note that Proposition 4.9 implies that $b_j = b_{2d-j}$. Let $r_j = \sum_{e \leq j} b_e$, and let $r = \dim M^{d/2}$. Choose a basis $x_1, x_2, \ldots, x_r$ for $M^{d/2}$ so that $x_{r_{j-1}+1}, \ldots, x_{r_j}$ lie in $P_j \cap M^{d/2}$; then the x_i also give a basis for $\mathrm{Gr}^{d/2,\bullet}$. Let T be the symmetric matrix with $T_{s,t} = \mathcal{Q}(x_s, x_t)$, so T represents the restriction of $\mathcal{Q}$ to $M^{d/2}$. Note that (HR) implies that the signature of T is

$$\sum_{i=0}^{d/2} (-1)^i (\dim M^i - \dim M^{i-1}).$$

We can divide T into blocks, representing how different pieces of the perverse filtration pair with each other. By Lemma 4.7, T vanishes northwest of the blocks along the main antidiagonal, which represent the pairing of complementary pieces of the perverse filtration.

The matrix representing the restriction of $\underline{\mathcal{Q}}$ to $\mathrm{Gr}^{d/2,\bullet}$ is obtained by setting the entries in the blocks below the antidiagonal to 0. In particular, $\underline{\mathcal{Q}}$ has the same signature as $\mathcal{Q}$, giving the result. □

Assuming Theorem 1.5 and 1.7 for all Lefschetz modules of degree at most $d-1$, Proposition 5.1 and 5.2 imply that we have a decomposition

$$\mathrm{Gr}^{d/2,\bullet} = \bigoplus_{\substack{j\leqslant d\\ i\leqslant j/2}} \bigoplus_{\substack{s\leqslant d-j\\ t\leqslant j-2i\\ s+t=d/2-i}} \eta^s * \ell^t \,\mathrm{Prim}^{i,j}\,.$$

This decomposition is close to being orthogonal with respect to $\underline{\mathcal{Q}}$: we have

$$\underline{\mathcal{Q}}(\eta^s * \ell^t \,\mathrm{Prim}^{i,j}, \eta^{s'} * \ell^{t'} \,\mathrm{Prim}^{i',j'}) = 0$$

unless $i=i'$, $j=j'$, $s+s'=d-j$, and $t+t'=j-2i$. We group the summand $\eta^s * \ell^t \,\mathrm{Prim}^{i,j}$ with the summand $\eta^{d-j-s} * \ell^{j-2i-t}\,\mathrm{Prim}^{i,j}$. Because the signature of a matrix of the form $\begin{pmatrix} 0 & U\\ U & 0\end{pmatrix}$, where U is symmetric, is always 0, a summand where $s \neq d-j-s$ (which is equivalent to $t \neq j-2i-t$) contributes 0 to the signature. In particular, the signature of $\underline{\mathcal{Q}}$ is the same as the signature of the restriction of $\underline{\mathcal{Q}}$ to $\mathrm{Gr}^{d/2,d}$. The decomposition of $\mathrm{Gr}^{d/2,d}$ is simpler: we have

$$\mathrm{Gr}^{d/2,d} = \bigoplus_{\substack{j\leqslant d \text{ even}\\ i\leqslant j/2}} \eta^{(d-j)/2} * \ell^{j/2-i}\,\mathrm{Prim}^{i,j}\,.$$

Lemma 5.4. We have

$$\sum_{i=0}^{d/2}(-1)^i(\dim M^i - \dim M^{i-1}) = \sum_{\substack{j\leqslant d \text{ even}\\ i\leqslant j/2}} (-1)^i \dim \mathrm{Prim}^{i,j}\,.$$

Proof. We decompose M as in (1.5) and compute the contribution to the left-hand side of the terms associated to $\mathrm{Prim}^{i,j}$. For each p, the dimension of the sum of the pieces associated to $\mathrm{Prim}^{i,j}$ in $\mathrm{Gr}^{p,\bullet}$ is $\dim \mathrm{Prim}^{i,j}$ times the number of pairs (s,t) with $s+t=p-i$, $0\leqslant s\leqslant d-j$, and $0\leqslant t\leqslant j-2i$.

The number of such pairs is 0 for $p<i$, increases by one when we increase p by one until p reaches $\min(d-j, j-2i)$ (which happens for some $p\leqslant d/2$), and then is constant until p reaches $d/2$. We see that the contribution to the left-hand side of the terms associated to $\mathrm{Prim}^{i,j}$ is

$$\sum_{0\leqslant q\leqslant \min(d-j,j-2i)} (-1)^{i+q}\dim \mathrm{Prim}^{i,j}\,.$$

Note that, because d is even, $d-j$ and $j-2i$ both have the same parity as j, so this sum is 0 if j is odd and is $(-1)^i \dim \mathrm{Prim}^{i,j}$ if j is even. $\square$

Proposition 5.5. Suppose that Theorem 1.5 and Theorem 1.7 hold for every Lefschetz module of degree at most $d-1$. Suppose that d is even. Then, for any $\eta \in \mathcal{K}_A$ and $\ell \in \mathcal{K}_B$, the restriction of $(-1)^{d/2}\underline{\mathcal{Q}}$ to $\mathrm{Prim}^{d/2,d}$ is positive definite.

Proof. Let n be the signature of $\mathcal{Q}$ on $\mathrm{Prim}^{d/2,d}$. By Proposition 5.3 and Lemma 5.4, the signature of $\mathcal{Q}$ on $\mathrm{Gr}^{d/2,\bullet}$ is

$$\sum_{i=0}^{d/2}(-1)^i(\dim M^i - \dim M^{i-1}) = \sum_{\substack{j\leqslant d \text{ even}\\ i\leqslant j/2}} (-1)^i \dim \mathrm{Prim}^{i,j}\,.$$

On the other hand, using Proposition 5.2 and the discussion above about the decomposition being nearly orthogonal with respect to $\mathcal{Q}$, the signature of $\mathcal{Q}$ on $\mathrm{Gr}^{d/2,\bullet}$ is

$$n + \sum_{\substack{j\leqslant d \text{ even}\\ i\leqslant j/2\\ (i,j)\neq(d/2,d)}} (-1)^i \dim \mathrm{Prim}^{i,j}\,.$$

Comparing these expressions gives that $n = (-1)^{d/2}\dim \mathrm{Prim}^{d/2,d}$. □

Proof of Theorem 1.5 and Theorem 1.7. We induct on d. Both statements are trivial when $d = 0$. If Theorems 1.5 and 1.7 hold for all Lefschetz modules of degree at most $d-1$, then Propositions 5.1, 5.2, and 5.5 imply that the conclusions of Theorem 1.5 and Theorem 1.7 hold for all $\eta \in \mathcal{K}_A$. Viewed as a subset of A^1, the action of B^1 on Gr is 0. Then the result follows, as $\eta \in \mathcal{K}_{A/B}$ if and only if there is $b \in B^1$ such that $\eta + b \in \mathcal{K}_A$. □

We now prove Theorem 1.10. We show that it is a formal algebraic consequence of Theorem 1.5, following ideas of Deligne [Del68]. See also [VdB04, dC13].

Proof of Theorem 1.10. Choose $\eta \in \mathcal{K}_{A/B}$. We produce a splitting as a graded R-module of the inclusion $P_0 \hookrightarrow M$ by the following composition:

$$M \xrightarrow{\eta^d} M \longrightarrow \mathrm{Gr}^{\bullet,2d} = M/P_{2d-1} \xrightarrow{(\eta^d)^{-1}} P_0.$$

This induces a direct sum decomposition $M = P_0 \oplus M' = \mathrm{Gr}^{\bullet,0} \oplus M'$ as graded R-modules. The perverse filtration on M restricts to a filtration on M' whose associated graded is $\oplus_{j=1}^{2d}\mathrm{Gr}^{\bullet,j}$. We have a splitting of the map $M' \to \mathrm{Gr}^{\bullet,2d}$ given by the composition

$$\mathrm{Gr}^{\bullet,2d} \xrightarrow{(\eta^d)^{-1}} \mathrm{Gr}^{\bullet,0} \longrightarrow M \xrightarrow{\eta^d} M \longrightarrow M'.$$

This induces a direct sum decomposition $M' = \mathrm{Gr}^{\bullet,2d} \oplus M''$ as graded R-modules. Through the same procedure, we can split off $\mathrm{Gr}^{\bullet,1}$ and $\mathrm{Gr}^{\bullet,2d-1}$, and so on. This proves the theorem. □

6. Duality, polarization, and simplicity

In this section, we prove Theorems 1.1, 1.2, 1.3, and 1.4. Our strategy is to use Theorem 1.5 and Theorem 1.7 to construct a bilinear form on N_α so that it is a Lefschetz module over $(B, \mathcal{K}_B)$. This proves Theorem 1.3. We then use this Lefschetz module and results of Looijenga–Lunts and

Verbitsky [LL97, Ver95] to prove Theorem 1.1. We then use this to deduce the uniqueness of the bilinear form, proving Theorem 1.2 and 1.4.

Before continuing, we will need to establish some elementary facts about Lefschetz modules. The following two lemmas are immediate.

Lemma 6.1. Let N be a Lefschetz module of degree e over $(B, \mathcal{K}_B)$, equipped with a bilinear form $\mathcal{Q}_N$. Let $k \in \mathbb{Z}$ be such that $N[-k]_i = 0$ for all $i < 0$. Then $N[-k]$, equipped with the bilinear form $(-1)^k \mathcal{Q}_N$, is a Lefschetz module of degree $e + 2k$ over $(B, \mathcal{K}_B)$.

Lemma 6.2. Let N_1 and N_2 be Lefschetz modules of degree e over $(B, \mathcal{K}_B)$, equipped with bilinear forms $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then $N_1 \oplus N_2$, equipped with $\mathcal{Q}_1 \oplus \mathcal{Q}_2$, is a Lefschetz module of degree e over $(B, \mathcal{K}_B)$.

We will also need the following result.

Lemma 6.3. Let N be a Lefschetz module of degree e over $(B, \mathcal{K}_B)$, equipped with a bilinear form $\mathcal{Q}_N$. Then any graded B-module summand of N, equipped with the restriction of $\mathcal{Q}_N$, is a Lefschetz module of degree e over $(B, \mathcal{K}_B)$.

Proof. Suppose we have a graded B-module decomposition $N = N_1 \oplus N_2$. For each $i \leqslant e/2$ and $\ell \in \mathcal{K}_B$, the map $N_1^i \to N_1^{e-i}$ given by multiplication by ℓ^{e-2i} is injective, so $\dim N_1^i \leqslant \dim N_1^{e-i}$. Similarly, $\dim N_2^i \leqslant \dim N_2^{e-i}$. As $\dim N^i = \dim N_1^i + \dim N_2^i$ and $\dim N^i = \dim N^{e-i}$ by (HL), we see that $\dim N_1^i = \dim N_1^{e-i}$, and so (HL) holds for N_1. For any ℓ, we have

$$\ker(\ell^{e-2i+1} \colon N_1^i \to N_1^{e-i+1}) = N_1^i \cap \ker(\ell^{e-2i+1} \colon N^i \to N^{e-i+1}).$$

In particular, by (HR) for N, $\mathcal{Q}_N$ is definite on this subspace of N_1^i. This implies (PD) and (HR) for N_1, and the lemma follows. $\square$

6.1. **Duality.** Choose $\eta \in \mathcal{K}_{A/B}$. By Theorem 1.5, we have a primitive decomposition with respect to η, as follows. Set $\mathrm{K}_\eta^j = \ker(\eta^{d-j+1} \colon \mathrm{Gr}^{\bullet,j} \to \mathrm{Gr}^{\bullet+d-j+1, 2d-j+2})$. Then we have a decomposition

$$\mathrm{Gr} \simeq \bigoplus_{0 \leqslant j \leqslant d} \bigoplus_{0 \leqslant i \leqslant d-j} \eta^i * \mathrm{K}_\eta^j . \tag{6.1}$$

Because $\eta *$ is a map of B-modules, this is an isomorphism of B-modules.

We equip each K_η^j with the symmetric bilinear form $(x, y) \mapsto \underline{\mathcal{Q}}(x, \eta^{d-j} * y)$. Recall that by Corollary 1.8, each K_η^j is a Lefschetz module of degree j over $(B, \mathcal{K}_B)$.

Proposition 6.4. Each N_α admits a B-invariant symmetric bilinear form $\mathcal{Q}_\alpha \colon N_\alpha \times N_\alpha \to \mathbb{R}$ so that it is a Lefschetz module of degree $d(\alpha)$ over $(B, \mathcal{K}_B)$.

Proof. Refine the decomposition given in (6.1) of Gr into a decomposition into indecomposable B-modules. By Theorem 1.10, Gr is isomorphic to M as graded B-modules, so by the Krull–Schmidt theorem, some summand must be isomorphic to a shift of N_α. For each i, $\eta^i * \mathrm{K}_\eta^j$ is either isomorphic to K_η^j as B-modules or is 0. The primitive decomposition then implies that there is some k and j such that $N_\alpha[-k]$ is a summand of K_η^j; we must have $2k + d(\alpha) = j$.

By Lemma 6.3, $N_\alpha[-k]$ is equipped with a bilinear form that gives it the structure of a Lefschetz module of degree $2k + d(\alpha)$ over $(B, \mathcal{K}_B)$. The result then follows from Lemma 6.1. □

Although we do not yet know the uniqueness of the symmetric bilinear form, and so we cannot formulate Theorem 1.4, we know enough to prove Theorem 1.3.

Proof of Theorem 1.3. This is immediate from Proposition 6.4 □

6.2. **Simplicity.** We now associate a Lie algebra to a Lefschetz module N of degree e over $(B, \mathcal{K}_B)$, using a construction introduced by Looijenga–Lunts and Verbitsky [LL97, Ver95].

For each $\ell \in \mathcal{K}_B$, the fact that multiplication by $\ell^{e-2i} \colon N^i \to N^{e-i}$ is an isomorphism means that there is a corresponding representation of the Lie algebra $\mathfrak{sl}_2$, where the raising operator corresponds to multiplication by ℓ. Let $\mathfrak{g}_N$ be the Lie subalgebra of $\mathrm{End}(N)$ generated by the raising and lowering operators associated to all $\ell \in \mathcal{K}_B$. The following result was proved independently by Looijenga–Lunts and Verbitsky.

Proposition 6.5 ([LL97, Proposition 1.6])**.** Let N be a Lefschetz module of degree e over $(B, \mathcal{K}_B)$. If $e > 0$, then $\mathfrak{g}_N$ is a semisimple Lie algebra.

The Lie algebra $\mathfrak{g}_N$ has a distinguished semisimple element H, the common semisimple element in all of the $\mathfrak{sl}_2$ triples that generate $\mathfrak{g}_N$. The action of H on N records the grading.

If N_1, N_2 are Lefschetz modules of degree e, then, by Lemma 6.2, $N = N_1 \oplus N_2$ is a Lefschetz module of degree e. The action of $\mathfrak{g}_N$ preserves N_1 and N_2, and we see from the construction of $\mathfrak{g}_N$ that there are surjective homomorphisms $\mathfrak{g}_N \to \mathfrak{g}_{N_1}$ and $\mathfrak{g}_N \to \mathfrak{g}_{N_2}$.

We will now show that, in many cases, morphisms of B-modules between Lefschetz modules of the same degree can be upgraded to morphisms of representations of these Lie algebras.

Proposition 6.6. Let N_1, N_2 be Lefschetz modules of degree e over $(B, \mathcal{K}_B)$. Let $\varphi \colon N_1 \to N_2$ be a map of abelian groups. Then φ is a map of graded B-modules if and only if it is a map of $\mathfrak{g}_{N_1 \oplus N_2}$ representations.

Proof. First suppose that φ is a map of $\mathfrak{g}_{N_1 \oplus N_2}$ representations. This implies that φ commutes with the action of any $\ell \in \mathcal{K}_B$. Because $\mathcal{K}_B$ is open, we see that φ commutes with the action of any $b \in B^1$. Because B is generated by B^1, we see that φ is a map of B-modules. Because the action of the distinguished semisimple element commutes with φ, we see that φ respects the grading.

Now suppose that φ is a map of graded B-modules. Let $\Gamma \subseteq N_1 \oplus N_2$ be the graph of φ. It is enough to show that Γ is a $\mathfrak{g}_{N_1 \oplus N_2}$-subrepresentation.

For each $\ell \in \mathcal{K}_B$, we can choose a decomposition of N_1 into indecomposable graded $\mathbb{R}[\ell]$-modules. Because N_1 satisfies (HL), each summand in the decomposition will be isomorphic to $\mathbb{R}[\ell]/(\ell^{f+1})[-k]$ for some f and k with $2k+f=e$. By (HL), every $\mathbb{R}[\ell]$-submodule of N_2 is of the form $\mathbb{R}[\ell]/(\ell^{f'+1})[-k']$ for some f' and k' with $2k'+f'=e$. There is no nonzero homomorphism of graded $\mathbb{R}[\ell]$-modules from $\mathbb{R}[\ell]/(\ell^{f+1})[-k]$ to $\mathbb{R}[\ell]/(\ell^{f'+1})[-k']$ unless $f'=f$ and $k'=k$, so the action of the raising and lowering operators on $N_1 \oplus N_2$ must preserve each $\mathbb{R}[\ell]/(\ell^{f+1})[-k]$ in Γ. This implies Γ is preserved by the corresponding $\mathfrak{sl}_2$.

For each $\ell \in \mathcal{K}_B$, the action of the $\mathfrak{sl}_2$ commutes with φ. As these $\mathfrak{sl}_2$ subgroups generate $\mathfrak{g}_{N_1 \oplus N_2}$, we see that $\mathfrak{g}_{N_1 \oplus N_2}$ preserves Γ. □

By Lemma 6.2, a direct summand of a Lefschetz module N of degree e is a Lefschetz module of degree e. Then Proposition 6.6 implies that this direct summand is a $\mathfrak{g}_N$ subrepresentation. Because $\mathfrak{g}_N$ is semisimple if $e > 0$, any $\mathfrak{g}_N$ subrepresentation is a $\mathfrak{g}_N$ summand, and so a graded B-module summand by Proposition 6.6. In particular, we have the following result.

Corollary 6.7. Let N be a Lefschetz module over $(B, \mathcal{K}_B)$ of degree e. Then N is an irreducible $\mathfrak{g}_N$ representation if and only if N is indecomposable as a graded B-module.

Proof of Theorem 1.1. By shifting both N_α and $N_\beta[-k]$, we can assume that $N_\beta[-k]$ is nonnegatively graded and so is a Lefschetz module. The statement is clear if $e=0$.

We assume that $e > 0$. If $d(\alpha)+2k = d(\beta)$, then N_α and $N_\beta[-k]$ are Lefschetz modules of the same degree. Let $\mathfrak{g}$ be the semisimple Lie algebra associated to $N_\alpha \oplus N_\beta[-k]$. Because they are indecomposable, Corollary 6.7 implies N_α and $N_\beta[-k]$ are irreducible representations of $\mathfrak{g}$. In particular, $\mathrm{Hom}_{\mathfrak{g}}(N_\alpha, N_\beta[-k]) = 0$ unless they are isomorphic, i.e., unless $\alpha = \beta$. By Proposition 6.6, this implies that $\mathrm{Hom}_B(N_\alpha, N_\beta[-k]) = 0$ unless $\alpha = \beta$. If $\alpha = \beta$, then we have $k=0$. Because they are irreducible representations of the semisimple Lie algebra $\mathfrak{g}$, we have $\mathrm{Hom}_{\mathfrak{g}}(N_\alpha, N_\alpha) = \mathrm{Hom}_B(N_\alpha, N_\alpha)$ is isomorphic to $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

Now suppose that $d(\alpha)+2k < d(\beta)$, and let $\varphi\colon N_\alpha \to N_\beta[-k]$ be a map of graded B-modules. Choose some $\ell \in \mathcal{K}_B$. By (HL), N_α is generated as a B-module by

$$\bigcup_{i \leqslant d(\alpha)/2} \ker(\ell^{d(\alpha)-2i+1}\colon N_\alpha^i \to N_\alpha^{d(\alpha)-i+1}).$$

For any $i \leqslant d(\alpha)/2$ and $x \in \ker(\ell^{d(\alpha)-2i+1}\colon N_\alpha^i \to N_\alpha^{d(\alpha)-i+1})$, we have $\ell^{d(\alpha)-i+1}\varphi(x) = 0$, so $\varphi(x) = 0$ by (HL) for N_β. This implies that $\varphi = 0$. □

Proof of Theorem 1.2. In the proof of Proposition 6.4, we have constructed a nondegenerate symmetric B-invariant bilinear form $\mathcal{Q}_\alpha$ which gives N_α the structure of a Lefschetz module. Let $\mathcal{Q}'$ be another symmetric B-invariant bilinear form on N_α which respects the grading. Because $\mathcal{Q}_\alpha$ is nondegenerate and $\mathcal{Q}'$ is B-invariant, there is an endomorphism $\phi\colon N_\alpha \to N_\alpha$ of graded B-modules such that, for all $x, y \in N_\alpha$, we have $\mathcal{Q}'(x,y) = \mathcal{Q}_\alpha(\phi(x), y)$. By Theorem 1.1, the ring of graded B-module endomorphisms of N_α is isomorphic to either $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. If it is isomorphic to $\mathbb{R}$, then $\mathcal{Q}'$ is a constant multiple of $\mathcal{Q}_\alpha$, as desired. Otherwise, we may assume that the endomorphism ring is isomorphic to either $\mathbb{C}$ or $\mathbb{H}$. By replacing $\mathcal{Q}'$ by $\mathcal{Q}' - \lambda\mathcal{Q}_\alpha$ for some $\lambda \in \mathbb{R}$, we may assume that ϕ is purely imaginary (and nonzero). In particular, $\phi^2 = c$ for some $c < 0$. We have

$$\mathcal{Q}_\alpha(x, \phi(y)) = \mathcal{Q}_\alpha(\phi(y), x) = \mathcal{Q}'(y, x) = \mathcal{Q}'(x, y) = \mathcal{Q}_\alpha(\phi(x), y),$$

so ϕ is self-adjoint with respect to $\mathcal{Q}_\alpha$. Choose some $\ell \in \mathcal{K}_B$. Suppose that N_α has degree e, and that the lowest nonzero graded piece of N_α is N_α^i. Choose some nonzero $x \in N_\alpha^i$. By Theorem 1.1, ϕ is an automorphism, so $\phi(x)$ is nonzero. We have

$$\mathcal{Q}_\alpha(\ell^{e-2i}\phi(x), \phi(x)) = \mathcal{Q}_\alpha(\ell^{e-2i}x, \phi^2(x)) = c\mathcal{Q}_\alpha(\ell^{e-2i}x, x).$$

As $c < 0$, this contradicts (HR). □

Proof of Theorem 1.4. In Proposition 6.4, we showed that N_α has the structure of a Lefschetz module over $(B, \mathcal{K}_B)$ for some B-invariant symmetric bilinear form $\mathcal{Q}_\alpha$. The definition of a Lefschetz module implies that (HR) holds for N_α when equipped with $\mathcal{Q}_\alpha$; we just need to prove the uniqueness part of Theorem 1.4. By Theorem 1.2, the choice of a B-invariant symmetric bilinear form is unique up to a constant. As $N_\alpha \neq 0$, $c\mathcal{Q}_\alpha$ does not satisfy (HR) for any $c < 0$, proving the uniqueness of ϵ_α in the statement of the theorem. □

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